

ON FINITE FORM OF ANDREW'S IDENTITY FOR BASIS
PARTITION POLYNOMIAL AND GENERALIZATION

ZhiZheng Zhang and HanFei Song*

Department of Mathematics,
Luoyang Normal University,
Luoyang, 471934, P. R. CHINA

E-mail : zhzhzhang-yang@163.com

*College of Mathematics and Information Science,
Henan Normal University,
Xinxiang, 453001, P. R. CHINA

E-mail : hanfeiSong00@163.com

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Abstract: Andrews gave a common generalization of the Rogers-Ramanujan series and the generating function for basis partitions. In this paper, we obtain a finite form and further generalization of this identity. As applications, some interesting identities are given.

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1. Introduction

Basis partitions are defined as successive ranks [6] or the “rank vector” of a partition and they were first considered by Gupta [10]. More precisely, an integer partition of n is a basis partition if, in the set of all partitions of n with a given rank vector, its weight is minimum.

A partition π of a positive integer n is a finite non-increasing sequence of positive integers $(\pi_1, \pi_2, \dots, \pi_l)$ such that $n = \pi_1 + \pi_2 + \dots + \pi_l$. We write $\pi = (\pi_1, \pi_2, \dots, \pi_l)$

and call π_i s is the parts of π . The size of π is the sum of all parts, which is denoted by $|\pi|$. The Ferrers graph of π is defined to be a left-justified arrangement of n dots in l rows consisting of $\pi_1, \pi_2, \dots, \pi_l$ dots. The Durfee square of π is the largest square subarray of dots in the Ferrers graph of π . If the Durfee square has side d , then every partition π with Durfee square of side d gives rise to a ‘successive rank vector’ $(r_1, r_2, \dots, r_d)$. Note that $r_k = x_k - y_k$, where x_k is the k th largest part of π and y_k is the k th largest part of the conjugate partition of π .

On the contrary, given a vector $\mathbf{r} = (r_1, \dots, r_d)$, there is a unique partition π_0 of minimal size called the basis partition with $\mathbf{r}$ as its successive rank vector. In [14], Nolan et al. showed that if $b(n)$ denotes the number of basis partitions of n , then the generating function is

$$\sum_{n=0}^{\infty} b(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q)_n}{(q)_n}. \quad (1)$$

In [11], Hirschhorn interpreted (1) as a weighted partition identity connecting $b(n)$ with the Rogers-Ramanujan partitions of n with weights as powers of 2. Recently, Andrews [1] studied a common generalization of the Rogers-Ramanujan series and the generating function for basis partitions and deduced the following identity.

Theorem 1.1. [1, Theorem 1]

$$\sum_{n=0}^{\infty} \frac{\lambda^n q^{n^2} (x)_n}{(q)_n} = \frac{1}{(\lambda q)_{\infty}} \left(1 + \sum_{n=1}^{\infty} \frac{(\lambda q)_{n-1} (1 - \lambda q^{2n}) (-1)^n q^{n(3n-1)/2} \lambda^n}{(q)_n} B_n(\lambda, x) \right). \quad (2)$$

where

$$B_n(\lambda, x) = \sum_{k=0}^n \frac{(q^{-n}, \lambda q^n, x)_k q^k}{(q)_k} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (x)_k (\lambda q^n)^k x^{n-k}.$$

Here,

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{(q)_n}{(q)_k (q)_{n-k}}.$$

Andrews called the $B_n(\lambda, x)$, the Basis Partition polynomial, abbreviated BsP-polynomial. It associated with Rogers-Ramanujan identities, the basis partition generating function and overpartitions.

In order to facilitate the subsequent application, we recall some standard notations for the basic hypergeometric series in [8]:

$$(a; q)_0 = 1, \quad (a; q)_n = (a)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = (a)_\infty = \prod_{k=0}^{\infty} (1 - aq^k),$$

and we also adopt the following compact notation for the multiple q -shifted factorial:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$

where n is an integer or ∞ . The basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r)_n}{(q, b_1, b_2, \dots, b_s)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n.$$

The remainder of the paper is structured as follows. In Section 2, we deduce a finite form of Andrews's identity (2) by applying the method of Carlitz inversion technique. An extension of the equation (2) be given in Section 3 and as applications, several of special cases will be discussed in Section 4.

2. A finite form of Andrews's identity (2)

In this section, we obtain a finite form of the identity (2). To begin our main result, we first list some results of Carlitz inversion for use later.

In [7], Carlitz gave a useful q -analogue of the result that Gould and Hsu [9] found a pair of reciprocal formulas.

Lemma 2.1. (*Carlitz inversion*) Let $\{a_i\}$ and $\{b_j\}$ be any two sequences of numbers and the sequences of q -polynomials

$$\phi(x, n) = \prod_{i=1}^n (a_i + q^x b_i), \quad (n = 1, 2, \dots)$$

be different from zero for nonnegative integers x and n with $\phi(x, 0) = 1$. Then there hold the following inverse relations:

$$f(n) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \phi(k, n) g(k), \quad (3)$$

$$g(n) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{a_{k+1} + q^k b_{k+1}}{\phi(n, k+1)} f(k). \quad (4)$$

Theorem 2.2. *We have*

$$\sum_{n=0}^m \begin{bmatrix} m \\ n \end{bmatrix} \lambda^n q^{n^2} (x)_n = \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{(\lambda)_k}{(\lambda)_{m+k+1}} (1 - \lambda q^{2k}) \lambda^k q^{k^2 + \binom{k}{2}} B_k(\lambda, x).$$

Proof. Setting $a_k = 1$, $b_k = -\lambda q^{k-1}$ in (3) and (4), then the following inverse relation holds.

$$f(n) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (\lambda q^k)_n g(k), \quad (5)$$

$$g(n) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1 - \lambda q^{2k}}{(\lambda q^n)_{k+1}} f(k). \quad (6)$$

and after elementary simplification by using (5) and (6), we have

$$\sum_{n=0}^m \begin{bmatrix} m \\ n \end{bmatrix} \frac{\lambda^n q^{n^2}}{(\lambda)_n} g(n) = \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1 - \lambda q^{2k}}{(\lambda)_{m+k+1}} \lambda^k q^{k^2} f(k) \quad (7)$$

Taking $g(n) = (\lambda, x)_n$ in (5), then

$$\begin{aligned} f(n) &= \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (\lambda q^k)_n (\lambda, x)_k \\ &= (\lambda)_n q^{\binom{n}{2}} \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (\lambda q^n, x)_k q^{\binom{k+1}{2} - kn}. \end{aligned}$$

Substituting $f(n)$ and $g(n)$ into the equation (7), we can obtain the desired identity. Note that when $m \rightarrow \infty$ in Theorem (2.2), we can immediately obtain the identity (2). Letting $x = 0$ in Theorem (2.2), then we obtain

$$\sum_{n=0}^m \begin{bmatrix} m \\ n \end{bmatrix} \lambda^n q^{n^2} = \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1 - \lambda q^{2k}}{(\lambda q^k)_{m+1}} \lambda^{2k} q^{2k^2 + \binom{k}{2}}. \quad (8)$$

Furthermore, setting $\lambda = 1$ or q in (8), the equality reduces to the finite forms of well-known Rogers-Ramanujan identities:

$$\begin{aligned} \sum_{n=0}^m \begin{bmatrix} m \\ n \end{bmatrix} q^{n^2} &= \frac{(q)_m}{(q)_{2m}} \sum_{k=-m}^m (-1)^k \begin{bmatrix} 2m \\ m+k \end{bmatrix} q^{\binom{k}{2} + 2k^2}, \\ \sum_{n=0}^m \begin{bmatrix} m \\ n \end{bmatrix} q^{n^2+n} &= \frac{(q)_m}{(q)_{2m+1}} \sum_{k=-m}^{m+1} (-1)^k \begin{bmatrix} 2m+1 \\ m+k \end{bmatrix} q^{\binom{k}{2} + 2k^2 - k}. \end{aligned}$$

3. An extension of Andrews's identity (2)

First, let us recall the definition of the Bailey pair and Bailey lemma.

The sequences (α_n, β_n) are called a Bailey pair relative to (a, q) if (α_n, β_n) satisfy

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}. \quad (9)$$

for all $n \geq 0$. Moreover, if only the β_n are given, then α_n can be determined using Bailey inversion:

$$\alpha_n = \frac{(-1)^n q^{\binom{n}{2}} (1 - aq^{2n})(a)_n}{(q)_n(1 - a)} \sum_{j=0}^n (q^{-n}, aq^n)_j q^j \beta_j, \quad (10)$$

Bailey lemma says that if (α_n, β_n) be a Bailey pair with respect to a , then so is the new Bailey pair (α'_n, β'_n) given by

$$\alpha'_n = \frac{(\rho)_n(\sigma)_n \left(\frac{aq}{\rho\sigma}\right)^n}{(aq/\rho)_n(aq/\sigma)_n} \alpha_n, \quad (11)$$

for all $n \geq 0$, and

$$\beta'_n = \sum_{j=0}^n \frac{(\rho)_j(\sigma)_j(aq/\rho\sigma)_{n-j}(aq/\rho\sigma)^j}{(aq/\rho)_n(aq/\sigma)_n(q)_{n-j}} \beta_j, \quad (12)$$

Substituting (11) and (12) into (9) and letting $n \rightarrow \infty$, we have

$$\sum_{n=0}^{\infty} (\rho)_n(\sigma)_n(aq/\rho\sigma)^n \beta_n = \frac{(aq/\rho, aq/\sigma)_{\infty}}{(aq, aq/\rho\sigma)_{\infty}} \sum_{n=0}^{\infty} \frac{(\rho)_n(\sigma)_n(aq/\rho\sigma)^n}{(aq/\rho)_n(aq/\sigma)_n} \alpha_n. \quad (13)$$

More details of the Bailey pair and Bailey lemma, see [2, 16].

Next, we obtain an extension of the equality (2) by using the Bailey pair.

Theorem 3.1. *We have*

$$\begin{aligned} & \frac{(\lambda q, \lambda ab/q)_{\infty}}{(\lambda a, \lambda b)_{\infty}} \sum_{n=0}^{\infty} \frac{(q/a, q/b, x)_n (\lambda ab/q)^n}{(q)_n} \\ &= \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{2n})(\lambda, q/a, q/b)_n (-\lambda ab/q)^n q^{\binom{n}{2}}}{(1 - \lambda)(q, \lambda a, \lambda b)_n} B_n(\lambda, x). \end{aligned} \quad (14)$$

Proof. Taking $\beta_n = \frac{(x)_n}{(q)_n}$ and using (10), we have

$$\alpha_n = \frac{(-1)^n q^{\binom{n}{2}} (1 - \lambda q^{2n}) (\lambda)_n}{(q)_n (1 - \lambda)} \sum_{j=0}^n \frac{(q^{-n}, \lambda q^n, x)_j}{(q)_j} q^j,$$

then (α_n, β_n) form a Bailey pair. Inserting the Bailey pair (α_n, β_n) into (13) with $a \rightarrow \lambda$, $\rho \rightarrow q/a$ and $\sigma \rightarrow q/b$, we find the right-hand side of this equation becomes

$$\frac{(\lambda a, \lambda b)_\infty}{(\lambda q, \lambda ab/q)_\infty} \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{2n}) (\lambda, q/a, q/b)_n (-\lambda ab/q)^n q^{\binom{n}{2}}}{(1 - \lambda) (q, \lambda a, \lambda b)_n} \sum_{j=0}^n \frac{(q^{-n}, \lambda q^n, x)_j}{(q)_j} q^j,$$

and the left-hand side of this equation becomes

$$\sum_{n=0}^{\infty} (q/a, q/b)_n (\lambda ab/q)^n \beta_n = \sum_{n=0}^{\infty} \frac{(q/a, q/b, x)_n (\lambda ab/q)^n}{(q)_n}.$$

We complete the proof of Theorem 3.1.

Remark: Note that Theorem 3.1 is also a corollary of a transformation formula from Liu's work. Liu [13] obtained the general transformation formula by using a q -expansion formula [12] which utilizes the Leibniz formula for the q -differential operator. If setting

$$A_n = \frac{(q/b, x)_n (b/q)^n}{(q)_n},$$

in [13, Theorem 1.6], we can also deduce Theorem 3.1.

Taking $a, b \rightarrow 0$, then Theorem 3.1 reduces to the identity (2). And, we can obtain a series of others identities associated with BsP-polynomial by using Theorem 3.1.

Setting $(a, b) = (1, 0)$, $(-1, 0)$, $(\sqrt{q}, -\sqrt{q})$ and $(-q, 0)$, respectively, in Theorem 3.1, we are led to the following theorem.

Theorem 3.2. *We have*

$$\sum_{n=0}^{\infty} (x)_n (-\lambda)^n q^{\binom{n}{2}} = \sum_{n=0}^{\infty} (1 - \lambda q^{2n}) \lambda^n q^{n^2-n} B_n(\lambda, x), \quad (15)$$

$$\sum_{n=0}^{\infty} \frac{(-q, x)_n \lambda^n q^{\binom{n}{2}}}{(q)_n} = \frac{(-\lambda)_{\infty}}{(\lambda)_{\infty}} \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{2n}) (-q, \lambda)_n (-\lambda)^n q^{n^2-n}}{(q, -\lambda)_n} B_n(\lambda, x), \quad (16)$$

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n (x)_n (-\lambda)^n}{(q)_n} = \frac{(\lambda^2 q; q^2)_{\infty}}{(\lambda^2; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{2n}) (\lambda)_n (q; q^2)_n \lambda^n q^{\binom{n}{2}}}{(q)_n (\lambda^2 q; q^2)_n} B_n(\lambda, x), \quad (17)$$

$$\sum_{n=0}^{\infty} \frac{(-1, x)_n \lambda^n q^{\binom{n+1}{2}}}{(q)_n} = \frac{(-\lambda q)_{\infty}}{(\lambda q)_{\infty}} \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{2n}) (-1, \lambda)_n (-\lambda)^n q^{n^2}}{(1 - \lambda)(q, -\lambda q)_n} B_n(\lambda, x). \quad (18)$$

Setting $(a, b, q) = (1, q, q^2)$, $(-q^2, -q, q^2)$ and $(q, 0, q^2)$, respectively, in Theorem 3.1, we obtain the following theorem.

Theorem 3.3. *We have*

$$\sum_{n=0}^{\infty} (x, q; q^2)_n (\lambda/q)^n = \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{4n}) (q; q^2)_n (-\lambda)^n q^{n^2-2n}}{(\lambda/q; q^2)_{n+1}} B_n(\lambda, x, q^2), \quad (19)$$

$$\sum_{n=0}^{\infty} \frac{(-1)_{2n} (x; q^2)_n (\lambda q)^n}{(q^2; q^2)_n} = \frac{(-\lambda q)_{\infty}}{(\lambda q)_{\infty}} \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{4n}) (-1)_{2n} (\lambda; q^2)_n (-\lambda)^n q^{n^2}}{(1 - \lambda)(-\lambda q)_{2n} (q^2; q^2)_n} B_n(\lambda, x, q^2), \quad (20)$$

$$\sum_{n=0}^{\infty} \frac{(q, x; q^2)_n (-\lambda)^n q^{n^2}}{(q^2; q^2)_n} = \frac{(\lambda q; q^2)_{\infty}}{(\lambda q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{4n}) (\lambda, q; q^2)_n \lambda^n q^{2n^2-n}}{(1 - \lambda)(\lambda q, q^2; q^2)_n} B_n(\lambda, x, q^2), \quad (21)$$

where

$$B_n(\lambda, x, q^2) = \sum_{k=0}^n \frac{(q^{-2n}, \lambda q^{2n}, x; q^2)_k q^{2k}}{(q^2; q^2)_k}.$$

4. Some applications of Theorems 3.2 and 3.3

In this section, we consider some special cases of Theorems 3.2 and 3.3. To this end, we need the following identity [13, Lemma 4.1]:

$$\sum_{k=0}^n \frac{(q^{-n}, \alpha q^n)_k q^k}{(cq)_k} = \alpha^n q^{n^2} \frac{(q)_n}{(cq)_n} \sum_{j=0}^n \frac{(c)_j \alpha^{-j} q^{j(1-n)}}{(q)_j}.$$

Setting $c = 1, x = 0$, we can obtain $B_n(\lambda, 0) = \lambda^n q^{n^2}$, $B_n(\lambda, 0, q^2) = \lambda^n q^{2n^2}$. So let $x = 0$ in Theorems 3.2 and 3.3, we obtain the following corollary.

Corollary 4.1. *We have*

$$\sum_{n=0}^{\infty} (-\lambda)^n q^{\binom{n}{2}} = \sum_{n=0}^{\infty} (1 - \lambda q^{2n}) \lambda^{2n} q^{2n^2 - n}, \quad (22)$$

$$\sum_{n=0}^{\infty} \frac{(-q)_n \lambda^n q^{\binom{n}{2}}}{(q)_n} = \frac{(-\lambda)_{\infty}}{(\lambda)_{\infty}} \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{2n})(-q, \lambda)_n (-1)^n \lambda^{2n} q^{2n^2 - n}}{(q, -\lambda)_n}, \quad (23)$$

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-\lambda)^n}{(q)_n} = \frac{(\lambda^2 q; q^2)_{\infty}}{(\lambda^2; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{2n})(\lambda)_n (q; q^2)_n \lambda^{2n} q^{n(3n-1)/2}}{(q)_n (\lambda^2 q; q^2)_n}, \quad (24)$$

$$\sum_{n=0}^{\infty} \frac{(-1)_n \lambda^n q^{\binom{n+1}{2}}}{(q)_n} = \frac{(-\lambda q)_{\infty}}{(\lambda q)_{\infty}} \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{2n})(-1, \lambda)_n (-1)^n \lambda^{2n} q^{2n^2}}{(1 - \lambda)(q, -\lambda q)_n}, \quad (25)$$

$$\sum_{n=0}^{\infty} (q; q^2)_n (\lambda/q)^n = \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{4n})(q; q^2)_n (-1)^n \lambda^{2n} q^{3n^2 - 2n}}{(\lambda/q; q^2)_{n+1}}, \quad (26)$$

$$\sum_{n=0}^{\infty} \frac{(-1)_{2n} (\lambda q)^n}{(q^2; q^2)_n} = \frac{(-\lambda q)_{\infty}}{(\lambda q)_{\infty}} \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{4n})(-1)_{2n} (\lambda; q^2)_n (-1)^n \lambda^{2n} q^{3n^2}}{(1 - \lambda)(-\lambda q)_{2n} (q^2; q^2)_n}, \quad (27)$$

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-\lambda)^n q^{n^2}}{(q^2; q^2)_n} = \frac{(\lambda q; q^2)_{\infty}}{(\lambda q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{4n})(\lambda, q; q^2)_n \lambda^{2n} q^{4n^2 - n}}{(1 - \lambda)(\lambda q, q^2; q^2)_n}. \quad (28)$$

Note. (23) can be obtained in [5, Entry 4.2.3] with $a = b = \lambda/q, c = 0$ and (25) can be obtained with $a = -\lambda/q, b = \lambda$ in [5, Entry 4.2.4], respectively. It is obvious that the Corollary 4.1 contains many identities as special cases.

(a). Setting $\lambda = q$ in (24), we conclude the following identity:

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q)^n}{(q)_n} = \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{3n(n+1)/2}. \quad (29)$$

Taking $a \rightarrow 0, b = q, t = -q$ in the following identity:

$$\sum_{n=0}^{\infty} \frac{(a)_n (b; q^2)_n t^n}{(q)_n (atb; q^2)_n} = \frac{(at, bt; q^2)_{\infty}}{(t, abt; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(a, b; q^2)_n (tq)^n}{(q^2, bt; q^2)_n}, \quad (30)$$

(see [3, p.29, Ex.4]), combining with (29), then

Corollary 4.2. *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{2n}}{(q^4; q^4)_n} = \frac{1}{\psi(q^2)} \sum_{n=0}^{\infty} q^{3\binom{n+1}{2}} = \frac{\psi(q^3)}{\psi(q^2)},$$

where the Gauss function,

$$\psi(q) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}.$$

(b). Setting $\lambda = 1$ or q^2 in (26), respectively, we obtain the following corollary.

Corollary 4.3. *We have*

$$\begin{aligned} \sum_{n=0}^{\infty} (q; q^2)_n q^{-n} &= \frac{1}{(1 - q^{-1})} \sum_{n=0}^{\infty} (-1)^n (1 - q^{4n}) q^{3n^2 - 2n}, \\ \sum_{n=0}^{\infty} (q; q^2)_n q^n &= \sum_{n=0}^{\infty} (-1)^n (1 + q^{2n+1}) q^{3n^2 + 2n}. \end{aligned}$$

Note. The second equality can be found in [4, Entry 9.5.2].

(c). Setting $\lambda = 1$ in (27), we deduce the following identity:

$$\sum_{n=0}^{\infty} \frac{(-1)_{2n} q^n}{(q^2; q^2)_n} = \frac{2(-q)_{\infty}}{(q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{3n^2}. \quad (31)$$

Letting $a \rightarrow 0$, $b = -1$, $c \rightarrow 0$, $t = q$ in the following identity:

$$\sum_{n=0}^{\infty} \frac{(a; q^2)_n (b)_{2n}}{(q^2; q^2)_n (c)_{2n}} t^n = \frac{(b)_{\infty} (at; q^2)_{\infty}}{(c)_{\infty} (t; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(c/b)_n (t; q^2)_n}{(q)_n (at; q^2)_n} b^n, \quad (32)$$

(see [4, (9.3.3)]), combining with (31), then

Corollary 4.4. *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n}{(q)_n} = \frac{1}{(q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{3n^2}.$$

(d). Setting $\lambda = 1$ or q^2 in (28), respectively, we obtain the following corollary.

Corollary 4.5. *We have*

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{(q^2; q^2)_n} &= \frac{1}{\psi(q)} \sum_{n=0}^{\infty} (1 + q^{2n}) q^{4n^2 - n} \\ &= \frac{(-q^3, -q^5, q^8; q^8)_{\infty}}{\psi(q)}, \\ \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2 + 2n}}{(q^2; q^2)_n} &= \frac{1}{\psi(q)} \sum_{n=0}^{\infty} (1 + q^{2n+1}) q^{4n^2 + 3n} \\ &= \frac{(-q, -q^7, q^8; q^8)_{\infty}}{\psi(q)}.\end{aligned}$$

Note. Replaced q by $-q$, then the results above become the Ramanujan-Slater Mod 8 identities, which can be found in [15, Eq.(1.11),(1.12)].

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