

**FRACTIONAL q - DERIVATIVE AND q - SERIES
TRANSFORMATIONS**

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(Received: Nov. 12, 2025 Accepted: Jun. 07, 2026 Published: Jun. 30, 2026)

Abstract: In this paper, making use of fractional q - derivative for the product of two functions, transformation formulas involving q - hypergeometric series have been established.

Keywords and Phrases: Fractional q -derivative, q -hypergeometric series, summation formula, transformation formula.

2020 Mathematics Subject Classification: Primary 33D15; Secondary 11B05, 05A19.

1. Introduction, Notations and Definitions

Throughout this paper we shall use the standard notation and terminology for q - series. As customary, the q -shifted factorials are given as,

$$(a; q)_{\infty} = \prod_{r=0}^{\infty} (1 - aq^r), \quad |q| < 1,$$
$$(a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}} = (1 - a)(1 - aq) \dots (1 - aq^{n-1}),$$

In general, the multi factorial is given by,

$$(a_1, a_2, \dots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_r; q)_n.$$

Here we introduce a basic (q -) hypergeometric series as,

$${}_{r+1}\Phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1}; q; z \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n z^n}{(b_1, b_2, \dots, b_r, q; q)_n},$$

where q is the base and z is the argument of the series. For the convergence of the series $\max. (|q|, |z|) < 1$.

Leibnitz's rule state that for any finite integer $n \geq 0$ and two functions $F(x)$ and $G(x)$ differentiable n times with respect to x , it holds

$$D_x^n \{F(x)G(x)\} = \sum_{k=0}^n \binom{n}{k} D_x^k (F(x)) D_x^{n-k} \{G(x)\},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Fractional q -derivative is defined as,

$$D_{q,x}^\lambda x^{\mu-1} = \frac{\Gamma_q(\mu)}{\Gamma_q(\mu-\lambda)} x^{\mu-\lambda-1}, \quad [1; (1.5), \text{ p. } 26]$$

where

$$\Gamma_q(\mu) = \frac{(q; q)_\infty}{(q^\mu; q)_\infty} (1-q)^{1-\mu}, \quad 0 < q < 1. \quad (1.1)$$

[2; (1.10.1), p. 20]

Basic analogue of Leibnitz theorem is given as;

If $F(x)$ and $G(x)$ are two regular functions such that

$$F(x) = \sum_{r=0}^{\infty} a_r x^r, \quad |x| < R_1$$

and

$$G(x) = \sum_{r=0}^{\infty} b_r x^r, \quad |x| < R_2$$

then for $|x| < R$ where $R = \min(R_1, R_2)$,

$$D_{q,x}^\lambda \{F(x)G(x)\} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n}{(q; q)_n} D_{q,x}^{\lambda-n} \{F(xq^n)\} D_{q,x}^n \{G(x)\}. \quad (1.2)$$

[1; Theorem 1.2]

Fractional q -derivatives and their applications to basic hypergeometric series have attracted to researchers in recent years. Several transformation formulas, product identities, and applications of fractional q -operators have been established by Singh and others [3, 4, 5, 6]. This research paper derives new transformation formulas involving fractional q -derivatives through the application of the fractional q -Leibnitz theorem. We shall make use of (1.1) and (1.2) to establish certain transformation formulas for q -series.

2. Transformation Formulas for q -series

In this section we shall establish certain transformation formulas for q -series by using (1.1) and (1.2).

(a) Taking $F(x) = x^{A-1}$ and $G(x) = x^{B-1}$ in (1.2) we get

$$\begin{aligned}
 D_{q,x}^\lambda \{x^{A-1} . x^{B-1}\} &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n}{(q; q)_n} D_{q,x}^{\lambda-n} \{x^{A-1} q^{n(A-1)}\} D_{q,x}^n \{x^{B-1}\} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n q^{n(A-1)}}{(q; q)_n} \\
 &\quad \times \frac{\Gamma_q(A)}{\Gamma_q(A-\lambda+n)} x^{A-\lambda+n-1} \frac{\Gamma_q(B)}{\Gamma_q(B-n)} x^{B-n-1} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n q^{n(A-1)} x^{A+B-\lambda-2}}{(q; q)_n} \\
 &\quad \times \frac{(q^{A-\lambda+n}; q)_\infty}{(q^n; q)_\infty} (1-q)^{n-\lambda} \frac{(q^{B-n}; q)_\infty}{(q^B; q)_\infty (1-q^n)} \\
 &= x^{A+B-\lambda-2} (1-q)^{-\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n q^{n(A-1)}}{(q; q)_n (q^{A-\lambda}; q)_n} \\
 &\quad \times \frac{(q^{A-\lambda}; q)_\infty}{(q^A; q)_\infty (q^B; q)_{-n}} \\
 &= \frac{x^{A+B-\lambda-2} (1-q)^{-\lambda} (q^{A-\lambda}; q)_\infty}{(q^A; q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{-\lambda}; q)_n q^{n(A-1)} q^{n(B-1)} (q^{1-B}; q)_n}{(q; q)_n (q^{A-\lambda}; q)_n} \\
 &= \frac{x^{A+B-\lambda-2} (1-q)^{-\lambda} (q^{A-\lambda}; q)_\infty}{(q^A; q)_\infty} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{1-B}; q^{(A+B-2)} \\ q^{A-\lambda} \end{matrix} \right] \quad (2.1)
 \end{aligned}$$

Interchanging A and B we get

$$= \frac{x^{A+B-\lambda-2} (1-q)^{-\lambda} (q^{B-\lambda}; q)_\infty}{(q^B; q)_\infty} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{1-A}; q; q^{(A+B-2)} \\ q^{B-\lambda} \end{matrix} \right] \quad (2.2)$$

Again, we have

$$\begin{aligned}
 D_q^\lambda(x^{A+B-2}) &= \frac{\Gamma_q(A+B-1)}{\Gamma_q(A+B-\lambda-1)} x^{A+B-\lambda-2} \\
 &= \frac{(1-q)^{2-A-B}(q^{A+B-\lambda-1}; q)_\infty}{(q^{A+B-1}; q)_\infty (1-q)^{2-A-B+\lambda}} x^{A+B-\lambda-2} \\
 &= \frac{(1-q)^{-\lambda}(q^{A+B-\lambda-1}; q)_\infty}{(q^{A+B-1}; q)_\infty} x^{A+B-\lambda-2} \quad (2.3)
 \end{aligned}$$

Equating (2.1), (2.2) and (2.3) we get

$$\begin{aligned}
 \frac{(q^{A-\lambda}; q)_\infty}{(q^A; q)_\infty} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{1-B}; q; q^{(A+B-2)} \end{matrix} \right] \\
 = \frac{(q^{B-\lambda}; q)_\infty}{(q^B; q)_\infty} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{1-A}; q; q^{(A+B-2)} \end{matrix} \right] \\
 = \frac{(q^{A+B-\lambda-1}; q)_\infty}{(q^{A+B-1}; q)_\infty}. \quad (2.4)
 \end{aligned}$$

(b) Taking $F(x) = x^{A-1}$ and $G(x) = \sum_{r=0}^{\infty} \frac{(a; q)_r}{(q; q)_r} x^r$ in (1.3) we get

$$\begin{aligned}
 &D_{q,x}^\lambda \left\{ (x^{A-1}) \cdot \left(\sum_{r=0}^{\infty} \frac{(a; q)_r}{(q; q)_r} x^r \right) \right\} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n}{(q; q)_n} D_{q,x}^{\lambda-n} (x^{A-1} q^{n(A-1)}) D_{q,x}^n \left(\sum_{r=0}^{\infty} \frac{(a; q)_r}{(q; q)_r} x^r \right) \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n q^{n(A-1)}}{(q; q)_n} \frac{\Gamma_q(A) x^{A-\lambda+n-1}}{\Gamma_q(A-\lambda+n)} \sum_{r=0}^{\infty} \frac{(a; q)_r \Gamma_q(r+1) x^{r-n}}{(q; q)_r \Gamma_q(1+r-n)} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n q^{n(A-1)} x^{A-\lambda-1} (q^{A-\lambda+n}; q)_\infty}{(q; q)_n (q^A; q)_\infty} \\
 &\quad \times (1-q)^{n-\lambda} \sum_{r=0}^{\infty} \frac{(a; q)_r x^r (1-q)^{-r} (q^{1+r-n}; q)_\infty}{(q; q)_r (q^{1+r}; q)_\infty (1-q)^{-r+n}} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n q^{n(A-1)} (1-q)^{-\lambda} (q^{A-\lambda}; q)_\infty}{(q; q)_n (q^A; q)_\infty (q^{A-\lambda}; q)_n} \sum_{r=0}^{\infty} \frac{(a; q)_r x^r}{(q; q)_r (q^{1+r}; q)_n} \\
 &= \frac{(1-q)^{-\lambda} (q^{A-\lambda}; q)_\infty x^{A-\lambda-1}}{(q^A; q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{-\lambda}; q)_n q^{n(A-1)}}{(q; q)_n (q^{A-\lambda}; q)_n} \sum_{r=0}^{\infty} \frac{(a; q)_r x^r q^{nr} (q^{-r}; q)_n}{(q; q)_r}
 \end{aligned}$$

Now, changing the order of summations we get,

$$= \frac{(q^{A-\lambda}; q)_\infty x^{A-\lambda-1}}{(q^A; q)_\infty (1-q)^\lambda} \sum_{r=0}^{\infty} \frac{(a; q)_r x^r}{(q; q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-r}, q^{-\lambda}; q; q^{A+r-1} \end{matrix} \right] \quad (2.5)$$

(c) Again, taking $F(x) = \sum_{r=0}^{\infty} \frac{(a; q)_r}{(q; q)_r} x^r$ and $G(x) = x^{A-1}$ in (1.3) we get

$$\begin{aligned} & G_{q,x}^\lambda \left\{ \left(\sum_{r=0}^{\infty} \frac{(a; q)_r x^r}{(q; q)_r} \right) \cdot (x^{A-1}) \right\} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n}{(q; q)_n} D_{q,x}^{\lambda-n} \left(\sum_{r=0}^{\infty} \frac{(a; q)_r q^{nr} x^r}{(q; q)_r} \right) D_{q,x}^n (x^{A-1}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n}{(q; q)_n} \sum_{r=0}^{\infty} \frac{(a; q)_r q^{nr}}{(q; q)_r} \frac{\Gamma_q(1+r) x^{r-\lambda+n}}{\Gamma_q(1+r-\lambda+n)} \frac{\Gamma_q(A)}{\Gamma_q(A-n)} x^{A-n-1} \\ &= x^{A-\lambda-1} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n}{(q; q)_n} \sum_{r=0}^{\infty} \frac{(a; q)_r x^r q^{nr} (q^{1+r-\lambda+n}; q)_\infty (1-q)^{-\lambda+n}}{(q; q)_r (q^{1+r}; q)_\infty} \\ &\quad \times \frac{(q^{A-n}; q)_\infty}{(q^A; q)_\infty} (1-q)^{-n} \\ &= x^{A-\lambda-1} (1-q)^{-\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (q^{-\lambda}; q)_n}{(q; q)_n (q; q)_\infty} \sum_{r=0}^{\infty} \frac{(a; q)_r x^r q^{nr} (q^{1+r-\lambda}; q)_\infty}{(q^{1+r-\lambda}; q)_n} \\ &\quad \times \frac{1}{(q^A; q)_{-n}} \\ &= \frac{x^{A-\lambda-1} (1-q)^{-\lambda}}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{-\lambda}; q)_n q^{(A-1)n} (q^{1-A}; q)_n}{(q; q)_n} \\ &\quad \times \sum_{r=0}^{\infty} \frac{(a; q)_r x^r q^{nr}}{(q^{1+r-\lambda}; q)_n} \frac{(q^{1-\lambda}; q)_\infty}{(q^{1-\lambda}; q)_r} \\ &= \frac{x^{A-\lambda-1} (1-q)^{-\lambda} (q^{1-A}; q)_\infty}{(q; q)_\infty} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(q^{-\lambda}; q)_n q^{(A-1)n} (q^{1-A}; q)_n}{(q; q)_n} \sum_{r=0}^{\infty} \frac{(a; q)_r x^r q^{nr}}{(q^{1-\lambda}; q)_r (q^{1+r-\lambda}; q)_n} \end{aligned}$$

Now, changing the order of summation we get,

$$= \frac{x^{A-\lambda-1}(1-q)^{-\lambda}(q^{1-A};q)_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty} \frac{(a;q)_r x^r}{(q^{1-\lambda};q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{1-A}; q; q^{A+r-1} \\ q^{1+r-\lambda} \end{matrix} \right] \quad (2.6)$$

Equating (2.5) and (2.6) we get

$$\begin{aligned} & \frac{(q^{A-\lambda};q)_\infty}{(q^A;q)_\infty} \sum_{r=0}^{\infty} \frac{(a;q)_r x^r}{(q;q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-r}, q^{-\lambda}; q; q^{A+r-1} \\ q^{A-\lambda} \end{matrix} \right] \\ &= \frac{(q^{1-A};q)_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty} \frac{(a;q)_r x^r}{(q^{1-\lambda};q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{1-A}; q; q^{A+r-1} \\ q^{1+r-\lambda} \end{matrix} \right] \end{aligned} \quad (2.7)$$

Taking $a = 0$ in (2.7) we get

$$\begin{aligned} & \frac{(q^{A-\lambda};q)_\infty}{(q^A;q)_\infty} \sum_{r=0}^{\infty} \frac{x^r}{(q;q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-r}, q^{-\lambda}; q; q^{A+r-1} \\ q^{A-\lambda} \end{matrix} \right] \\ &= \frac{(q^{1-A};q)_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty} \frac{x^r}{(q^{1-\lambda};q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{1-A}; q; q^{A+r-1} \\ q^{1+r-\lambda} \end{matrix} \right] \end{aligned} \quad (2.8)$$

Putting $\frac{x}{a}$ for x and then taking $a \rightarrow \infty$ in (2.7) we get,

$$\begin{aligned} & \frac{(q^{A-\lambda};q)_\infty}{(q^A;q)_\infty} \sum_{r=0}^{\infty} \frac{(-1)^r q^{r(r-1)/2} x^r}{(q;q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-r}, q^{-\lambda}; q; q^{A+r-1} \\ q^{A-\lambda} \end{matrix} \right] \\ &= \frac{(q^{1-A};q)_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty} \frac{(-1)^r q^{r(r-1)/2} x^r}{(q^{1-\lambda};q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{1-A}; q; q^{A+r-1} \\ q^{1+r-\lambda} \end{matrix} \right] \end{aligned} \quad (2.9)$$

Equating the coefficients of x^r in (2.7), (2.8) and (2.9) we get

$$\begin{aligned} & \frac{(q^{A-\lambda};q)_\infty}{(q^A;q)_\infty (q;q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-r}, q^{-\lambda}; q; q^{A+r-1} \\ q^{A-\lambda} \end{matrix} \right] \\ &= \frac{(q^{1-A};q)_\infty}{(q;q)_\infty (q^{1-\lambda};q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{1-A}; q; q^{A+r-1} \\ q^{1+r-\lambda} \end{matrix} \right] \end{aligned} \quad (2.10)$$

3.

(a) Taking $F(x) = \sum_{r=0}^{\infty} \frac{x^r}{(q;q)_r}$ and $G(x) = \sum_{s=0}^{\infty} \frac{(a;q)_s x^s}{(q;q)_s}$ in (1.3) and proceeding as

above we get,

$$\begin{aligned} D_q^\lambda \left\{ \left(\sum_{r=0}^{\infty} \frac{x^r}{(q; q)_r} \right) \left(\sum_{s=0}^{\infty} \frac{(a; q)_s x^s}{(q; q)_s} \right) \right\} \\ = \frac{x^{-\lambda} (q^{1-\lambda}; q)_\infty}{(q; q)_\infty (1-q)^\lambda} \sum_{r,s=0}^{\infty} \frac{(a; q)_s x^{r+s}}{(q; q)_s (q^{1-\lambda}; q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{-s}; q; q^{r+s} \\ q^{1+r-\lambda} \end{matrix} \right]. \end{aligned} \quad (3.1)$$

(b) Again, taking $F(x) = \sum_{s=0}^{\infty} \frac{(a; q)_s x^s}{(q; q)_s}$ and $G(x) = \sum_{r=0}^{\infty} \frac{x^r}{(q; q)_r}$ in (1.3) and proceeding as above we get,

$$\begin{aligned} D_q^\lambda \left\{ \left(\sum_{s=0}^{\infty} \frac{(a; q)_s x^s}{(q; q)_s} \right) \left(\sum_{r=0}^{\infty} \frac{x^r}{(q; q)_r} \right) \right\} \\ = \frac{x^{-\lambda} (q^{1-\lambda}; q)_\infty}{(1-q)^\lambda (q; q)_\infty} \sum_{r,s=0}^{\infty} \frac{(a; q)_s x^{r+s}}{(q^{1-\lambda}; q)_s (q; q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{-r}; q; q^{r+s} \\ q^{1+s-\lambda} \end{matrix} \right]. \end{aligned} \quad (3.2)$$

Equating (3.1) and (3.2) we get

$$\begin{aligned} \sum_{r,s=0}^{\infty} \frac{(a; q)_s x^{r+s}}{(q; q)_s (q^{1-\lambda}; q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{-s}; q; q^{r+s} \\ q^{1+r-\lambda} \end{matrix} \right] \\ = \sum_{r,s=0}^{\infty} \frac{(a; q)_s x^{r+s}}{(q; q)_r (q^{1-\lambda}; q)_s} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{-r}; q; q^{r+s} \\ q^{1+s-\lambda} \end{matrix} \right]. \end{aligned} \quad (3.3)$$

Taking the coefficient of x^{r+s} of both sides in (3.3) we get,

$${}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{-s}; q; q^{r+s} \\ q^{1+r-\lambda} \end{matrix} \right] = \frac{(q; q)_s (q^{1-\lambda}; q)_r}{(q; q)_r (q^{1-\lambda}; q)_s} {}_2\Phi_1 \left[\begin{matrix} q^{-\lambda}, q^{-r}; q; q^{r+s} \\ q^{1+s-\lambda} \end{matrix} \right]. \quad (3.4)$$

Choosing different functions for $F(x)$ and $G(x)$, one can establish large number of results of above type.

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