

δ_s^α -CLOSED SET AND ITS APPLICATIONS

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Abstract: This paper deals with a new type of crisp subset, viz., δ_s^α -closed set of a set X endowed with fuzzy topology. Using this newly defined set, we introduce and study δ_s^α -continuity between two fuzzy topological spaces. Lastly, we show that under this function δ_s^α -almost compactness remains invariant.

Keywords and Phrases: α -shading, δ_s^α -closed set, δ_s^α -Urysohn space, δ_s^α -almost compact space.

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1. Introduction

In [7], α -shading in a fuzzy topological space is introduced. Afterwards, different types of compactness are introduced using this concept as a basic tool which is a generalization of fuzzy compactness introduced by Chang [5]. Using α -shading in [4], δ_s^α -almost compactness is introduced and studied for crisp subsets in a fuzzy topological space. In this paper a class of crisp subsets, called δ_s^α -closed subsets is introduced. Then establish that δ_s^α -almost compactness of a space is inherited by such subsets.

2. Preliminaries

Throughout this paper, (X, τ) or simply by X , we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [5]. An ordinary subset A of X is said to be a crisp subset of X where the underlying structure of the set X being a fuzzy

topology. Fuzzy set is introduced by L.A. Zadeh in 1965 as follows : A fuzzy set A is a mapping from a nonempty set X into the closed interval $I = [0, 1]$ of the real line, i.e., $A \in I^X$ [9]. For a fuzzy set A , the fuzzy closure [5] and fuzzy interior [5] of A in X are denoted by clA and $intA$ respectively. The support [9] of a fuzzy set A in X will be denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [8] in X with the singleton support $\{x\} \subseteq X$ and the value t ($0 < t \leq 1$) at x will be denoted by x_t . For a fuzzy set A , the complement [9] of A in X will be denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A and B in X , we write $A \leq B$ if $A(x) \leq B(x)$, for each $x \in X$ [9] while we write AqB if A is quasi-coincident (q -coincident, for short) with B [8], i.e., if there exists $x \in X$ such that $A(x) + B(x) > 1$; the negation of these statements are written as $A \not\leq B$ and $A \not q B$ respectively. A fuzzy set A is called fuzzy regular open [1] if $A = int(clA)$. A fuzzy set B is called a fuzzy neighbourhood (fuzzy nbd, for short) [8] of a fuzzy point x_t if there is a fuzzy open set U in X such that $x_t q U \leq B$. If, in addition, B is fuzzy open in X , then B is called fuzzy open nbd of x_t [8]. A fuzzy set B is called a quasi-neighbourhood (q -nbd, for short) [8] of a fuzzy set A if there is a fuzzy open set U in X such that $UqA \leq B$. If, in addition, B is fuzzy open in X , then B is called fuzzy open q -nbd of A [8]. A fuzzy point x_t is said to be a fuzzy δ -cluster point of a fuzzy set B in an fts X if every fuzzy regular open set U in X such that $x_t q U$, UqB [6]. The union of all fuzzy δ -cluster points of B is called the fuzzy δ -closure of B and is denoted by $\delta cl B$ [6]. A fuzzy set B in an fts X is fuzzy δ -closed if and only if $B = \delta cl B$ [6]. The complement of a fuzzy δ -closed set is called fuzzy δ -open [6]. The union of all fuzzy δ -open sets contained in a fuzzy set B is called fuzzy δ -interior of B , denoted by $\delta int B$ [6]. A fuzzy set $B \in I^X$ is fuzzy δ -open if and only if $B = \delta int B$ [6]. For any fuzzy set $B \in I^X$, $\delta cl(1_X \setminus B) = 1_X \setminus \delta int B$ [6].

3. δ_s^α -Closed Set : Some Properties

In this section a new type of crisp subset of an fts X is introduced and studied. Also a new type of Urysohn space is introduced which helps to establish a strong relationship between δ_s^α -closed set and δ_s^α -almost compactness.

We first recall some definitions from [2, 3, 4].

Definition 3.1. [2] A fuzzy set B in an fts (X, τ) is said to be fuzzy δ -semiopen if $B \leq cl(\delta int B)$. The complement of this set is called fuzzy δ -semiclosed.

Definition 3.2. [3] A fuzzy set A in an fts X is said to be a fuzzy δ -semi- q -nbd of a fuzzy point x_t in X if there is a fuzzy δ -semiopen set B in X such that $x_t q B \leq A$.

Definition 3.3. [3] A fuzzy point x_t in an fts X is called a fuzzy δ -semichuster

point of a fuzzy set A in X if every fuzzy δ -semi- q -nbd U of x_t , UqA . The union of all fuzzy δ -semicluster points of a fuzzy set A is called fuzzy δ -semiclosure of A , denoted by $\delta scl A$. For any $A \in I^X$, $\delta scl(\delta scl A) = \delta scl A$.

Result 3.4. [3] Let (X, τ) be an fts and $B \in I^X$. Then a fuzzy point $x_t \in \delta scl B$ if and only if every fuzzy δ -semiopen set V in X with $x_t q V$, $V q B$.

Result 3.5. [3] Let (X, τ) be an fts and $C, D \in I^X$ with $C \not q D$. If C is fuzzy δ -semiopen, then $C \not q \delta scl D$.

Result 3.6. [3] The union (resp., intersection) of any collection of fuzzy δ -semiopen (resp., fuzzy δ -semiclosed) sets in an fts is also so.

Result 3.7. [3] Let (X, τ) be an fts and $B \in I^X$. Then $\delta scl(1_X \setminus B) = 1_X \setminus \delta sint B$, $\delta sint(1_X \setminus B) = 1_X \setminus \delta scl B$.

Result 3.8. [4] For a fuzzy δ -semiopen set $U \in I^X$, $\delta scl(\delta sint(\delta scl U)) = \delta scl U$.

Definition 3.9. Let (X, τ) be an fts and B , a crisp subset of X . Let $\mathcal{U}$ be a collection of fuzzy sets in X . Then $\mathcal{U}$ is called an α -shading (where $0 < \alpha < 1$) of B if for each $x \in B$, there is some $V_x \in \mathcal{U}$ such that $V_x(x) > \alpha$ [7]. If, in addition, the members of $\mathcal{U}$ are fuzzy open (resp., fuzzy δ -semiopen) sets, then $\mathcal{U}$ is called a fuzzy open [7] (resp., fuzzy δ -semiopen) α -shading of B .

Definition 3.10. [4] Let B be a crisp subset of an fts (X, τ) . Then B is called δ_s^α -almost compact if each fuzzy δ -semiopen α -shading $\mathcal{U}$ of B has a finite δ_s -proximate α -subshading, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{\delta scl U : U \in \mathcal{U}_0\}$ is again an α -shading of B . If, in particular $B = X$, we get the definition of δ_s^α -almost compact space X .

Let us now introduce a new concept of limit points of a crisp subset of an fts.

Definition 3.11. Let A be a crisp subset of an fts (X, τ) . A point $x \in X$ is said to be an δ_s^α -limit point of A if for every fuzzy δ -semiopen set B in X with $B(x) > \alpha$, there exists $y \in A \setminus \{x\}$ such that $(\delta scl B)(y) > \alpha$.

The set of all δ_s^α -limit points of A will be denoted by $[A]_{\delta_s^\alpha}^\alpha$.

For a crisp subset A of X , the δ_s^α -closure of A , to be denoted by $\delta_s^\alpha-cl A$, is defined by $\delta_s^\alpha-cl A = A \cup [A]_{\delta_s^\alpha}^\alpha$.

Definition 3.12. A crisp subset A of an fts X is said to be δ_s^α -closed if it contains all of its δ_s^α -limit points. The complement of a δ_s^α -closed set is called δ_s^α -open set.

Remark 3.13. It is clear from definition that for any crisp subset A of X , $A \subseteq \delta_s^\alpha-cl A$ and $\delta_s^\alpha-cl A = A$ if and only if $\delta_s^\alpha-cl A \subseteq A$. Also $A \subseteq B \subseteq X \Rightarrow [A]_{\delta_s^\alpha}^\alpha \subseteq [B]_{\delta_s^\alpha}^\alpha$.

Theorem 3.14. A δ_s^α -closed subset of a δ_s^α -almost compact space is δ_s^α -almost

compact.

Proof. Let a crisp subset A of a δ_s^α -almost compact space X be δ_s^α -closed. Then for any $a \notin A$, there is a fuzzy δ -semiopen set B_a in X such that $B_a(a) > \alpha$, but $(\delta scl B_a)(b) \leq \alpha$, for every $b \in A$. Let $\mathcal{U} = \{B_a : a \notin A\}$. Let $\mathcal{V}$ be a fuzzy δ -semiopen α -shading of A . Then $\mathcal{U} \cup \mathcal{V}$ is a fuzzy δ -semiopen α -shading of X . By δ_s^α -almost compactness of X , there exists a finite subcollection $\{C_1, C_2, \dots, C_n\}$ of $\mathcal{U} \cup \mathcal{V}$ such that for every $x \in X$, there exists C_i ($1 \leq i \leq n$) such that $(\delta scl C_i)(x) > \alpha$. Since by construction, for every member B_a of $\mathcal{U}$, $(\delta scl B_a)(b) \leq \alpha$, for every $b \in A$, we discard the members of $\mathcal{U}$ which are in that finite subcollection and get our desired result.

To get the converse of the above theorem, we have to introduce the following concept.

Definition 3.15. An fts (X, τ) is said to be δ_s^α -Urysohn if for any two distinct crisp points a, b of X , there exist a fuzzy open set A and a fuzzy δ -semiopen set B in X with $A(a) > \alpha$, $B(b) > \alpha$ and $\min((\delta scl A)(c), (\delta scl B)(c)) \leq \alpha$, for all $c \in X$.

Theorem 3.16. A δ_s^α -almost compact set in a δ_s^α -Urysohn space X is δ_s^α -closed.

Proof. Let A be a δ_s^α -almost compact set in a δ_s^α -Urysohn space X . Let $a \in X \setminus A$. Then for each $b \in A$, there is a fuzzy open set C_b and a fuzzy δ -semiopen set D_b in X such that $C_b(a) > \alpha$, $D_b(b) > \alpha$ and $\min((\delta scl C_b)(t), (\delta scl D_b)(t)) \leq \alpha$, for all $t \in X \dots (1)$.

Let $\mathcal{D} = \{D_b : b \in A\}$. Then $\mathcal{D}$ is a fuzzy δ -semiopen α -shading of A . By δ_s^α -almost compactness of A , there are finitely many points $b_1, b_2, \dots, b_n$ in A such that $\mathcal{D}_0 = \{\delta scl D_{b_1}, \delta scl D_{b_2}, \dots, \delta scl D_{b_n}\}$ forms an α -shading of A . Let $C = C_{b_1} \cap \dots \cap C_{b_n}$. Then C being a fuzzy open set in X is fuzzy δ -semiopen in X such that $C(a) > \alpha$. We claim that $(\delta scl C)(b) \leq \alpha$, for each $b \in A$. If not, there is some $x \in A$ such that $(\delta scl C)(x) > \alpha$ and so $(\delta scl D_{b_k})(x) > \alpha$, for some k ($1 \leq k \leq n$). Also $(\delta scl C_{b_k})(x) > \alpha$. Hence $\min\{(\delta scl C_{b_k})(x), (\delta scl D_{b_k})(x)\} > \alpha$, contradicts (1).

Corollary 3.17. A crisp subset A of a δ_s^α -almost compact, δ_s^α -Urysohn space X is δ_s^α -almost compact if and only if A is δ_s^α -closed set in X .

Theorem 3.18. Every cover of a δ_s^α -almost compact space X by δ_s^α -open sets has a finite subcover.

Proof. Let $\mathcal{A} = \{A_\beta : \beta \in \Lambda\}$ be a cover of X by δ_s^α -open sets of X . Then for each $a \in X$, there is $A_a \in \mathcal{A}$ such that $a \in A_a$. Now $X \setminus A_a$ is δ_s^α -closed set in X . So there is a fuzzy δ -semiopen set B_a in X such that $B_a(a) > \alpha$, but $(\delta scl B_a)(b) \leq \alpha$, for each $b \in X \setminus A_a \dots (1)$. Let $\mathcal{B} = \{B_a : a \in X\}$. Then $\mathcal{B}$ is a fuzzy δ -semiopen α -shading of X . By hypothesis, there is a finite subset $\{a_1, a_2, \dots, a_n\}$ of X such

that $\{\delta scl B_{a_i} : i = 1, 2, \dots, n\}$ is an α -shading of X ... (2).

We claim that $\mathcal{A}_0 = \{A_{a_i} : i = 1, 2, \dots, n\}$ is a finite subcover of $\mathcal{A}$. If not, then there is some $b \in X \setminus \bigcup_{i=1}^n A_{a_i} = \bigcap_{i=1}^n (X \setminus A_{a_i})$. Using (1), $(\delta scl B_{a_i})(b) \leq \alpha$, for $i = 1, 2, \dots, n$ implies that $(\bigcup_{i=1}^n \delta scl B_{a_i})(b) \leq \alpha$, contradicts (2).

Theorem 3.19. *Let (X, τ) be a δ_s^α -almost compact space. Then every collection of δ_s^α -closed sets in X with finite intersection property has non-empty intersection.*

Proof. Let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be a collection of δ_s^α -closed sets in a δ_s^α -almost compact space X having finite intersection property. On the contrary, let $\bigcap_{i \in \Lambda} U_i = \phi$.

Then $X \setminus \bigcap_{i \in \Lambda} U_i = \bigcup_{i \in \Lambda} (X \setminus U_i) = X \Rightarrow \mathcal{V} = \{X \setminus U_i : i \in \Lambda\}$ is a δ_s^α -open cover of X . Then by Theorem 3.13, there is a finite subset Λ_0 of Λ such that $\bigcup_{i \in \Lambda_0} (X \setminus U_i) = X \Rightarrow \bigcap_{i \in \Lambda_0} U_i = \phi$, contradicts our hypothesis.

It is clear from definitions that δ_s^α -almost compactness implies α -almost compactness. To get the reverse implication we have to introduce the following concept.

Definition 3.20. *Let (X, τ) be an fts. Then X is called δ_s^α -regular, if for each point $y \in X$ and each fuzzy open set V_y in X with $V_y(y) > \alpha$, there is a fuzzy δ -semiopen set U_y in X with $U_y(y) > \alpha$ such that $\delta scl U_y \leq V_y$.*

Theorem 3.21. *Let (X, τ) be an fts. Then the following are statements are equivalent :*

- (a) X is δ_s^α -regular.
- (b) For each fuzzy closed set A and each point $y \in X$ with $A(y) < 1 - \alpha$, there is a fuzzy δ -semiopen set B in X with that $(\delta scl B)(y) < 1 - \alpha$ and $A \leq B$.
- (c) For each fuzzy closed set A and each point $y \in X$ with $A(y) < 1 - \alpha$, there are fuzzy δ -semiopen sets B and C in X such that $C(y) > \alpha$, $A \leq B$ and $B \not\leq C$.

Proof. (a) $\Rightarrow$ (b) : Let $y \in X$ and A be a fuzzy closed set in X with $A(y) < 1 - \alpha$. Put $B = 1_X \setminus A$. Then $B(y) > \alpha$ where B is fuzzy open set in X . By (a), there is a fuzzy δ -semiopen set U in X with $U(y) > \alpha$ and $\delta scl U \leq B = 1_X \setminus A$. Then $A \leq 1_X \setminus \delta scl U = \delta sint(1_X \setminus U) = V$ (say). Then V is a fuzzy δ -semiopen set in X . Now, $\delta scl V = \delta scl(\delta sint(1_X \setminus U)) = \delta scl(1_X \setminus \delta scl U) = 1_X \setminus \delta sint(\delta scl U) \leq 1_X \setminus U$. Thus $(\delta scl V)(y) \leq (1_X \setminus U)(y) < 1 - \alpha$.

(b) $\Rightarrow$ (a) : Let $y \in X$ and $A \in \tau$ with $A(y) > \alpha$. Let $B = 1_X \setminus A$. Then $B \in \tau^c$ with $B(y) < 1 - \alpha$. By (b), there exists a fuzzy δ -semiopen set C

in X with $(\delta scl C)(y) < 1 - \alpha$ and $B \leq C$. So $(1_X \setminus \delta scl C)(y) > \alpha$. Put $D = 1_X \setminus \delta scl C$. Then D is a fuzzy δ -semiopen set in X with $D(y) > \alpha$. Now $\delta scl D = \delta scl(1_X \setminus \delta scl C) = 1_X \setminus \delta sint(\delta scl C) \leq 1_X \setminus C \leq 1_X \setminus B = A$.

(b) $\Rightarrow$ (c) : Let $y \in X$ and A , a fuzzy closed set in X with $A(y) < 1 - \alpha$. By (b), there exists a fuzzy δ -semiopen set B in X such that $(\delta scl B)(y) < 1 - \alpha$ and $A \leq B$. Then $y_{1-\alpha} \notin \delta scl B$. Hence by Result 3.4, there is a fuzzy δ -semiopen set C in X such that $y_{1-\alpha} q C$ and $C \not\leq B$, i.e., $C(y) + 1 - \alpha > 1 \Rightarrow C(y) > \alpha$.

(c) $\Rightarrow$ (b) : Let $y \in X$, and A , a fuzzy closed set in X with $A(y) < 1 - \alpha$. Then by (c), there are two fuzzy δ -semiopen sets B, C in X with $C(y) > \alpha, A \leq B$ and $B \not\leq C$. Now $C(y) > \alpha \Rightarrow y_{1-\alpha} q C$. Now $B \not\leq C$, by Result 3.5, $\delta scl B \not\leq C \Rightarrow (\delta scl B)(y) \leq 1 - C(y) < 1 - \alpha$.

Theorem 3.22 *A δ_s^α -almost compact crisp subset of a δ_s^α -regular space is α -compact.*

Proof. Let (X, τ) be a δ_s^α -regular space and $Y(\subseteq X)$ be δ_s^α -almost compact set in X . Let $\mathcal{U}$ be a fuzzy open α -shading of Y . Then for each $a \in Y$, there exists $B_a \in \mathcal{U}$ such that $B_a(a) > \alpha$. By our hypothesis, there exists a fuzzy δ -semiopen set C_a in X with $C_a(a) > \alpha$ and $\delta scl C_a \leq B_a \dots (1)$.

Let $\mathcal{V} = \{C_a : a \in Y\}$. Then $\mathcal{V}$ is a fuzzy δ -semiopen α -shading of Y . By δ_s^α -almost compactness of Y , there is a finite subset Y_0 of Y such that $\mathcal{V}_0 = \{\delta scl C_a : a \in Y_0\}$ is an α -shading of Y . Then by (1), $\mathcal{U}_0 = \{B_a : a \in Y_0\}$ is a finite α -subshading of $\mathcal{U}$. Hence Y is α -compact (and hence α -almost compact).

Theorem 3.23. *Let (X, τ) be an fts and $Y(\subseteq X)$. Then Y is δ_s^α -almost compact if and only if every family of fuzzy δ -semiopen sets, the δ -semiinteriors of whose δ -semiclosures form an α -shading of Y has a finite subfamily, the δ -semiclosures of whose members form an α -shading of Y .*

Proof. The proof follows from Result 3.8.

Theorem 3.24. *Let (X, τ) be an fts and $Y(\subseteq X)$. Then Y is δ_s^α -almost compact if and only if for every collection $\{U_i : i \in \Lambda\}$ of fuzzy δ -semiopen sets with the property that for each finite subset Λ_0 of Λ , there is $y \in Y$ such that $\inf_{i \in \Lambda_0} U_i(y) \geq 1 - \alpha$, we have $\inf_{i \in \Lambda} (\delta scl U_i)(z) \geq 1 - \alpha$, for some $z \in Y$.*

Proof. Let Y be δ_s^α -almost compact set in an fts (X, τ) . If possible, let $\{U_i : i \in \Lambda\}$ be a collection of fuzzy δ -semiopen sets in X with the property that for each finite subset Λ_0 of Λ , there is $y \in Y$ such that $\inf_{i \in \Lambda_0} U_i(y) \geq 1 - \alpha$, but $\inf_{i \in \Lambda} (\delta scl U_i)(y) < 1 - \alpha$, for all $y \in Y$. Then $\alpha < (1_X \setminus \bigcap_{i \in \Lambda} \delta scl U_i)(y) =$

$[\bigcup_{i \in \Lambda} (1_X \setminus \delta scl U_i)](y)$, for each $y \in Y$ which shows that $\{1_X \setminus \delta scl U_i : i \in \Lambda\}$ is a fuzzy δ -semiopen α -shading of Y . By hypothesis, there is a finite subset Λ_0 of Λ such that $\{\delta scl(1_X \setminus \delta scl U_i) : i \in \Lambda_0\} = \{1_X \setminus \delta sint(\delta scl U_i) : i \in \Lambda_0\}$ is an α -shading of Y . So $\alpha < [\bigcup_{i \in \Lambda_0} (1_X \setminus \delta sint(\delta scl U_i))](y) = [1_X \setminus (\bigcap_{i \in \Lambda_0} \delta sint(\delta scl U_i))](y)$, for each $y \in Y$. Then $(\bigcap_{i \in \Lambda_0} U_i)(y) \leq [\bigcap_{i \in \Lambda_0} \delta sint(\delta scl U_i)](y) < 1 - \alpha$, for each $y \in Y$, which contradicts our assumption .

Conversely, suppose that Y is not δ_s^α -almost compact though the given condition holds. Then we have a fuzzy δ -semiopen α -shading $\mathcal{G} = \{G_i : i \in \Lambda\}$ of Y such that for every finite subset Λ_0 of Λ , $\{\delta scl G_i : i \in \Lambda_0\}$ is not an α -shading of Y . Then there is some $y \in Y$ such that $\sup_{i \in \Lambda_0} (\delta scl G_i)(y) \leq \alpha$, i.e., $1_X \setminus \sup_{i \in \Lambda_0} (\delta scl G_i)(y) = \inf_{i \in \Lambda_0} (1_X \setminus \delta scl G_i)(y) \geq 1 - \alpha$. So $\{1_X \setminus \delta scl G_i : i \in \Lambda\}$ is a family of fuzzy δ -semiopen sets in X with the given property. So there exists some $a \in Y$ such that $\inf_{i \in \Lambda} [\delta scl(1_X \setminus \delta scl G_i)](a) \geq 1 - \alpha \Rightarrow \sup_{i \in \Lambda} G_i(a) \leq \sup_{i \in \Lambda} (\delta sint(\delta scl G_i))(a) = 1 - \inf_{i \in \Lambda} (1_X \setminus \delta sint(\delta scl G_i))(a) = 1 - \inf_{i \in \Lambda} [\delta scl(1_X \setminus \delta scl G_i)](a) \leq \alpha$ which goes against our assumption that $\mathcal{G}$ is a fuzzy δ -semiopen α -shading of Y .

Definition 3.25. Let (X, τ) be an fts and $Y (\subseteq X)$. A family $\{G_i : i \in \Lambda\}$ of fuzzy sets in X is said to have δ_s^α -interiorly finite intersection property (δ_s^α -IFIP, for short) in a subset Y of X , if for each finite subset Λ_0 of Λ , there exists $y \in Y$ such that $[\bigcap_{i \in \Lambda_0} \delta sint G_i](y) \geq 1 - \alpha$.

Theorem 3.26. Let (X, τ) be an fts and Y be a crisp subset of X . Then Y is δ_s^α -almost compact if and only if for every family $\mathcal{G} = \{G_i : i \in \Lambda\}$ of fuzzy δ -semiclosed sets in X with δ_s^α -IFIP in Y , there is some $y \in Y$ such that $\inf_{i \in \Lambda} G_i(y) \geq 1 - \alpha$.

Proof. Let $\mathcal{G} = \{G_i : i \in \Lambda\}$ be a family of fuzzy δ -semiclosed sets in X with δ_s^α -IFIP in Y where Y is δ_s^α -almost compact subset of X . If possible, let for each $y \in Y$, $\inf_{i \in \Lambda} G_i(y) < 1 - \alpha$, i.e., $(\bigcap_{i \in \Lambda} G_i)(y) < 1 - \alpha$ and hence $[\bigcup_{i \in \Lambda} (1_X \setminus G_i)](y) > \alpha$.

So $\mathcal{F} = \{1_X \setminus G_i : i \in \Lambda\}$ is a fuzzy δ -semiopen α -shading of Y . By assumption, there exists a finite subfamily Λ_0 of Λ such that $[\bigcup_{i \in \Lambda_0} \delta scl(1_X \setminus G_i)](y) > \alpha$, i.e.,

$1 - (\bigcap_{i \in \Lambda_0} \delta sint G_i)(y) > \alpha$, i.e., $(\bigcap_{i \in \Lambda_0} \delta sint G_i)(y) < 1 - \alpha$, for each $y \in Y$, which shows that $\mathcal{G}$ does not have δ_s^α -IFIP in Y , which contradicts our assumption.

Conversely, suppose that $\mathcal{G} = \{G_i : i \in \Lambda\}$ is a fuzzy δ -semiopen α -shading of Y . Then $\mathcal{V} = \{1_X \setminus G_i : i \in \Lambda\}$ is a family of fuzzy δ -semiclosed sets in X with $\inf_{i \in \Lambda} (1_X \setminus G_i)(y) < 1 - \alpha$, for each $y \in Y$. Then by our assumption, $\mathcal{V}$ does not have δ_s^α -IFIP in Y . Then there is some finite subset Λ_0 of Λ such that for each $y \in Y$, $[\bigcap_{i \in \Lambda_0} \delta sint(1_X \setminus G_i)](y) < 1 - \alpha \Rightarrow 1 - (\bigcup_{i \in \Lambda_0} \delta scl G_i)(y) < 1 - \alpha$, for each $y \in Y \Rightarrow (\bigcup_{i \in \Lambda_0} \delta scl G_i)(y) > \alpha$, for each $y \in Y \Rightarrow Y$ is δ_s^α -almost compact.

4. δ_s^α -Continuous Function and its Applications

In this section a new class of functions is introduced under which δ_s^α -almost compactness remains invariant.

Definition 4.1. Let (X, τ) and (Y, τ_1) be two fts's. A function $f : (X, \tau) \rightarrow (Y, \tau_1)$ is said to be δ_s^α -continuous if for each crisp point $x \in X$ and each fuzzy δ -semiopen set A in Y with $A(f(x)) > \alpha$, there exists a fuzzy δ -semiopen set B in X with $B(x) > \alpha$ such that $\delta scl_\tau B \leq f^{-1}(\delta scl_{\tau_1} A)$.

Theorem 4.2. If $f : (X, \tau) \rightarrow (Y, \tau_1)$ is δ_s^α -continuous, then the following statements are true :

- (a) $f([A]_{\delta_s}^\alpha) \subseteq [f(A)]_{\delta_s}^\alpha$, for every $A \subseteq X$.
- (b) $[f^{-1}(A)]_{\delta_s}^\alpha \subseteq f^{-1}([A]_{\delta_s}^\alpha)$, for every $A \subseteq Y$.
- (c) $f^{-1}(A)$ is δ_s^α -closed in X , for each δ_s^α -closed set A in Y .
- (d) $f^{-1}(A)$ is δ_s^α -open in X , for each δ_s^α -open set A in Y .

Proof. (a). Let $x \in [A]_{\delta_s}^\alpha$ and B be any fuzzy δ -semiopen set in Y with $B(f(x)) > \alpha$. Then by definition, there is a fuzzy δ -semiopen set C in X with $C(x) > \alpha$ and $\delta scl_\tau C \leq f^{-1}(\delta scl_{\tau_1} B)$. Now $x \in [A]_{\delta_s}^\alpha$ and C is a fuzzy δ -semiopen set in X with $C(x) > \alpha \Rightarrow \delta scl C(y) > \alpha$, for some $y \in A \setminus \{x\}$. Then $\alpha < \delta scl_\tau C(y) \leq (f^{-1}(\delta scl_{\tau_1} B))(y) = (\delta scl_{\tau_1} B)(f(y))$ where $f(y) \in f(A) \setminus \{f(x)\} \Rightarrow f(x) \in [f(A)]_{\delta_s}^\alpha$. Hence (a) follows.

(b) Using (a), $f([f^{-1}(A)]_{\delta_s}^\alpha) \subseteq [ff^{-1}(A)]_{\delta_s}^\alpha \subseteq [A]_{\delta_s}^\alpha \Rightarrow [f^{-1}(A)]_{\delta_s}^\alpha \subseteq f^{-1}([A]_{\delta_s}^\alpha)$.

(c) By hypothesis, $[A]_{\delta_s}^\alpha = A$. Using (b), $[f^{-1}(A)]_{\delta_s}^\alpha \subseteq f^{-1}([A]_{\delta_s}^\alpha) = f^{-1}(A) \Rightarrow [f^{-1}(A)]_{\delta_s}^\alpha = f^{-1}(A) \Rightarrow f^{-1}(A)$ is δ_s^α -closed set in X .

(d) Follows from (c).

Let us now recall the following definition and remark from [3] for ready references.

Definition 4.3. [3] A function $f : (X, \tau) \rightarrow (Y, \tau_1)$ is called fuzzy δ -semiopen if $f(U)$ is fuzzy δ -semiopen set in Y for every fuzzy δ -semiopen set U in X .

Remark 4.4. [3] If $f : (X, \tau) \rightarrow (Y, \tau_1)$ is a bijective fuzzy δ -semiopen function,

then $f(U)$ is fuzzy δ -semiclosed set in Y for every fuzzy δ -semiclosed set U in X .

Theorem 4.5. *If a bijective function $f : (X, \tau) \rightarrow (Y, \tau_1)$ is a fuzzy δ -semiopen, then $f(A)$ is δ_s^α -closed set in (Y, τ_1) , for every δ_s^α -closed set A in (X, τ) .*

Proof. Let A be a δ_s^α -closed set in (X, τ) and let $b \in Y \setminus f(A)$. Then there exists a unique $a \in X$ such that $f(a) = b$. As $b \notin f(A)$, $a \notin A$. Now, A is δ_s^α -closed set in X . So there exists a fuzzy δ -semiopen set B in X such that $B(a) > \alpha$ and $(\delta scl_\tau B)(t) \leq \alpha$, for each $t \in A \dots (1)$.

By definition, $f(B)$ is a fuzzy δ -semiopen set in Y , and also $(f(B))(b) = B(a) > \alpha$. Let $c \in f(A)$. Then as f is bijective, there exists unique $d \in A$ such that $f(d) = c$. As f is fuzzy δ -semiopen function, by Remark 4.4, $\delta scl_{\tau_1} f(B) \leq \delta scl_{\tau_1} (f(\delta scl_\tau B)) = f(\delta scl_\tau B)$. So $(\delta scl_{\tau_1} f(B))(c) \leq f(\delta scl_\tau B)(c) = \delta scl_\tau B(d) \leq \alpha$ (by (1)). So b is not a δ_s^α -limit point of $f(A)$. Hence the proof.

From Theorem 4.2 (c) and Theorem 4.5, we can state the following corollary :

Corollary 4.6. *Let a bijective function $f : (X, \tau) \rightarrow (Y, \tau_1)$ be a fuzzy δ_s^α -continuous and fuzzy δ -semiopen function. Then $f^{-1}(A)$ is δ_s^α -closed set in X if and only if A is δ_s^α -closed set in Y .*

Theorem 4.7. *Let $f : (X, \tau) \rightarrow (Y, \tau_1)$ be fuzzy δ_s^α -continuous function. If $A(\subseteq X)$ is δ_s^α -almost compact subset of X , then $f(A)$ is δ_s^α -almost compact subset of Y .*

Proof. Let $\mathcal{B} = \{B_i : i \in \Lambda\}$ be a fuzzy δ -semiopen α -shading of $f(A)$, where A is δ_s^α -almost compact set in X . Then for each $a \in A$, $f(a) \in f(A)$ and so there exists $B_a \in \mathcal{B}$ such that $B_a(f(a)) > \alpha$. By hypothesis, there exists a fuzzy δ -semiopen set C_a in X such that $C_a(a) > \alpha$ and $f(\delta scl_\tau C_a) \leq \delta scl_{\tau_1} B_a$. Then $\mathcal{C} = \{C_a : a \in A\}$ is a fuzzy δ -semiopen α -shading of A . By δ_s^α -almost compactness of A , there are finitely many points $a_1, a_2, \dots, a_n$ of A such that $\mathcal{C}_0 = \{\delta scl_\tau C_{a_i} : i = 1, 2, \dots, n\}$ is again an α -shading of A . We propose to show that $\mathcal{B}_0 = \{\delta scl_{\tau_1} B_{a_i} : i = 1, 2, \dots, n\}$ is an α -shading of $f(A)$. Indeed, $y \in f(A) \Rightarrow$ there exists $x \in A$ such that $y = f(x)$. Since $\mathcal{C}_0$ is an α -shading of A , there is an C_{a_j} (for some $j, 1 \leq j \leq n$) such that $(\delta scl_\tau C_{a_j})(x) > \alpha$ and so $(\delta scl_{\tau_1} B_{a_j})(y) \geq f(\delta scl_\tau C_{a_j})(y) \geq \delta scl_\tau C_{a_j}(x) > \alpha$. Hence $f(A)$ is δ_s^α -almost compact subset of Y .

5. Conclusion

This paper is a continuation of [4]. Here we study a new type of limit points of a crisp subset of a topological space X which helps us to introduce a continuous-like function under which δ_s^α -almost compactness remains invariant.

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