

**SOME RELATION THEORETIC NONUNIQUE FIXED POINT
THEOREMS FOR CERTAIN CONTRACTION MAPPINGS IN A
NON-COMPLETE METRIC SPACE**

J. B. Dhage and B. C. Dhage

Kasubai, Gurukul Colony, Thodga Road,
Ahmedpur - 413515, Dist. Latur,
Maharashtra, INDIA

E-mail : jbdhage@gmail.com, bcdhage@gmail.com

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Abstract: In this paper, we establish some nonunique fixed point results for the orbital relationally continuous self-mappings of orbital relationally complete metric space satisfying a more generalized contraction condition than given in the literature. Our nonunique fixed point results include some interesting similar results in an orbital partially complete metric space as the special cases. Our abstract results are also illustrated with the help of a couple of numerical examples.

Keywords and Phrases: Relational complete metric space, Relational contraction mapping, Nonunique fixed point theorem, Successive iteration method.

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1. Introduction

Ciric [3] proved some nonunique fixed point results for a certain class of contraction mappings on a metric space into itself under weaker version of continuity and completeness of a metric space. Thereafter, several authors extended and generalized the above results to other classes of contraction mappings in a metric space which is not necessarily complete. See for example, Achari [1], Dhage [4,5,6, Pachpatte [15] and references therein. Similarly, Alam and Imdad [2] proved a basic relation theoretic fixed point theorem for relational contraction mappings in relational complete metric spaces. Later, this result is generalized to a wider class

of mappings with fixed point by Dubey *et al.* [14] and Samet and Turinici [16]. All the above fixed point results are about the existence of a unique fixed point and to the best of our knowledge there is no result for existence of nonunique fixed point so far relational metric space is concerned. In this paper we blend the above two approaches and prove a couple of relation theoretic nonunique fixed point theorems for the self-mappings of a metric space under weaker suitable conditions. Before going to the main results we give some preliminary definitions.

Let (X, d) be a metric space and let $\mathcal{T} : X \rightarrow X$ be a mapping. Then $\mathcal{T}$ -orbit of $\mathcal{T}$ at a point $x \in X$ is a set $O(x, \mathcal{T})$ in X defined by

$$O(x, \mathcal{T}) = \{x, \mathcal{T}x, \mathcal{T}^2x, \dots, \mathcal{T}^n x, \dots\} = \{\mathcal{T}^n x\}_{n=0}^{\infty}. \quad (1)$$

A mapping $\mathcal{T}$ is called $\mathcal{T}$ -orbitally continuous if and only if sequence $\{x_n\} \subseteq O(x, \mathcal{T})$ converges to x implies $\mathcal{T}x_n \rightarrow \mathcal{T}x$. X is called $\mathcal{T}$ -orbitally complete if every Cauchy sequence $\{x_n\}$ in $O(x, \mathcal{T})$ converges to a point in X . It is known that every continuous mapping is $\mathcal{T}$ -orbitally continuous and every complete metric space X is $\mathcal{T}$ -orbitally complete, but the reverse implication need not hold. See Dhage *et al.* [13] and references therein.

A binary relation or simply relation $\mathbf{R}$ in X is a subset of the Cartesian product $X \times X$ which is full if $\mathbf{R} = X \times X$ and which is empty if $\mathbf{R} = \emptyset$. $\mathbf{R}$ is called partial relation or proper relation if $\emptyset \neq \mathbf{R} \neq X \times X$. We introduce a partial relation $\mathbf{R}$ in X and two elements x and y in X are said to be relatable if they are related to each other under the relation $\mathbf{R}$. In this case we write $x \mathbf{R} y$ or $(x, y) \in \mathbf{R}$. A metric space X together with a partial relation $\mathbf{R}$ is called a relational metric space and it is denoted by $(X, d, \mathbf{R})$. A sequence $\{x_n\}$ of points in X is called **relatonic** or **term-wise related** if $(x_n, x_{n+1}) \in \mathbf{R}$ and **inverse relatonic** if $(x_{n+1}, x_n) \in \mathbf{R}$ for each $n \in \mathbb{N}$. We say the metric space X is $\mathbf{R}$ -regular or PR-closed or simply closed with respect to the relation $\mathbf{R}$, if $\{x_n\}$ is a relatonic (resp. inverse relatonic) sequence of points in X converging to a point x^* , then $(x_n, x^*) \in \mathbf{R}$ (resp. $(x^*, x_n) \in \mathbf{R}$) for all $n \in \mathbb{N}$. A relatonic sequence $\{x_n\}$ in $(X, d, \mathbf{R})$ is called $\mathbf{R}$ -Cauchy if $\lim_{m, n \rightarrow \infty} d(x_m, x_n) = 0$. Again, $(X, d, \mathbf{R})$ is called $\mathbf{R}$ -complete if every $\mathbf{R}$ -Cauchy sequence converges to a point in X . A mapping $\mathcal{T} : X \rightarrow X$ is called relational $\mathbf{R}$ -continuous at a point $x \in \mathbb{R}^n$ if and only if $x_n \rightarrow x$ implies $\mathcal{T}x_n \rightarrow \mathcal{T}x$ for every sequence $\{x_n\}$ of points relatable to x in $\mathbb{R}^n$. A few details of above notions appear in Anam and Imdad [2] and Dhage and Dhage [9].

2. Nonunique fixed point results

Before going to the main nonunique fixed point theorems we state a couple of mathematical concepts which we need in what follows.

Definition 1. A mapping $\mathcal{T} : X \rightarrow X$ is called $\mathcal{T}$ -orbital $\mathbf{R}$ -continuous at a point $x \in X$ if and only if it is $\mathbf{R}$ -continuous on $O(x, \mathcal{T})$ for each $x \in X$. Again, $(X, d, \mathbf{R})$ is called $\mathcal{T}$ -orbital $\mathbf{R}$ -complete if $O(x, \mathcal{T})$ is $\mathbf{R}$ -complete for each $x \in X$.

Remark 1. It is clear that

Completeness of $X \Rightarrow \mathcal{T}$ -orbital completeness of $X \Rightarrow \mathcal{T}$ -orbital $\mathbf{R}$ -completeness of X

and

Continuity of $\mathcal{T} \Rightarrow \mathcal{T}$ -orbital continuity of $\mathcal{T} \Rightarrow \mathcal{T}$ -orbital $\mathbf{R}$ -continuity of $\mathcal{T}$,

however the reverse implications may not hold.

Definition 2. A mapping $\mathcal{T} : X \rightarrow X$ is called a **relational $\mathbf{R}$ -relatone** if it preserves the binary relation $\mathbf{R}$ in X , that is, if $x, y \in X$ with $(x, y) \in \mathbf{R}$, then $(\mathcal{T}x, \mathcal{T}y) \in \mathbf{R}$.

We consider the following notations in what follows. We denote

- (i) $F_{\mathcal{T}}(X) = \{x \in X : \mathcal{T}x = x\}$,
- (ii) $L_{\mathcal{T}}(X) = \{x \in X : (x, \mathcal{T}x) \in \mathbf{R}\}$, and
- (iii) $U_{\mathcal{T}}(X) = \{x \in X : (\mathcal{T}x, x) \in \mathbf{R}\}$.

Theorem 1. Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital $\mathbf{R}$ -continuous and $\mathbf{R}$ -relatone self-mapping of a $\mathbf{R}$ -regular and $\mathcal{T}$ -orbital $\mathbf{R}$ -complete metric space X . Suppose that there exists a real number b such that

$$\begin{aligned}
 0 &\leq \min \{d(\mathcal{T}x, \mathcal{T}y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \\
 &\quad \frac{d(y, \mathcal{T}y)[1 + d(x, \mathcal{T}x)]}{1 + d(x, y)}, \frac{d(x, \mathcal{T}x)[1 + d(y, \mathcal{T}y)]}{1 + d(x, y)} \} \\
 &\quad + b \min \{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\} \\
 &\leq q \max \left\{ d(x, y) \bigcup [\min \{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\}] \right\}
 \end{aligned} \tag{2}$$

for all relatable elements $x, y \in X$, where $0 \leq q < 1$. If there exists an element $x_0 \in L_{\mathcal{T}}(X) \neq \emptyset$ or $x_0 \in U_{\mathcal{T}}(X) \neq \emptyset$, then $F_{\mathcal{T}}(X) \neq \emptyset$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to an element $x^* \in F_{\mathcal{T}}(X)$ relationally. Moreover, $\mathcal{T}$ has a unique relatable fixed point provided $b > q$.

Proof. Suppose first that $L_{\mathcal{T}}(S) \neq \emptyset$, then there exists an element $x_0 \in S$ such

that $(x_0, \mathcal{T}x_0) \in \mathbf{R}$. Define a sequence $\{x_n\}_{n=0}^\infty = \{\mathcal{T}^n x_0\}_{n=0}^\infty$ of iterations of $\mathcal{T}$ at $x_0 \in X$. Since $\mathcal{T}$ is relatone, we have $(x_n, x_{n+1}) \in \mathbf{R}$ for each n , $n = 0, 1, \dots$. If $x_r = x_{r+1}$ for some $r \in \mathbb{N}$, then the conclusion of the theorem follows immediately. Therefore, we assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. We show that $\{x_n\}$ is $\mathbf{R}$ -Cauchy sequence. Let $x = x_{n-1}$ and $y = x_n$ in the inequality (2), we obtain

$$\begin{aligned} 0 &\leq \min \left\{ d(x_n, x_{n+1}), d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_{n-1}, x_n)} \right\} \\ &\quad + b \min \{ d(x_{n-1}, x_{n+1}), 0 \} \\ &\leq q \max \left\{ d(x_{n-1}, x_n) \bigcup [\min \{ d(x_{n-1}, x_{n+1}), 0 \}] \right\}, \end{aligned}$$

$$\text{i.e.,} \quad \min \left\{ d(x_n, x_{n+1}), d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_{n-1}, x_n)} \right\} \leq q d(x_{n-1}, x_n).$$

Since $d(x_{n-1}, x_n) \leq q d(x_{n-1}, x_n)$ is not possible, because $q < 1$, we have that

$$\min \left\{ d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_{n-1}, x_n)} \right\} \leq q d(x_{n-1}, x_n).$$

Now there are two possibilities:

$$\min \left\{ d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_{n-1}, x_n)} \right\} = d(x_n, x_{n+1})$$

or

$$\min \left\{ d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_{n-1}, x_n)} \right\} = \frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_{n-1}, x_n)}.$$

In the first possibility, we get

$$d(x_n, x_{n+1}) \leq q d(x_{n-1}, x_n).$$

In the second possibility, we obtain

$$\frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_{n-1}, x_n)} \leq q d(x_{n-1}, x_n)$$

or

$$\frac{1 + d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} \leq q,$$

that is,

$$1 + d(x_n, x_{n+1}) \leq q + q d(x_{n-1}, x_n) \leq 1 + q d(x_{n-1}, x_n)$$

which implies that

$$d(x_n, x_{n+1}) \leq q d(x_{n-1}, x_n).$$

Thus, in both the cases, we obtain

$$d(x_n, x_{n+1}) \leq q d(x_{n-1}, x_n), \quad (3)$$

for each $n \in \mathbb{N}$.

By repeated application of the above inequality (3) yields

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1) \quad (4)$$

for each $n \in \mathbb{N}$. Next for any positive integer p , by triangle inequality, we set

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + \dots + d(x_{n+p-1}, x_{n+p}) \\ &\leq (q^n + \dots + q^{n+p-1}) d(x_0, x_1) \\ &= q^n \left(\frac{1 - q^{p-1}}{1 - q} \right) d(x_0, x_1) \\ &\leq \left(\frac{q^n}{1 - q} \right) d(x_0, x_1) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (5)$$

This shows that $\{x_n\}$ is **R**-Cauchy sequence in $O(x, \mathcal{T})$. Since X is $\mathcal{T}$ -orbital **R**-complete, $\{x_n\}$ converges to a point u in X . From $\mathcal{T}$ -orbital **R**-continuity of $\mathcal{T}$, it follows that

$$u = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \mathcal{T}x_n = \mathcal{T} \left(\lim_{n \rightarrow \infty} x_n \right) = \mathcal{T}u.$$

Thus u is a fixed point of $\mathcal{T}$. Similarly, if $U_{\mathcal{T}}(X) \neq \emptyset$, then following similar arguments, it can be shown that $\mathcal{T}$ has a fixed point. Further assume that $b > q$ and let $v(\neq u)$ be another relatable fixed point of $\mathcal{T}$. Then from condition (2) we get a contradiction whence $\mathcal{T}$ has a unique relatable fixed point. This completes the proof.

On taking $b = 0$ in contraction condition (2) we obtain the following fixed point result.

Corollary 1. Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital $\mathbf{R}$ -continuous and $\mathbf{R}$ -relatone self-mapping of $\mathbf{R}$ -regular and $\mathcal{T}$ -orbital $\mathbf{R}$ -complete metric space X . Suppose that $\mathcal{T}$ satisfies the contraction condition

$$\begin{aligned} & \min \{d(\mathcal{T}x, \mathcal{T}y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \\ & \quad \frac{d(y, \mathcal{T}y)[1 + d(x, \mathcal{T}x)]}{1 + d(x, y)}, \frac{d(x, \mathcal{T}x)[1 + d(y, \mathcal{T}y)]}{1 + d(x, y)}\} \\ & \leq q \max \left\{ d(x, y) \bigcup [\min\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\}] \right\} \end{aligned} \quad (6)$$

for all relatable elements $x, y \in X$, where $0 \leq q < 1$. Furthermore, if there exists an element $x_0 \in L_{\mathcal{T}}(X) \neq \emptyset$ or $x_0 \in U_{\mathcal{T}}(X) \neq \emptyset$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^* relationally.

Again taking $b = -1$ in contraction condition (2), we obtain the following improvement of the fixed point result for Ciric [3] type mappings in $\mathcal{T}$ -orbital $\mathbf{R}$ -complete metric spaces.

Corollary 2. Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital $\mathbf{R}$ -continuous and $\mathbf{R}$ -relatone self-mapping of a $\mathbf{R}$ -regular and $\mathcal{T}$ -orbital $\mathbf{R}$ -complete metric space X . Suppose that $\mathcal{T}$ satisfies the contraction condition

$$\begin{aligned} & 0 \leq \min \{d(\mathcal{T}x, \mathcal{T}y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \\ & \quad \frac{d(y, \mathcal{T}y)[1 + d(x, \mathcal{T}x)]}{1 + d(x, y)}, \frac{d(x, \mathcal{T}x)[1 + d(y, \mathcal{T}y)]}{1 + d(x, y)}\} \\ & \quad - \min\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\} \\ & \leq q \max \left\{ d(x, y) \bigcup [\min\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\}] \right\} \end{aligned} \quad (7)$$

for all relatable elements $x, y \in X$, where $0 \leq q < 1$. If there exists an element $x_0 \in L_{\mathcal{T}}(X) \neq \emptyset$ or $x_0 \in U_{\mathcal{T}}(X) \neq \emptyset$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^* relationally.

Remark 2. The sufficient contraction condition for the existence of nonunique fixed point given by Dhage and Dhage [10] is as follows: there exists a real number $b \in \mathbb{R}$ such that

$$\begin{aligned} & 0 \leq \min\{d(\mathcal{T}x, \mathcal{T}y), d(x, \mathcal{T}x), d(y, \mathcal{T}y)\} \\ & \quad + b \min\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\} \leq q d(x, y) \end{aligned} \quad (8)$$

for all relatable elements $x, y \in X$, where $0 \leq q < 1$. Therefore, the results of this paper include the nonunique fixed point theorems of [10] as special cases.

Next, we introduce a partial order $\preceq$ in X which is a reflexive, anti-symmetric and transitive relation, so that non-empty set X now becomes a **partially ordered metric space** and it is denoted by $(X, d, \preceq)$. A mapping $\mathcal{T}$ is called **partially continuous** at a point $x \in X$ if and only if $x_n \rightarrow x$ implies $\mathcal{T}x_n \rightarrow \mathcal{T}x$ for every sequence $\{x_n\}$ of points comparable to x in X . $\mathcal{T}$ is called partial continuous on X if it is partial continuous at each point $x \in X$. A **partially ordered complete metric space** X is one in which every monotonic Cauchy sequence converges to a point in X . The details of these partial notions appear in Dhage [8] and Dhage *et al.* [9] and references therein. We consider the following definitions in what follows.

Definition 3. A mapping $\mathcal{T} : X \rightarrow X$ is called $\mathcal{T}$ -orbital partially continuous at a point $x \in X$ if and only if it is partial continuous on $O(x, \mathcal{T})$ for each $x \in X$. Again, $(X, d, \preceq)$ is called $\mathcal{T}$ -orbital partially complete if $O(x, \mathcal{T})$ is partial complete for each $x \in X$.

Remark 3. Note that every partial order is a relation but the converse is not necessarily true, so every $\mathcal{T}$ -orbital partially continuous mapping $\mathcal{T} : X \rightarrow X$ is $\mathcal{T}$ -orbital relationally continuous, but the converse may not be true. Similarly, every $\mathcal{T}$ -orbital partially complete metric space is $\mathcal{T}$ -orbital relationally complete, however the converse need not hold.

Definition 4. A mapping $\mathcal{T} : X \rightarrow X$ is called a **monotone nondecreasing** or simply **nondecreasing** if it preserves the partial order $\preceq$ in X , that is, if $x, y \in X$ with $x \preceq y$, then $\mathcal{T}x \preceq \mathcal{T}y$.

Now, we obtain a nonunique fixed point theorem for the $\mathcal{T}$ -orbital partial continuous monotone nondecreasing mappings on mappings $\mathcal{T}$ -orbital partial complete metric space X .

Theorem 2. Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital partial continuous and monotone nondecreasing self-mapping of a regular and $\mathcal{T}$ -orbital partial complete metric space X . Suppose that there exists a real number b such that the contraction condition (2) holds for all comparable elements $x, y \in X$. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $F_{\mathcal{T}}(X) \neq \emptyset$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to an element $x^* \in F_{\mathcal{T}}(X)$ monotonically. Moreover, $\mathcal{T}$ has a unique comparable fixed point provided $b > q$.

Proof. The proof is obvious and the conclusion follows by an application of Theorem 1 in view of Remark 3.

Note that Theorem 2 includes two more nonunique fixed point results for the mapping satisfying the contraction conditions of the type (6) and (7) in the partially ordered metric spaces. When $b = 0$, we obtain the following result.

Corollary 3. *Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital $\mathbf{R}$ -continuous and $\mathbf{R}$ -relatone self-mapping of a $\mathbf{R}$ -regular and $\mathcal{T}$ -orbital $\mathbf{R}$ -complete metric space X . Suppose that $\mathcal{T}$ satisfies the contraction condition (6) holds for all comparable elements $x, y \in X$. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $F_{\mathcal{T}}(X) \neq \emptyset$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to an element $x^* \in F_{\mathcal{T}}(X)$ monotonically.*

As a consequence of Corollary 3, we obtain the following result which is also new to the ordered metric fixed point theory.

Corollary 4. *Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital partial continuous and monotone nondecreasing self-mapping of a regular and $\mathcal{T}$ -orbital partial complete metric space X . Suppose that $\mathcal{T}$ satisfies the contraction condition (7) for all comparable elements $x, y \in X$. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $F_{\mathcal{T}}(X) \neq \emptyset$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to an element $x^* \in F_{\mathcal{T}}(X)$ monotonically.*

The contraction condition (7) is a two-way generalization of the contraction condition given in the following applicable hybrid fixed point theorem of Dhage [8] in the ordered metric fixed point theory.

Corollary 5. *Let $\mathcal{T}$ be a nondecreasing self-mapping of a regular partially complete metric space $(X, d, \preceq)$ satisfying the contraction condition*

$$d(\mathcal{T}x, \mathcal{T}y) \leq q d(x, y) \quad (9)$$

for all comparable elements $x, y \in X$. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a unique comparable fixed point x^* in X and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^* monotonically.

Example 1. Let $X = [0, 1]$ and let $<$ be a relation defined in $[0, 1]$. Then the triplet $([0, 1], |\cdot|, <)$ becomes a relationally $\mathbf{R}$ -complete metric space, where $\mathbf{R} = <$. Define a mapping $\mathcal{T} : [0, 1] \rightarrow [0, 1]$ by

$$\mathcal{T}x = \begin{cases} x^2 & \text{if } x \in [0, \frac{1}{2}), \\ x^3 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases} \quad (10)$$

Note that $\mathcal{T}$ is not continuous on $[0, 1)$. However, $\mathcal{T}$ is $\mathcal{T}$ -orbital relationally $\mathbf{R}$ -continuous on $[0, 1]$ and $[0, 1]$ is $\mathcal{T}$ -orbital relationally $\mathbf{R}$ -complete metric space. Moreover the condition (8) is satisfied with $b = 0$ and $q = \frac{1}{4}$. Also we have $U_{\mathcal{T}}(X) \neq \emptyset$. Therefore, by Corollary 3, $\mathcal{T}$ has two fixed points $u = 0$ and $v = 1$.

Example 2. Let $X = [0, 1) \cup (1, 2]$ with usual metric in $\mathbb{R}$ and define a relation $\mathbf{R} = <$ in X so that the triplet $(X, |\cdot|, <)$ becomes a relational metric space. Define a mapping $f : X \rightarrow X$ by

$$f(x) = \begin{cases} x^2 + \frac{x}{2} & \text{if } x \in [0, 1), \\ \frac{x^2}{2} & \text{if } x \in (1, 2]. \end{cases} \quad (11)$$

Clearly, X is not a complete metric space, but it is a relationally f -orbitally complete metric space. Again f is not a continuous mapping, but is a relationally f -orbitally continuous mapping on X . Next, f satisfies the contraction condition (2) with $b = -1$ and $q = \frac{3}{4}$. Finally, there exists an element $x_0 = \frac{1}{3}$ such that $x_0 = \frac{3}{4} > \frac{5}{18} = f(x_0)$. Therefore f has a fixed point in X . In fact f has three fixed points in X namely, $0, \frac{1}{2}$ and 2 .

3. Local nonunique fixed point results

A closed ball $B[x, r]$ in a metric space (X, d) centered at a point $x \in X$ of radius r is defined by $B[x, r] = \{y \in X \mid d(x, y) \leq r\}$. Is it clear that a closed ball in a relationally $\mathbf{R}$ -complete metric space is $\mathbf{R}$ -complete.

Theorem 3. Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital $\mathbf{R}$ -continuous and $\mathbf{R}$ -relatone self-mapping of $\mathbf{R}$ -regular and $\mathcal{T}$ -orbital $\mathbf{R}$ -complete metric space X . Suppose that there exists a real number b such that the contraction condition (2) holds. . If there is an element $x_0 \in L_{\mathcal{T}}(X)$ or $x_0 \in U_{\mathcal{T}}(X)$ such that

$$d(x_0, \mathcal{T}x_0) \leq (1 - q)r, \quad r > 0, \quad (12)$$

then $\mathcal{T}$ has a local fixed point $x^* \in B[x_0, r]$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^* relationally. The fixed point x^* is unique relatable if $b > q$.

Proof. Suppose first that there exists an element $x_0 \in L_{\mathcal{T}}(X)$, then $(x_0, \mathcal{T}x_0) \in \mathbf{R}$. Define a sequence $\{x_n\}_{n=0}^{\infty} = \{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of iterations of $\mathcal{T}$ at $x_0 \in X$. Since $\mathcal{T}$ is relatone, we have $(x_n, x_{n+1}) \in \mathbf{R}$ for each $n, n = 0, 1, \dots$. If $x_r = x_{r+1}$ for some $r \in \mathbb{N}$, then the conclusion of the theorem follows immediately. Therefore, we assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. We show that $\{x_n\}$ is $\mathbf{R}$ -Cauchy sequence. Let $x = x_{n-1}$ and $y = x_n$ in the inequality (2), we obtain

$$d(x_n, x_{n+1}) = d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) \leq qd(x_{n-1}, x_n) \quad (13)$$

for all $n \in \mathbb{N}$. By repeated application of the above inequality (13) yields

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1) \quad (14)$$

for each $n \in \mathbb{N}$. Next for any positive integer p , by triangle inequality, we set

$$\begin{aligned}
 d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + \dots + d(x_{n+p-1}, x_{n+p}) \\
 &\leq (q^n + \dots + q^{n+p-1})d(x_0, x_1) \\
 &= q^n \left(\frac{1 - q^{p-1}}{1 - q} \right) d(x_0, x_1) \\
 &\leq \left(\frac{q^n}{1 - q} \right) d(x_0, x_1) \\
 &\rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{15}$$

This shows that $\{x_n\}$ is an **R**-Cauchy sequence in $O(x, \mathcal{T})$. Next, we show that $O(x, \mathcal{T}) \subset B[x_0, r]$. Now from inequality (12) we obtain

$$d(x_0, x_1) = d(x_0, \mathcal{T}x_0) \leq (1 - q)r \leq r$$

and so $x_1 \in B[x_0, r]$. Similarly, by induction, we have

$$\begin{aligned}
 d(x_0, x_n) &\leq d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{n-1}, x_n) \\
 &\leq d(x_0, x_1) + qd(x_0, x_1) + \dots + q^{n-1}d(x_0, x_1) \\
 &= (1 + q + q^2 + \dots + q^{n-1})d(x_0, x_1) \\
 &= \frac{1 - q^n}{1 - q} (1 - q)r \\
 &\leq \frac{1}{1 - q} \cdot (1 - q)r \\
 &= r
 \end{aligned}$$

showing that $x_n \in B[x_0, r]$ for all $n \in \mathbb{N}$. As a result we have that $O(x, \mathcal{T}) \subset B[x_0, r]$. Since $B[x_0, r]$ is a closed subset of X , it is $\mathcal{T}$ -orbital **R**-complete. Hence $\{x_n\}$ converges to a point u in $B[x_0, r]$. From $\mathcal{T}$ -orbital **R**-continuity of $\mathcal{T}$, it follows that

$$u = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \mathcal{T}x_n = \mathcal{T} \left(\lim_{n \rightarrow \infty} x_n \right) = \mathcal{T}u.$$

Thus u is a local fixed point of $\mathcal{T}$. Similarly, if there is an element $x_0 \in U_{\mathcal{T}}(X)$ satisfying (2), then following the similar arguments, it can be shown that $\mathcal{T}$ has a local fixed point. Furthermore, assume that $b > q$ and let $v (\neq u)$ be another relatable fixed point of $\mathcal{T}$. Then from condition (2) we get a contradiction whence $\mathcal{T}$ has a unique relatable fixed point. This completes the proof.

On taking $b = 0$ in contraction condition (2) we obtain the following fixed point result.

Corollary 6. *Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital $\mathbf{R}$ -continuous and $\mathbf{R}$ -relatone self-mapping of $\mathbf{R}$ -regular and $\mathcal{T}$ -orbital $\mathbf{R}$ -complete metric space X . Suppose that $\mathcal{T}$ satisfies the contraction condition (6). If there is an element $x_0 \in L_{\mathcal{T}}(X)$ or $x_0 \in U_{\mathcal{T}}(X)$ such that the condition (12) holds for some real number $r > 0$, then $\mathcal{T}$ has a local fixed point $x^* \in B[x_0, r]$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^* relationally.*

Again taking $b = -1$ in contraction condition (2), we obtain the following fixed point result Ćirić [3] type mappings in metric spaces.

Corollary 7. *Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital $\mathbf{R}$ -continuous and $\mathbf{R}$ -relatone self-mapping of $\mathbf{R}$ -regular and $\mathcal{T}$ -orbital $\mathbf{R}$ -complete metric space X . Suppose that $\mathcal{T}$ satisfies the contraction condition (7). If there is an element $x_0 \in L_{\mathcal{T}}(X)$ or $x_0 \in U_{\mathcal{T}}(X)$ such that the condition (12) holds for some real number $r > 0$, then $\mathcal{T}$ has a local fixed point $x^* \in B[x_0, r]$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^* relationally.*

Corollary 8. *Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital partially continuous and monotone nondecreasing self-mapping of regular and $\mathcal{T}$ -orbital partially complete metric space X . Suppose that there exists a real number b such that the contraction condition (2) holds for all comparable elements $x, y \in X$. If there is an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (12) for some real number $r > 0$, then $\mathcal{T}$ has a local fixed point $x^* \in B[x_0, r]$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^* monotonically. Moreover, the fixed point x^* is unique comparable if $b > q$.*

Corollary 9. *Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital partially continuous and monotone nondecreasing self-mapping of a regular and $\mathcal{T}$ -orbital partially complete metric space X . Suppose that the mapping $\mathcal{T}$ satisfies the contraction condition (6) for all comparable elements $x, y \in X$. If there is an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (12) for some real number $r > 0$, then $\mathcal{T}$ has a local fixed point $x^* \in B[x_0, r]$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^* monotonically.*

Corollary 10. *Let $\mathcal{T}$ be a $\mathcal{T}$ -orbital partially continuous and monotone nondecreasing self-mapping of a regular and $\mathcal{T}$ -orbital partially complete metric space X . Suppose that the mapping $\mathcal{T}$ satisfies the contraction condition (7) for all comparable elements $x, y \in X$. If there is an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (12) for some real number $r > 0$, then $\mathcal{T}$ has a local fixed*

point $x^* \in B[x_0, r]$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^* monotonically.

Again, Corollary 9 includes the following widely used local hybrid fixed point theorem of Dhage [7] in the subject of nonlinear differential and integral equations as a special case.

Corollary 11. *Let $(X, d, \preceq)$ be a regular partially complete metric space and let $\mathcal{T} : X \rightarrow X$ be a nondecreasing mapping satisfying the contraction condition (8). If there is an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (12) for some real number $r > 0$, then $\mathcal{T}$ has a unique comparable fixed point $x^* \in B[x_0, r]$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^* monotonically.*

Remark 4. *The hybrid fixed point results of his paper also remain true if we replace the relational and partial continuity of the mappings $\mathcal{T}$ on a relational or partially ordered metric space X with the following weaker ones: A mapping $\mathcal{T} : X \rightarrow X$ is called relational **R**-continuous at a point $x \in X$ if and only if $x_n \rightarrow x$ and $(x_n, x) \in \mathbf{R}$ or $(x, x_n) \in \mathbf{R}$ for sufficiently large n , implies $\mathcal{T}x_n \rightarrow \mathcal{T}x$ as $n \rightarrow \infty$. Similarly, a mapping $\mathcal{T} : X \rightarrow X$ is called partially continuous at a point $x \in X$ if and only if $x_n \rightarrow x$ and x_n and x are comparable for sufficiently large n , implies $\mathcal{T}x_n \rightarrow \mathcal{T}x$ as $n \rightarrow \infty$.*

4. Discussion

From the foregoing discussion we observe that our nonunique fixed point results of the mappings in relational as well as partially ordered metric spaces are the multi-directional generalizations of the fixed point mappings satisfying the contraction condition (9). In the first place the condition (9) is generalized in both the left as well as right direction. Next, the continuity of the mappings is weakened to orbital relational and partial continuity in view of Remark 4. Similarly, the completeness of the metric spaces is relaxed to orbital relational and partial completeness of the metric space. Our approach to continuity of the mappings is different from that of Alam and Imdad [2], Tomar *et al.* [17]. Thus the hybrid fixed point results of this paper are the generalizations of the several well-known fixed point theorems for mappings in the relational and partially ordered metric spaces under weaker conditions. Finally, the nonunique fixed point theorems of this paper may be extended to multi-metric spaces as done Dhage *et al.* [13] with appropriate modifications.

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