

FIXED POINT THEOREMS AND COMMON FIXED POINT THEOREMS IN B-METRIC SPACE WITH GRAPH

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Abstract: In this paper, we investigate common fixed point theorems for a pair of self-mappings defined on a complete (b)-metric space endowed with a directed graph structure. By employing the notion of a strong comparison function together with graph-based (A)-type contractive conditions, we establish new sufficient conditions ensuring the existence and uniqueness of a common fixed point. The obtained results extend, generalize, and unify several existing fixed point theorems in the literature, including those of M. Ozturk and E. Girgin [6], as well as related results in metric and graph-endowed spaces. Illustrative examples are presented to demonstrate the applicability and effectiveness of the established theorems.

Keywords and Phrases: Common fixed point, b-metric space, directed graph, A-type contraction.

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1. Introduction

The Banach contraction principle, introduced by Banach in 1922 [4], is one of the cornerstones of modern fixed point theory. It asserts that every contraction mapping on a complete metric space admits a unique fixed point. Owing to its simplicity and wide applicability, this celebrated theorem has inspired extensive research leading to numerous generalizations and extensions in different mathematical frameworks.

Among these extensions, the notion of a (b)-metric space has received considerable attention. Introduced as a natural generalization of metric spaces, a (b)-metric space replaces the classical triangle inequality by a weaker inequality involving a constant ($s \geq 1$). This relaxed structure enlarges the class of admissible spaces and has proved useful in studying the existence and uniqueness of fixed points for mappings that do not necessarily satisfy standard contractive conditions. Several authors have investigated fixed point problems in (b)-metric spaces and established important theoretical and applied results [6].

Another active direction of research concerns metric spaces endowed with graphs. The incorporation of graph structures into fixed point theory allows one to combine metric properties with relational information represented by edges of a graph. This approach has attracted significant interest because many problems arising in applied mathematics, computer science, and network theory naturally possess graph-theoretic features. In a pioneering work, Jachymski [5] demonstrated that the Banach contraction principle remains valid in metric spaces endowed with a graph. Subsequently, Abbas et al. [3] developed graph-preserving contractive conditions and obtained several common fixed point results in graph-based settings.

Fixed point theory has also been investigated in generalized fuzzy metric spaces and fuzzy 2-metric spaces. Tripathy et al. [8] established a fixed point theorem in generalized fuzzy metric spaces, whereas Tripathy et al. [9] obtained Banach- and Kannan-type fixed point results in fuzzy 2-metric spaces.

Motivated by these developments, we introduce the notion of an $((A_\varphi, G))$ -contraction and investigate its properties in the setting of complete (b)-metric spaces endowed with a directed graph. By combining graph-theoretic techniques with generalized contractive conditions, we establish new existence and uniqueness results for common fixed points. Several examples are provided to illustrate the applicability and effectiveness of the obtained results.

2. Preliminaries

In this section, we present some basic definitions, concepts, and results that will be used throughout the paper.

Definition 1. [1] Let X be a nonempty set. A function $d : X \times X \rightarrow \mathbb{R}_+$ is called a metric if it satisfies the following conditions:

1. $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
- 3.

$$d(x, y) \leq d(x, z) + d(z, y), \quad \forall x, y, z \in X.$$

Definition 2. [1] Let $\mathbb{R}_+$ denote the set of all non-negative real numbers, and let A be the set of all functions $\alpha : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ satisfying:

1. α is continuous on $\mathbb{R}_+^3$;
2. There exists $k \in [0, 1)$ such that for all $a, b \in \mathbb{R}_+$,

$$a \leq \alpha(a, b, b) \quad \text{or} \quad a \leq \alpha(b, a, b) \quad \text{or} \quad a \leq \alpha(b, b, a)$$

implies

$$a \leq kb.$$

The mappings belonging to A are called A -contractions.

Definition 3. [1] A self-map $T : X \rightarrow X$ on a metric space (X, d) is called an A -contraction if

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty))$$

for all $x, y \in X$ and some $\alpha \in A$.

Definition 4. [5] A mapping $T : X \rightarrow X$ is called a Banach G -contraction if the following conditions hold:

1. T preserves the edges of G , that is,

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G), \quad \forall x, y \in X;$$

2. There exists $\eta \in (0, 1)$ such that

$$(x, y) \in E(G) \Rightarrow d(Tx, Ty) \leq \eta d(x, y), \quad \forall x, y \in X.$$

Definition 5. [5] A mapping $T : X \rightarrow X$ is called orbitally G -continuous if for any $x, y \in X$ and any sequence $\{n_p\}_{p \in \mathbb{N}}$ of positive integers,

$$T^{n_p}x \rightarrow y \quad \text{and} \quad (T^{n_p}x, T^{n_p+1}x) \in E(G)$$

imply

$$T(T^{n_p}x) \rightarrow Ty \quad \text{as } p \rightarrow \infty.$$

Definition 6. [2] A graph G is called a (C) -graph if for every sequence $\{x_n\}_{n \geq 0}$ in X satisfying $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$, there exists a subsequence $\{x_{n_k}\}_{k \geq 0}$ such that $(x_{n_k}, x) \in E(G)$ for all sufficiently large k .

Definition 7. [7] Let $\Phi = \{\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+\}$ be a class of functions satisfying:

1. $t_1 \leq t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$;
2. $\varphi^n(t) \rightarrow 0$ for all $(t > 0)$;
3. $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for every $(t > 0)$.

If (i) and (ii) hold, then φ is called a comparison function.

If (i), (ii) and (iii) hold, then φ is called a strong comparison function.

Remark 1. [7] Every strong comparison function is a comparison function.

Remark 2. [7] If $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a comparison function, then

$$\varphi(t) < t, \quad \forall t > 0,$$

$$\varphi(0) = 0,$$

and φ is right continuous at 0.

Definition 8. [6] A b -metric space is a pair (X, d_b) , where $d_b : X \times X \rightarrow \mathbb{R}_+$ satisfies:

1. $d_b(x, y) = 0 \iff x = y$;
2. $d_b(x, y) = d_b(y, x), \quad \forall x, y \in X$;
3. There exists $s \geq 1$ such that

$$d_b(x, z) \leq s[d_b(x, y) + d_b(y, z)],$$

for all $x, y, z \in X$.

Definition 9. [6] A graph $G = (V, E)$ is said to be weakly connected if for every pair of vertices $x, y \in V$, there exists an undirected path joining x and y when the orientation of edges is ignored.

Definition 10. [6] Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a strong comparison function.

We define A_φ to be the set of all mappings $\alpha : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ satisfying:

1. α is continuous on $\mathbb{R}_+^3$;

2. For all $u, v \in \mathbb{R}_+$,

$$u \leq \alpha(u, v, v)$$

or

$$u \leq \alpha(v, u, v)$$

or

$$u \leq \alpha(v, v, u)$$

implies

$$u \leq \varphi(v).$$

Remark 3. If $\varphi(t) = kt$, $k \in (0, 1)$, then A_φ reduces to the classical class A .

Definition 11. [6] Let (X, d_b) be a b -metric space endowed with a graph G .

A self-map $T : X \rightarrow X$ is called an (A_φ, G) -contraction if:

1.

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G), \quad \forall x, y \in X;$$

2. There exists a function $\alpha \in A_\varphi$ such that

$$d_b(Tx, Ty) \leq \alpha(d_b(x, y), d_b(x, Tx), d_b(y, Ty)),$$

for all

$$(x, y) \in E(G).$$

2. Main Results

In this section, we present a common fixed point theorem for (A_φ, G) -contractions in b -metric spaces endowed with a graph.

Theorem 1. Let (X, d_b) be a complete b -metric space endowed with a graph G , and let $S, T : X \rightarrow X$ be two self-mappings satisfying the following conditions:

1. G is weakly connected and a (C) -graph.

2. The mappings S and T preserve the edges of G , that is,

$$(x, y) \in E(G) \implies (Sx, Sy) \in E(G) \quad \text{and} \quad (x, y) \in E(G) \implies (Tx, Ty) \in E(G),$$

for all $x, y \in X$.

3. There exists a function $\alpha \in A_\varphi$ such that

$$d_b(Sx, Ty) \leq \alpha(d_b(x, y), d_b(x, Sx), d_b(y, Ty)),$$

for all $(x, y) \in E(G)$.

4. The set

$$X_{S,T} = \{x \in X : (x, Sx) \in E(G) \text{ and } (Sx, TSx) \in E(G)\}$$

is nonempty.

Then the mappings S and T have a unique common fixed point in X .

Proof. Let $x_0 \in X_{S,T}$ and define a sequence $\{x_n\}$ in X by

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1}, \quad \forall n \geq 0.$$

Since S and T preserve the edges of G , it follows that

$$(x_n, x_{n+1}) \in E(G), \quad \text{for all } n \geq 0.$$

Using the (A_φ, G) -contractive condition, we obtain

$$\begin{aligned} d_b(x_{2n+1}, x_{2n+2}) &= d_b(Sx_{2n}, Tx_{2n+1}) \\ &\leq \alpha(d_b(x_{2n}, x_{2n+1}), d_b(x_{2n}, Sx_{2n}), d_b(x_{2n+1}, Tx_{2n+1})), \end{aligned}$$

for all $(x_{2n}, x_{2n+1}) \in E(G)$.

By applying the above inequality recursively, we get

$$d_b(x_n, x_{n+1}) \leq \alpha^n(d_b(x_0, x_1), d_b(x_0, Sx_0), d_b(x_1, Tx_1)).$$

Since $\alpha^n(t, t, t) \rightarrow 0$ as $n \rightarrow \infty$, it follows that

$$\lim_{n \rightarrow \infty} d_b(x_n, x_{n+1}) = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X . Therefore, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$d_b(x_m, x_n) < \varepsilon.$$

Since (X, d_b) is complete, there exists $x^* \in X$ such that

$$x_n \rightarrow x^* \quad \text{as } n \rightarrow \infty.$$

Next, we show that x^* is a common fixed point of S and T . Taking limits as $n \rightarrow \infty$, we obtain

$$d_b(x_{2n}, x_{2n+1}) = d_b(x_{2n}, Sx_{2n}) \rightarrow d_b(x^*, Sx^*),$$

and

$$d_b(x_{2n+1}, x_{2n+2}) = d_b(x_{2n+1}, Tx_{2n+1}) \rightarrow d_b(x^*, Tx^*).$$

Using the contractive condition together with the properties of the function α , we obtain

$$d_b(x^*, Sx^*) = 0 \quad \text{and} \quad d_b(x^*, Tx^*) = 0,$$

which implies that

$$Sx^* = x^* \quad \text{and} \quad Tx^* = x^*.$$

Hence, x^* is a common fixed point of S and T .

For uniqueness, suppose that $y^* \neq x^*$ is another common fixed point of S and T . Then

$$d_b(x^*, y^*) = d_b(Sx^*, Ty^*) \leq \alpha(d_b(x^*, y^*), d_b(x^*, Sx^*), d_b(y^*, Ty^*)).$$

Since $Sx^* = x^*$ and $Ty^* = y^*$, the above inequality reduces to

$$d_b(x^*, y^*) \leq \alpha(d_b(x^*, y^*), 0, 0).$$

Using the fact that $\alpha(t, t, t) < t$ for all $t > 0$, we obtain a contradiction unless

$$d_b(x^*, y^*) = 0,$$

which implies that $x^* = y^*$. Therefore, the common fixed point is unique.

Hence, the mappings S and T have a unique common fixed point in X .

Theorem 2. *Let (X, d_b) be a complete b -metric space endowed with a directed graph $G = (X, E(G))$, and let $S, T : X \rightarrow X$ be two self-maps. Assume that the following conditions hold:*

- (i) *G is a μ -graph, that is, for any sequence $\{x_n\}$ in X converging to some $x \in X$, there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x) \in E(G)$ or $(x, x_{n_k}) \in E(G)$ for all $k \in \mathbb{N}$.*

(ii) There exists a sequence $\{x_n\}$ in X such that

$$(x_{2n}, Sx_{2n}) \in E(G) \implies (x_{2n+1}, Sx_{2n+2}) \in E(G),$$

and

$$(x_{2n+1}, Tx_{2n+1}) \in E(G) \implies (x_{2n+3}, Tx_{2n+3}) \in E(G),$$

for all $n \in \mathbb{N}$.

(iii) There exists a function $\alpha \in A_\varphi$ such that

$$d_b(Sx, Ty) \leq \alpha(d_b(x, y), d_b(x, Sx), d_b(y, Ty)),$$

for all $(x, y) \in E(G)$.

(iv) The set

$$X_{S,T} = \{x \in X : (x, Sx) \in E(G) \text{ and } (Sx, TSx) \in E(G)\}$$

is nonempty.

Then the mappings S and T have a common fixed point in X .

Proof. Step I: Construction of the sequence

Let $x_0 \in X_{S,T}$ and define a sequence $\{x_n\}$ in X by

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad n = 0, 1, 2, \dots$$

Then,

$$(x_{2n}, x_{2n+1}) = (x_{2n}, Sx_{2n}) \in E(G), \quad (x_{2n+1}, x_{2n+2}) = (x_{2n+1}, Tx_{2n+1}) \in E(G),$$

for all $n \in \mathbb{N}$.

By condition (iii), for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d_b(x_{2n+1}, x_{2n+2}) &= d_b(Sx_{2n}, Tx_{2n+1}) \\ &\leq \alpha(d_b(x_{2n}, x_{2n+1}), d_b(x_{2n}, Sx_{2n}), d_b(x_{2n+1}, Tx_{2n+1})) \\ &= \alpha(d_b(x_{2n}, x_{2n+1}), d_b(x_{2n}, x_{2n+1}), d_b(x_{2n+1}, x_{2n+2})). \end{aligned}$$

By the definition of $\alpha \in A_\varphi$, it follows that

$$d_b(x_{2n+1}, x_{2n+2}) \leq \varphi(d_b(x_{2n}, x_{2n+1})).$$

Similarly, we obtain

$$d_b(x_{2n+2}, x_{2n+3}) \leq \varphi(d_b(x_{2n+1}, x_{2n+2})),$$

and by iteration,

$$d_b(x_n, x_{n+1}) \leq \varphi^n(d_b(x_0, x_1)), \quad \forall n \in \mathbb{N}.$$

Since $\varphi^n(t) \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} d_b(x_n, x_{n+1}) = 0.$$

Let $m > n$. By the b -metric inequality,

$$d_b(x_n, x_m) \leq s \sum_{k=n}^{m-1} d_b(x_k, x_{k+1}) \leq s \sum_{k=n}^{\infty} \varphi^k(d_b(x_0, x_1)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence in (X, d_b) . Since X is complete, there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = x^*.$$

Step II: Existence of common fixed point

From $(x_{2n}, x_{2n+1}) \in E(G)$ and the μ -graph property of G , there exists a subsequence $\{x_{2n_k}\}$ such that either $(x_{2n_k}, x^*) \in E(G)$ or $(x^*, x_{2n_k+1}) \in E(G)$ for all k .

Assume $(x^*, x_{2n_k+1}) \in E(G)$. Since T is continuous, we obtain

$$Tx^* = \lim_{k \rightarrow \infty} Tx_{2n_k+1} = \lim_{k \rightarrow \infty} x_{2n_k+2} = x^*.$$

Hence, $Tx^* = x^*$.

Now, using the contractive condition,

$$d_b(Sx^*, x^*) = d_b(Sx^*, Tx^*) \leq \alpha(d_b(x^*, x^*), d_b(x^*, Sx^*), d_b(x^*, Tx^*)) = 0,$$

which implies that $Sx^* = x^*$.

Therefore, x^* is a common fixed point of S and T .

Step III: Uniqueness

Suppose that y^* is another common fixed point of S and T , with $x^* \neq y^*$. Then,

$$\begin{aligned} d_b(x^*, y^*) &= d_b(Sx^*, Ty^*) \\ &\leq \alpha(d_b(x^*, y^*), d_b(x^*, Sx^*), d_b(y^*, Ty^*)) \\ &= \alpha(d_b(x^*, y^*), 0, 0) < d_b(x^*, y^*), \end{aligned}$$

a contradiction. Hence $x^* = y^*$.

Therefore, S and T have a unique common fixed point in X .

Corollary 1. *Let (X, d_b) be a complete b -metric space endowed with a weakly connected C -graph G . If the mappings $S, T : X \rightarrow X$ satisfy all the hypotheses of Theorem 1 and, in addition, both S and T are continuous on X , then the sequence $\{x_n\}$ constructed in the proof of Theorem 1 converges to a unique common fixed point of S and T , without requiring any additional graph-based assumption.*

Corollary 2. *Let (X, d_b) be a complete b -metric space and let G be a connected graph on X . Suppose that the mappings $S, T : X \rightarrow X$ satisfy all the hypotheses of Theorem 1. If, in addition, the function $\alpha \in A_\varphi$ satisfies*

$$\lim_{t \rightarrow 0} \alpha(t, t, t) = 0,$$

then the mappings S and T admit a unique common fixed point in X .

3. Application

In this section, we present some applications of the obtained results to illustrate their applicability and effectiveness.

1. Scalar contraction on a graph (Toy Example)

The following toy example is presented to illustrate the applicability of Theorem 1 in a simple and transparent setting. Although it is not intended to model a specific real-world phenomenon, it provides a convenient framework for explicitly verifying all the assumptions of the theorem. In particular, it demonstrates how the graph structure, the edge-preserving property, and the (A_φ, G) -contractive condition guarantee the existence and uniqueness of a common fixed point.

Let $X = \mathbb{R}$ and equip X with the usual metric

$$d_b(x, y) = |x - y|.$$

Then (X, d_b) is a complete b -metric space with coefficient $s = 1$.

Fix a radius $r > 0$ and define a directed graph $G = (X, E(G))$ by

$$(x, y) \in E(G) \iff |x - y| \leq r.$$

Let $c \in (0, 1)$ and define two self-mappings $S, T : X \rightarrow X$ by

$$S(x) = cx, \quad T(x) = cx, \quad x \in \mathbb{R}.$$

We verify the assumptions of Theorem 1.

1. Completeness and graph properties

Since $(\mathbb{R}, |\cdot|)$ is complete, (X, d_b) is a complete b -metric space. Moreover, the graph G is weakly connected and satisfies the required (C) -property.

2. Edge preservation

Suppose that $(x, y) \in E(G)$. Then

$$|x - y| \leq r.$$

Hence

$$d_b(Sx, Sy) = |cx - cy| = c|x - y| \leq cr \leq r,$$

since $0 < c < 1$. Therefore,

$$(Sx, Sy) \in E(G).$$

Similarly,

$$(Tx, Ty) \in E(G).$$

Thus both mappings preserve the edges of G .

3. (A_φ, G) -contractive condition

Choose the strong comparison function

$$\varphi(t) = kt, \quad 0 < k < 1.$$

Define $\alpha : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ by $\alpha(u, v, w) = cu$. Then for every $(x, y) \in E(G)$,

$$d_b(Sx, Ty) = |cx - cy| = c|x - y| = \alpha\left(d_b(x, y), d_b(x, Sx), d_b(y, Ty)\right).$$

Since $0 < c < 1$, the mapping α belongs to the class A_φ , and the function φ is a strong comparison function satisfying

$$\varphi^n(t) \rightarrow 0 \quad (n \rightarrow \infty).$$

4. Non-emptiness of $X_{S,T}$

Choose $x_0 = 0$. Then

$$S(0) = 0, \quad T(0) = 0.$$

Consequently,

$$(0, S0) \in E(G) \quad \text{and} \quad (S0, TS0) \in E(G),$$

which implies that

$$X_{S,T} \neq \emptyset.$$

Therefore, all assumptions of Theorem 1 are satisfied. Hence, Theorem 1 guarantees the existence of a unique common fixed point of S and T . In fact,

$$x^* = 0$$

is the unique common fixed point since

$$S(0) = T(0) = 0.$$

2. Averaging on a Finite Directed Network

The following example illustrates an application of Theorem 1 to a finite networked system. It represents an attenuated averaging process on a directed graph and demonstrates how the proposed graph-based contractive framework can be used to study consensus-type dynamics. Although this example is mainly illustrative, it highlights the relevance of the theoretical results in network interactions and distributed processes.

Let $V = \{1, 2, \dots, n\}$ be a finite set of nodes and let $X = \mathbb{R}^n$ denote the space of node states. Define

$$d_b(x, y) = \|x - y\|_\infty,$$

where $\|\cdot\|_\infty$ is the sup-norm. Then (X, d_b) is a complete b -metric space with coefficient $s = 1$.

Let $G = (V, E)$ be a directed weakly connected graph. Extend G to a graph G_X on X by defining

$$(x, y) \in E(G_X) \iff \exists (i, j) \in E \text{ such that } |x_i - y_j| \leq r,$$

for a fixed radius $r > 0$.

Let $W = (w_{ij}) \in \mathbb{R}^{n \times n}$ be a row-stochastic matrix, that is,

$$w_{ij} \geq 0, \quad \sum_{j=1}^n w_{ij} = 1, \quad i = 1, 2, \dots, n,$$

and assume that

$$w_{ii} > 0, \quad i = 1, 2, \dots, n.$$

Choose $\beta \in (0, 1)$ and define

$$S(x) = \beta Wx, \quad T(x) = \beta Wx, \quad x \in \mathbb{R}^n.$$

We verify the assumptions of Theorem 1.

1. Edge preservation

Suppose that $(x, y) \in E(G_X)$. Then there exist $i, j \in V$ such that

$$|x_i - y_j| \leq r.$$

Using the row-stochasticity of W , we obtain

$$\|W(x - y)\|_\infty \leq \|x - y\|_\infty.$$

Consequently,

$$\|Sx - Sy\|_\infty = \beta \|W(x - y)\|_\infty \leq \beta \|x - y\|_\infty \leq \beta r \leq r,$$

since $0 < \beta < 1$. Therefore,

$$(Sx, Sy) \in E(G_X).$$

Similarly,

$$(Tx, Ty) \in E(G_X).$$

Thus both mappings preserve the edges of G_X .

2. (A_φ, G) -contractive condition

Define

$$\alpha(u, v, w) = \beta u.$$

Then for every $(x, y) \in E(G_X)$,

$$d_b(Sx, Ty) = \|\beta Wx - \beta Wy\|_\infty \leq \beta \|x - y\|_\infty.$$

Equivalently,

$$d_b(Sx, Ty) \leq \alpha\left(d_b(x, y), d_b(x, Sx), d_b(y, Ty)\right).$$

Choose

$$\varphi(t) = \beta t, \quad 0 < \beta < 1.$$

Then φ is a strong comparison function satisfying

$$\varphi^n(t) \rightarrow 0 \quad (n \rightarrow \infty).$$

3. Non-emptiness of $X_{S,T}$

Take $x_0 = 0$. Then

$$S(0) = 0, \quad T(0) = 0,$$

which implies

$$(0, S0) \in E(G_X), \quad (S0, TS0) \in E(G_X).$$

Hence,

$$X_{S,T} \neq \emptyset.$$

Therefore, all assumptions of Theorem 1 are satisfied. Consequently, Theorem 1 guarantees the existence of a unique common fixed point of S and T .

Moreover,

$$x^* = 0$$

is the unique common fixed point since

$$S(0) = T(0) = 0.$$

The associated iteration process

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1},$$

converges to x^* . This example may be interpreted as a consensus-type averaging dynamics on a directed network with attenuation parameter β , where contractivity ensures asymptotic convergence.

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