

**DISCRETE EXPONENTIAL STABILITY OF A HERMITE
APPROXIMATION FOR A NON-HOMOGENEOUS DAMPED
EULER–BERNOULLI BEAM**

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Abstract: This paper studies a non-homogeneous damped Euler–Bernoulli beam subject to boundary feedback controls. A conforming cubic Hermite finite element semi-discretization is proposed. We prove, in the discrete weak setting, that the semi-discrete scheme is dissipative. Under a strictly positive distributed viscous damping assumption, we establish a mesh-independent exponential decay estimate by means of a Galerkin-admissible Lyapunov functional. We also derive an a priori error estimate of order two in the natural energy norm. Numerical experiments illustrate the decay of the discrete energy and the observed convergence rate.

Keywords and Phrases: Euler–Bernoulli beam, Hermite finite elements, dissipativity, discrete stability, a priori error estimate.

2020 Mathematics Subject Classification: 74K10, 35B35, 35L35, 47D06, 65N15, 65N30.

1. Introduction and semigroup formulation

In this paper, we consider a damped flexible Euler–Bernoulli beam of unit length, clamped at one end and subject to feedback controls applied at its free end. The vibratory motion of the beam is modeled by the following system of partial differential equations:

$$\rho(x)w_{tt}(x, t) + (r(x)w_{xx}(x, t))_{xx} + \mu(x)w_t(x, t) = 0, \quad t \geq 0, \quad 0 < x < 1, \quad (1.1)$$

$$w(0, t) = w_x(0, t) = 0, \quad t \geq 0, \quad (1.2)$$

$$r(1)w_{xx}(1, t) = -\beta w_{xt}(1, t), \quad t \geq 0, \quad (1.3)$$

$$(rw_{xx})_x(1, t) = \alpha w_t(1, t), \quad t \geq 0, \quad (1.4)$$

with the initial conditions

$$w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x). \quad (1.5)$$

Let us recall that the variable coefficients $\rho(\cdot)$ and $r(\cdot)$, which are positive functions of class $C^4([0, 1])$, respectively denote the mass density and the flexural rigidity of the beam, while $\mu(\cdot) \in C^1([0, 1])$ denotes the distributed viscous damping coefficient. The constants $\alpha \geq 0$ and $\beta \geq 0$ are the boundary feedback gains associated with the linear and rotational velocities, respectively. For $\mu(\cdot) \equiv 0$, system (1.1)–(1.4) was studied in [3], where the Riesz basis property and exponential stability were established under suitable assumptions. Related Galerkin and dissipative finite element approaches for beam models can be found in [1, 2, 6, 8].

Our aim is to develop a conforming finite element method that preserves the dissipativity of system (1.1)–(1.4). The analysis below is carried out directly in the discrete variational setting, so that no pointwise strong boundary condition is imposed on the finite element solution.

Before going into detail on the study of the problem (1.1)–(1.4) with the initial conditions (1.5), we briefly recall some well-posedness results according to the theory of C_0 -semigroups of contractions [9, 10]. To this end, we recall that the total mechanical energy of the system (1.1)–(1.4) is given by the function

$$\mathcal{E} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$$

defined by

$$\mathcal{E}(t) = \frac{1}{2} \int_0^1 (\rho(x)w_t^2(x, t) + r(x)w_{xx}^2(x, t)) dx \quad (1.6)$$

on the Hilbert space

$$\mathbb{X} = \mathcal{V} \times L^2(0, 1) \quad \text{where} \quad \mathcal{V} = \{w \in H^2(0, 1) : w(0) = w'(0) = 0\} \quad (1.7)$$

with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{X}}$ having the corresponding norm $\| \cdot \|_{\mathbb{X}}$.

By performing twice an integration by parts of the energy $\mathcal{E}(\cdot)$ given by (1.6) and taking into account the boundary conditions (1.3)-(1.4), we obtain the dissipativity identity (1.8) below:

$$\frac{d}{dt}\mathcal{E}(t) = -(\alpha w_t^2(1, t) + \beta w_{xt}^2(1, t) + \int_0^1 \mu(x)w_t^2(x, t)dx) \leq 0; \quad (1.8)$$

which allows us to deduce that $\mathcal{E}(\cdot)$ is a decaying function in time and therefore defines a Lyapunov function. Furthermore, in the form of an evolution problem, the problem (1.1)-(1.4) is written for $y(t) = (w(\cdot, t), w_t(\cdot, t))^T$ with $t \geq 0$

$$\begin{cases} \frac{d}{dt}y(t) = \Gamma y(t), \\ y(0) = y_0 = (w_0, w_1)^T \in \mathbb{X} \end{cases} \quad (1.9)$$

where the linear operator Γ is defined on

$$\mathcal{D}(\Gamma) \subset \mathbb{X} := \left\{ \begin{array}{l} (f, g)^T \in (\mathcal{V} \cap H^4(0, 1)) \times \mathcal{V} : \\ r(1)f''(1) = -\beta g'(1), \quad (rf'')'(1) = \alpha g(1) \end{array} \right\} \quad (1.10)$$

by

$$\Gamma(f, g)^T := \left(g, -\frac{1}{\rho}((rf'')'' + \mu g) \right)^T, \quad \forall (f, g)^T \in \mathcal{D}(\Gamma). \quad (1.11)$$

Now let us state the following Theorem 1.1 establishing the well-posedness of the problem (1.1)-(1.4) in the sense of C_0 -semigroups of contractions.

Theorem 1.1. *Let Γ be the closed linear operator defined by (1.10) – (1.11), with dense domain $\mathcal{D}(\Gamma)$. Then Γ is an infinitesimal generator of a C_0 -semigroup of contractions on $\mathbb{X}$, denoted by $\{T(t)\}_{t \geq 0}$ and defined by $T(t) = e^{\Gamma t}$.*

As an immediate consequence of Theorem 1.1, we have:

Theorem 1.2. *For any initial condition $y_0 \in \mathbb{X}$, the evolution problem (1.9) has a unique generalized solution $y(t) = T(t)y_0 \in C([0, \infty); \mathbb{X})$.*

The operator formulation above is used only for the continuous well-posedness result. In the finite element analysis below, the boundary feedback terms are incorporated through the variational formulation, and the strong boundary conditions in the definition of $\mathcal{D}(\Gamma)$ are not imposed pointwise on the Galerkin approximation. The paper is organized as follows. In Section 2, we construct a dissipative semi-discrete scheme using the cubic Hermite finite element method. We then prove a discrete exponential stability result under strictly positive internal damping, using only admissible Galerkin test functions. We conclude the section with an a priori error estimate for the semi-discrete Galerkin approximation. Numerical simulations are presented in Section 3 to illustrate the theoretical results.

2. Semi-discrete finite element scheme

2.1. Space discretization

Let $\mathcal{V}_h \subset \mathcal{V}$ be an arbitrarily chosen finite-dimensional space such that the cubic Hermite polynomial functions $\phi_j, j = 1, \dots, N$ (with $N = 2n$) form a basis of $\mathcal{V}_h$. The Galerkin finite element approximation of the problem (1.1)-(1.4) is given by the problem denoted **Problem G^h** and defined as follows:

Problem G^h : Find $w_h \in C^2([0, \infty); \mathcal{V}_h)$ such that

$$\begin{aligned} \int_0^1 \rho(x)(w_h)_{tt}\phi_j dx + \int_0^1 r(x)(w_h)_{xx}(\phi_j)_{xx} dx + \int_0^1 \mu(x)(w_h)_t \phi_j dx \\ + \alpha(w_h)_t(1)\phi_j(1) + \beta(w_h)_{xt}(1)(\phi_j)_x(1) = 0 \end{aligned} \quad (2.1)$$

for all $\phi_j \in \mathcal{V}_h$ and $t > 0$, with initial conditions

$$w_h(\cdot, 0) = w_{h,0} \in \mathcal{V}_h, \quad (2.2)$$

$$(w_h)_t(\cdot, 0) = w_{h,1} \in \mathcal{V}_h. \quad (2.3)$$

In fact, equation (2.1) can be written as a second-order ODE in time. Expanding the semi-discrete solution in the Hermite basis, we write

$$(w_h)(x, t) = \sum_{i=1}^N W_i(t)\phi_i(x) \quad (2.4)$$

where W is the vector representation of the function w_h defined as the following:

$$W := [W_1 \quad W_2 \quad \dots \quad W_N]^T.$$

Thus, equation (2.1) is equivalent to the following vectorial equation:

$$MW_{tt} + SW_t + KW = 0, \quad (2.5)$$

where M , S and K are the mass, damping and stiffness matrices, respectively, and are defined by:

$$M_{ij} = \int_0^1 \rho(x) \phi_i(x) \phi_j(x) dx, \quad (2.6)$$

$$S_{ij} = \int_0^1 \mu(x) \phi_i(x) \phi_j(x) dx + \alpha \phi_i(1) \phi_j(1) + \beta (\phi_i)_x(1) (\phi_j)_x(1), \quad (2.7)$$

$$K_{ij} = \int_0^1 r(x) (\phi_i)_{xx}(x) (\phi_j)_{xx}(x) dx. \quad (2.8)$$

Since the matrices M and K are symmetric positive definite, they are invertible. Consequently, problem (2.5) has a unique solution by the standard theory of finite-dimensional linear differential equations. Hence Problem G^h , defined by (2.1)-(2.3), admits a unique solution

$$w_h \in C^2([0, \infty); \mathcal{V}_h).$$

2.2. Dissipativity of the semi-discrete scheme

In this subsection, we show the dissipativity of the semi-discrete scheme. We recall that the time-dependent energy functional $\mathcal{E}(\cdot)$ along the trajectory $w \in C([0, \infty); \mathcal{V})$ is given by

$$\mathcal{E}(t, w) := \frac{1}{2} \int_0^1 \{ \rho(x) w_t^2(x, t) + r(x) w_{xx}^2(x, t) \} dx. \quad (2.9)$$

Theorem 2.1. *Let $w_h \in C^2([0, \infty); \mathcal{V}_h)$ be a solution of (2.1) – (2.3). Then for all $t > 0$,*

$$\frac{d}{dt} \mathcal{E}_h(t, w_h) = - \left\{ \int_0^1 \mu(x) (w_h)_t^2(x, t) dx + \alpha [(w_h)_t(1)]^2 + \beta [(w_h)_{xt}(1)]^2 \right\} \leq 0. \quad (2.10)$$

Proof. Considering (2.9), we proceed to the temporal derivation of the energy functional $\mathcal{E}_h(t, w_h)$. Then,

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_h(t, w_h) &= \frac{d}{dt} \left\{ \frac{1}{2} \int_0^1 \{ \rho(x) w_{ht}^2(x, t) + r(x) w_{hxx}^2(x, t) \} dx \right\} \\ &= \int_0^1 \rho(x) (w_h)_{tt} (w_h)_t dx + \int_0^1 r(x) (w_h)_{xxt} (w_h)_{xx} dx. \end{aligned}$$

Since $(w_h)_t \in \mathcal{V}_h$, we multiply (2.1) by the coefficients of $(w_h)_t$ in the Hermite basis and sum over the basis functions. Equivalently, we test the Galerkin equation with $v_h = (w_h)_t$. This gives

$$\begin{aligned} & \int_0^1 \rho(x)(w_h)_{tt}(w_h)_t dx + \int_0^1 r(x)(w_h)_{xx}(w_h)_{xxt} dx \\ &= - \int_0^1 \mu(x)(w_h)_t^2 dx - \alpha [(w_h)_t(1)]^2 - \beta [(w_h)_{xt}(1)]^2 \end{aligned}$$

with $\alpha \geq 0$, $\beta \geq 0$, and $\mu(\cdot) \geq 0$. Hence, the relation (2.10) is obtained.

2.3. Discrete exponential stability in the weak Galerkin setting

The previous dissipativity identity is not sufficient by itself to prove exponential decay. We therefore give a Lyapunov proof that is entirely based on the semi-discrete variational formulation (2.1). In particular, we do not use the strong boundary conditions (1.3)-(1.4) as pointwise identities for w_h .

We assume the following uniform coefficient bounds:

$$\begin{aligned} 0 < \rho_0 \leq \rho(x) \leq \rho_1, \quad 0 < r_0 \leq r(x) \leq r_1, \\ 0 < \mu_0 \leq \mu(x) \leq \mu_1, \quad x \in [0, 1]. \end{aligned} \quad (2.11)$$

We also assume $\alpha \geq 0$ and $\beta \geq 0$. For $u, v \in \mathcal{V}$, set

$$\begin{aligned} m(u, v) &= \int_0^1 \rho(x)u(x)v(x) dx, \\ a(u, v) &= \int_0^1 r(x)u_{xx}(x)v_{xx}(x) dx, \\ d(u, v) &= \int_0^1 \mu(x)u(x)v(x) dx + \alpha u(1)v(1) + \beta u_x(1)v_x(1). \end{aligned}$$

Then (2.1) may be written in the compact weak form

$$m((w_h)_{tt}, v_h) + a(w_h, v_h) + d((w_h)_t, v_h) = 0, \quad \forall v_h \in \mathcal{V}_h. \quad (2.12)$$

The discrete energy is

$$\mathcal{E}_h(t) = \frac{1}{2} [m((w_h)_t, (w_h)_t) + a(w_h, w_h)]. \quad (2.13)$$

Using $v_h = (w_h)_t$ in (2.12), we recover

$$\frac{d}{dt} \mathcal{E}_h(t) = -d((w_h)_t, (w_h)_t) \leq 0. \quad (2.14)$$

We shall use the following Poincaré–trace estimates, which follow from standard Sobolev and trace inequalities [5, 7]: there exist constants $C_0, C_1, C_2 > 0$, independent of h , such that for all $v \in \mathcal{V}$,

$$\|v\|_{L^2(0,1)}^2 \leq C_0 \|v_{xx}\|_{L^2(0,1)}^2, \quad |v(1)|^2 \leq C_1 \|v_{xx}\|_{L^2(0,1)}^2, \quad |v_x(1)|^2 \leq C_2 \|v_{xx}\|_{L^2(0,1)}^2. \quad (2.15)$$

These constants are mesh-independent because $\mathcal{V}_h \subset \mathcal{V}$.

Theorem 2.2. *Assume (2.11). Let w_h be the solution of the semi-discrete problem (2.12). Then there exist constants $M \geq 1$ and $\omega > 0$, independent of h , such that*

$$\mathcal{E}_h(t) \leq M e^{-\omega t} \mathcal{E}_h(0), \quad t \geq 0. \quad (2.16)$$

Proof. Let $\theta > 0$ be chosen later and define the modified energy

$$\mathcal{L}_h(t) = \mathcal{E}_h(t) + \theta m((w_h)_t, w_h) + \frac{\theta}{2} d(w_h, w_h). \quad (2.17)$$

By (2.15) and the coefficient bounds, there exist constants

$$\gamma = \frac{\rho_1 C_0}{r_0} > 0, \quad \delta = \frac{\mu_1 C_0 + \alpha C_1 + \beta C_2}{r_0} > 0,$$

independent of h , such that

$$m(v, v) \leq \gamma a(v, v), \quad d(v, v) \leq \delta a(v, v), \quad \forall v \in \mathcal{V}.$$

Hence, by Cauchy's inequality,

$$|m((w_h)_t, w_h)| \leq m((w_h)_t, (w_h)_t)^{1/2} m(w_h, w_h)^{1/2} \leq \sqrt{\gamma} \mathcal{E}_h(t),$$

and

$$0 \leq \frac{\theta}{2} d(w_h, w_h) \leq \theta \delta \mathcal{E}_h(t).$$

We choose $\theta > 0$ small enough both to satisfy $\theta < \mu_0/\rho_1$ and to ensure the positivity of the lower equivalence constant below, for instance $\theta < 1/(2\sqrt{\gamma})$. Then there exist constants $c_1, c_2 > 0$, independent of h , such that

$$c_1 \mathcal{E}_h(t) \leq \mathcal{L}_h(t) \leq c_2 \mathcal{E}_h(t), \quad t \geq 0. \quad (2.18)$$

Differentiating (2.17), using (2.14), and then using (2.12) with the admissible test function $v_h = w_h$, gives

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_h(t) &= -d((w_h)_t, (w_h)_t) + \theta m((w_h)_{tt}, w_h) + \theta m((w_h)_t, (w_h)_t) + \theta d(w_h, (w_h)_t) \\ &= -d((w_h)_t, (w_h)_t) + \theta m((w_h)_t, (w_h)_t) - \theta a(w_h, w_h). \end{aligned}$$

Since $\mu(x) \geq \mu_0$ and $\rho(x) \leq \rho_1$, we have

$$d((w_h)_t, (w_h)_t) \geq \mu_0 \|(w_h)_t\|_{L^2(0,1)}^2 \geq \frac{\mu_0}{\rho_1} m((w_h)_t, (w_h)_t).$$

With this choice of θ , we obtain

$$\frac{d}{dt} \mathcal{L}_h(t) \leq -\left(\frac{\mu_0}{\rho_1} - \theta\right) m((w_h)_t, (w_h)_t) - \theta a(w_h, w_h) \leq -\kappa \mathcal{E}_h(t),$$

where $\kappa = 2 \min\{\mu_0/\rho_1 - \theta, \theta\} > 0$. Combining this estimate with (2.18) yields

$$\frac{d}{dt} \mathcal{L}_h(t) \leq -\frac{\kappa}{c_2} \mathcal{L}_h(t).$$

Gronwall's lemma gives $\mathcal{L}_h(t) \leq e^{-\kappa t/c_2} \mathcal{L}_h(0)$. A final use of (2.18) proves (2.16) with $M = c_2/c_1$ and $\omega = \kappa/c_2$. The constants M and ω are independent of the mesh parameter h . The coercive decay constant κ is obtained from the strictly positive internal damping through μ_0/ρ_1 , whereas the boundary gains α and β enter only through the equivalence constants associated with the modified energy. Thus, (2.16) is a mesh-uniform sufficient exponential bound and is not intended to provide an optimal physical decay rate.

Remark 2.3. *The proof above relies essentially on the strict positivity of the internal damping coefficient μ . When $\mu \equiv 0$, boundary damping alone requires a different discrete observability or multiplier argument. The corresponding continuous case was studied in [3] by a Riesz basis approach; however, a mesh-independent discrete observability result for the Galerkin approximation is not established here. In that case, no uniform exponential stability claim is made without additional numerical viscosity, filtering, or a separate mesh-independent observability proof.*

2.4. A priori error estimates

We now derive an a priori estimate for the semi-discrete Galerkin approximation (2.1). The analysis is performed for the full damped problem, including both the internal damping and the boundary damping terms.

Let $I_h : \mathcal{V} \cap H^4(0, 1) \rightarrow \mathcal{V}_h$ denote the cubic Hermite interpolation operator. We set

$$\eta_h = w - I_h w, \quad \chi_h = w_h - I_h w.$$

Thus $w_h - w = \chi_h - \eta_h$. Since the Hermite interpolant preserves the nodal values and the first derivatives at the end-points, we have

$$\eta_h(1, t) = (\eta_h)_x(1, t) = (\eta_h)_t(1, t) = (\eta_h)_{xt}(1, t) = 0. \quad (2.19)$$

We assume that, for a fixed $T > 0$,

$$w \in C([0, T]; H^4(0, 1) \cap \mathcal{V}), \quad (2.20)$$

$$w_t \in L^2(0, T; H^4(0, 1) \cap \mathcal{V}), \quad (2.21)$$

$$w_{tt} \in L^2(0, T; H^2(0, 1)). \quad (2.22)$$

The standard Hermite interpolation estimates, see for instance [4], give

$$\|\eta_h\|_{H^2(0,1)} \leq Ch^2 \|w\|_{H^4(0,1)}, \quad (2.23)$$

$$\|(\eta_h)_t\|_{H^2(0,1)} \leq Ch^2 \|w_t\|_{H^4(0,1)}, \quad (2.24)$$

$$\|(\eta_h)_{tt}\|_{L^2(0,1)} \leq Ch^2 \|w_{tt}\|_{H^2(0,1)}. \quad (2.25)$$

Subtracting the weak formulation of the exact solution from (2.12), we obtain, for all $v_h \in \mathcal{V}_h$,

$$m((\chi_h)_{tt}, v_h) + a(\chi_h, v_h) + d((\chi_h)_t, v_h) = m((\eta_h)_{tt}, v_h) + a(\eta_h, v_h) + d((\eta_h)_t, v_h). \quad (2.26)$$

In view of (2.19), the boundary part of $d((\eta_h)_t, v_h)$ vanishes. Taking $v_h = (\chi_h)_t$ in (2.26), we get

$$\frac{d}{dt} \mathcal{E}_h(t, \chi_h) + d((\chi_h)_t, (\chi_h)_t) = m((\eta_h)_{tt}, (\chi_h)_t) + a(\eta_h, (\chi_h)_t) + d((\eta_h)_t, (\chi_h)_t), \quad (2.27)$$

where

$$\mathcal{E}_h(t, \chi_h) = \frac{1}{2} [m((\chi_h)_t, (\chi_h)_t) + a(\chi_h, \chi_h)].$$

Integrating (2.27) over $(0, t)$ and integrating the term $a(\eta_h, (\chi_h)_t)$ by parts in time yield

$$\begin{aligned} \mathcal{E}_h(t, \chi_h) &\leq \mathcal{E}_h(0, \chi_h) + a(\eta_h(t), \chi_h(t)) - a(\eta_h(0), \chi_h(0)) \\ &\quad + \int_0^t m((\eta_h)_{ss}, (\chi_h)_s) ds - \int_0^t a((\eta_h)_s, \chi_h) ds + \int_0^t d((\eta_h)_s, (\chi_h)_s) ds. \end{aligned}$$

Using Cauchy's inequality, Young's inequality, (2.15), and the coefficient bounds (2.11), we obtain

$$\mathcal{E}_h(t, \chi_h) \leq C\mathcal{E}_h(0, \chi_h) + C \int_0^t \mathcal{E}_h(s, \chi_h) ds + C\mathcal{R}_h(t), \quad (2.28)$$

where

$$\mathcal{R}_h(t) = \|\eta_h\|_{C([0,T];H^2(0,1))}^2 + \|(\eta_h)_t\|_{L^2(0,T;H^2(0,1))}^2 + \|(\eta_h)_{tt}\|_{L^2(0,T;L^2(0,1))}^2.$$

By Gronwall's lemma and (2.23)-(2.25),

$$\mathcal{E}_h(t, \chi_h)^{1/2} \leq C [\mathcal{E}_h(0, \chi_h)^{1/2} + h^2 \mathcal{N}(w, T)], \quad 0 \leq t \leq T, \quad (2.29)$$

with

$$\mathcal{N}(w, T) = \|w\|_{C([0,T];H^4(0,1))} + \|w_t\|_{L^2(0,T;H^4(0,1))} + \|w_{tt}\|_{L^2(0,T;H^2(0,1))}.$$

The triangle inequality and the interpolation estimates now give the following theorem.

Theorem 2.4. *Let $\mathcal{V}_h$ be the cubic Hermite finite element space and assume (2.20)-(2.22). Then the semi-discrete solution w_h of (2.1) satisfies*

$$\mathcal{E}_h(t, w_h - w)^{1/2} \leq C [\mathcal{E}_h(0, w_h - I_h w)^{1/2} + h^2 \mathcal{N}(w, T)], \quad 0 \leq t \leq T, \quad (2.30)$$

where $C > 0$ is independent of h . In particular, if $w_{h,0} = I_h w_0$ and $w_{h,1} = I_h w_1$, then

$$\mathcal{E}_h(t, w_h - w)^{1/2} \leq Ch^2 \mathcal{N}(w, T), \quad 0 \leq t \leq T. \quad (2.31)$$

Remark 2.5. *The estimate (2.31) is an order-two estimate in the natural energy norm. Therefore, a numerical convergence table intended to validate (2.31) should report the energy norm, or else an additional argument should be supplied if only the $L^2(0, 1)$ error is reported.*

3. Numerical simulations

In this section, we illustrate the dissipativity and the decay of the numerical solution and report a convergence test. The theoretical results above concern the spatially semi-discrete Galerkin scheme. For the fully discrete computations, the second-order matrix system (2.5) is rewritten as a first-order system and advanced in time by the Crank–Nicolson method. This implicit second-order method is used with a fixed time step, specified below for each experiment, so that the observed decay and convergence are not left dependent on an unspecified time integrator.

3.1. Numerical stability

The following spatially varying parameters are considered:

$$\rho(x) = r(x) = 1 + x^2, \quad \mu(x) = 1 + x^3.$$

The spatial discretization is defined by $h = 1/75$, and the time interval is $t \in [0, 30]$. The time step used for the time integration is $\Delta t = 10^{-3}$, and the number of degrees of freedom is $N = 150$. Two sets of damping parameters are examined:

$$\text{Case 1: } \alpha = 5, \beta = 3, \quad \text{Case 2: } \alpha = 0.5, \beta = 1.$$

These configurations illustrate the influence of the damping parameters on the rate of decay of the discrete energy.

Figures 1 and 2 show, on a semi-logarithmic scale, that the associated discrete energy decreases with an exponential trend. The decay is faster in Case 1, where the damping parameters are larger. This behavior is consistent with the dissipativity identity (2.14) and with the exponential estimate of Theorem 2.2 under the strictly positive internal damping assumption.

The semi-logarithmic plots display $\mathcal{E}_h(t)$ on a logarithmic scale together with a least-squares affine fit of $\log(\mathcal{E}_h(t))$ over the indicated time interval. If the fitted slope is denoted by s , the observed decay rate is $\omega_{\text{fit}} = -s$. The fitted values are reported in Table 1. The fitted quantities ω_{fit} are observed decay rates computed on the indicated time intervals. They should not be identified with the theoretical constant ω in Theorem 2.2, which is obtained from a Lyapunov estimate and provides a mesh-independent sufficient decay rate rather than an optimal spectral decay rate. The difference between the two fitted values reflects the influence of the boundary feedback gains and of the excited discrete modes on the observed decay, and does not contradict Theorem 2.2. Figure 3 illustrates the corresponding decay of the deflection profiles.

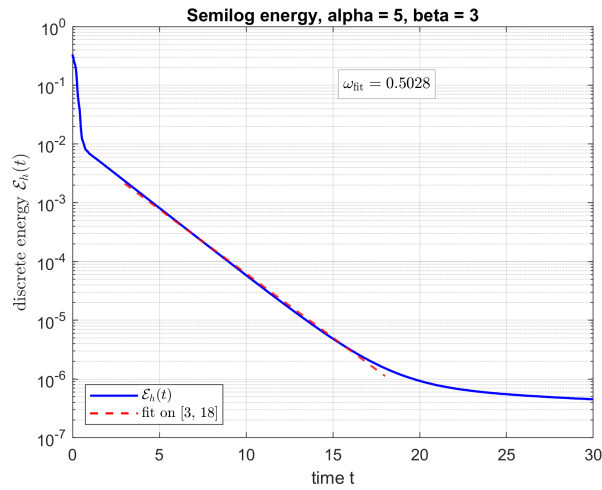


Figure 1: Semi-logarithmic energy decay for Case 1: $\alpha = 5$, $\beta = 3$.

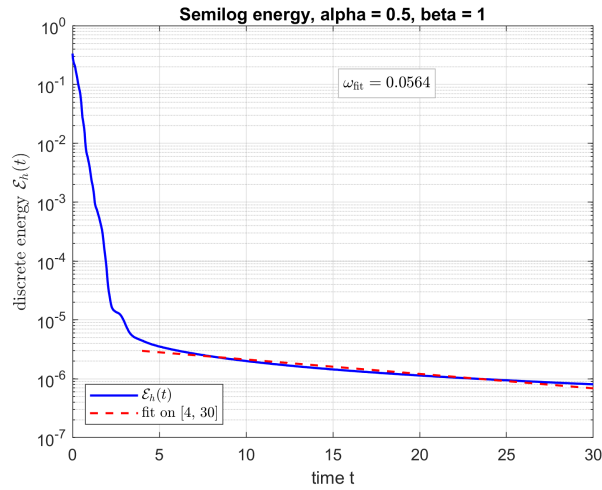


Figure 2: Semi-logarithmic energy decay for Case 2: $\alpha = 0.5$, $\beta = 1$.

α	β	fit interval	fitted slope s	ω_{fit}
5	3	[3, 18]	-0.50280	0.50280
0.5	1	[4, 30]	-0.05640	0.05640

Table 1: Least-squares fitted decay rates obtained from the semi-logarithmic energy curves.

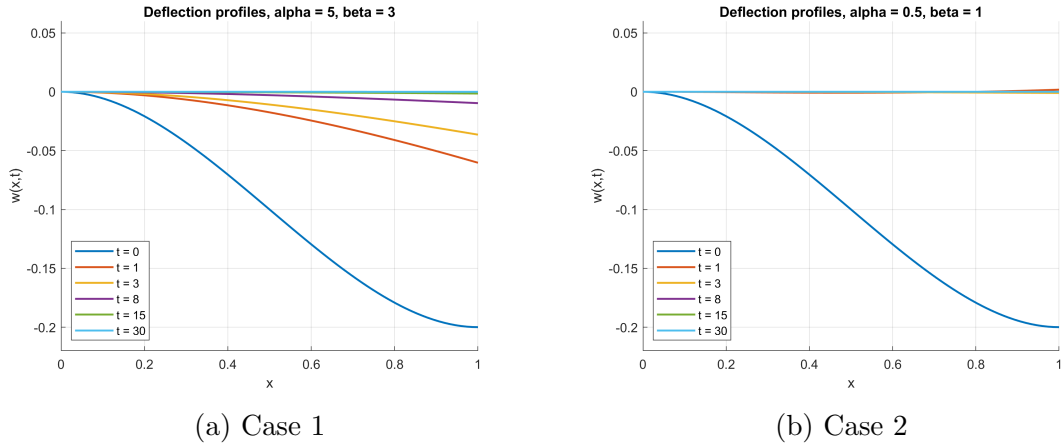


Figure 3: Deflection profiles at selected times.

3.2. Numerical order of convergence

We investigate the spatial convergence of the method in the natural energy norm. To isolate the spatial error from the use of a numerical reference solution, we use the manufactured smooth solution

$$w_{\text{ex}}(x, t) = e^{-t} g(x).$$

In the final test we take

$$g(x) = \frac{1}{4} x^2 (1 + x)^2.$$

This choice satisfies the clamped conditions $g(0) = g'(0) = 0$, but gives non-zero traces at $x = 1$. The feedback boundary contribution is therefore not eliminated in the manufactured computation. The corresponding volume source term

$$f(x, t) = \rho(x)(w_{\text{ex}})_{tt}(x, t) + (r(x)(w_{\text{ex}})_{xx}(x, t))_{xx} + \mu(x)(w_{\text{ex}})_t(x, t)$$

is inserted together with the boundary residuals

$$\ell_b(t; v_h) = b_0(t)v_h(1) + b_1(t)(v_h)_x(1),$$

where

$$\begin{aligned} b_0(t) &= -(r(w_{\text{ex}})_{xx})_x(1, t) + \alpha(w_{\text{ex}})_t(1, t), \\ b_1(t) &= r(1)(w_{\text{ex}})_{xx}(1, t) + \beta(w_{\text{ex}})_{xt}(1, t). \end{aligned}$$

This manufactured-solution test therefore exercises both the interior terms and the boundary feedback terms in the discrete variational equation. The mesh size is $h = 1/n$ with

$$n = 8, 16, 32, 64, 128.$$

The final time is $T = 1$ and the time step is $\Delta t = 10^{-4}$, small enough for the temporal error to be negligible relative to the spatial error in the range of meshes reported below. The error is measured by

$$\|w_h - w_{\text{ex}}\|_{\mathcal{E}} = \left(\frac{1}{2} \int_0^1 \rho(x) ((w_h)_t - (w_{\text{ex}})_t)^2 + r(x) ((w_h)_{xx} - (w_{\text{ex}})_{xx})^2 dx \right)^{1/2}.$$

Δt	h	$\ w_h - w_{\text{ex}}\ _{\mathcal{E}}$	OC
10^{-4}	$1/8$	1.050×10^{-3}	—
10^{-4}	$1/16$	2.624×10^{-4}	2.0000
10^{-4}	$1/32$	6.560×10^{-5}	2.0001
10^{-4}	$1/64$	1.640×10^{-5}	2.0001
10^{-4}	$1/128$	4.099×10^{-6}	2.0000

Table 2: Energy-norm error and observed order of convergence for the manufactured-solution test.

Table 2 shows an observed convergence rate close to two in the energy norm. This is consistent with the order-two estimate stated in Theorem 2.4.

4. Conclusion

We constructed a conforming cubic Hermite finite element semi-discretization for a non-homogeneous damped Euler–Bernoulli beam with boundary feedback. The scheme preserves dissipativity at the semi-discrete level. Under a strictly positive internal damping assumption, we proved mesh-independent exponential decay using only Galerkin-admissible test functions and derived an order-two error estimate in the natural energy norm. The numerical experiments now provide semi-logarithmic energy plots with fitted decay rates and an energy-norm manufactured-solution test, including non-trivial boundary feedback residuals, showing the expected second-order spatial convergence.

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