

**RELATIVE (α, β, γ) -ORDER OF COMPOSITE ENTIRE
FUNCTIONS RELATED TO MAXIMUM TERMS**

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Abstract: In this paper, we discuss maximum term related to some growth properties of composite entire functions with respect to another entire function using the relative (α, β, γ) -order and relative (α, β, γ) -lower order and derive some results with their left and right factors. We also obtain the integral representations of relative (α, β, γ) -order and relative (α, β, γ) -lower order of entire functions using maximum terms.

Keywords and Phrases: Entire function, maximum term, composition, relative (α, β, γ) - order, relative (α, β, γ) -lower order.

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1. Introduction

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let $f = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ of f on $|z| = r$ is defined as $M_f(r) = \max_{|z|=r} |f(z)|$ and the maximum term $\mu(r, f)$ is defined

as $\mu_f(r) = \max_{n \geq 0} (|a_n| r^n)$. For more details on the standard notations and definitions of the theory of entire functions, e.g. see [8, 11, 12]. For $x \in [0, \infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers, define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$, with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$, and $\exp^{[-1]} x = \log x$. To start our paper, we just recall the following definition:

Definition 1. The order ρ_f and the lower order λ_f of an entire function f are defined as

$$\rho_f = \overline{\lim}_{r \rightarrow +\infty} \frac{\log \log M_f(r)}{\log r} \text{ and } \lambda_f = \underline{\lim}_{r \rightarrow +\infty} \frac{\log \log M_f(r)}{\log r}.$$

1.1. Notation and hypotheses

Let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \rightarrow +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a) + \alpha(b) + c$ for all $a, b \geq R_0$ and fixed $c \in (0, +\infty)$. Further we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x + O(1)) = (1 + o(1))\alpha(x)$ as $x \rightarrow +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a) + \alpha(b)$ for all $a, b \geq R_0$, i.e., α is subadditive. Clearly $L_3 \subset L_1$.

Particularly, for $\alpha \in L_3$ one can easily verify that $\alpha(mr) \leq m\alpha(r)$, $m (\geq 2)$ is an integer. We note that up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0, +\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0, 1]$,

$$\begin{aligned} \alpha(tx) &= \alpha(tx + (1-t) \cdot 0) \\ &\geq t\alpha(x) + (1-t)\alpha(0) \geq t\alpha(x). \end{aligned}$$

If we take $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$, then

$$\begin{aligned} \alpha(a+b) &= \frac{a}{a+b}\alpha(a+b) + \frac{b}{a+b}\alpha(a+b) \\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right) + \alpha\left(\frac{b}{a+b}(a+b)\right) \\ &= \alpha(a) + \alpha(b), \quad a, b \geq 0. \end{aligned}$$

As a non-decreasing subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$. This yields that $\alpha(r) \sim \alpha(r + R_0)$ as $r \rightarrow +\infty$. Throughout the present paper we take $\alpha, \alpha_1, \alpha_2, \alpha_3 \in L_1, \beta \in L_2, \gamma \in L_3$.

Sheremeta [9], introduced the generalized order (α, β) of an entire function which considerably extended the definition of φ -order of entire function introduced by Chyzhykov et al. [5]. Later, Heittokangas et al. [6] introduced the concept of φ -order of entire function considering φ as subadditive function. Belaïdi et al. [1] extended this concept and have introduced the definitions of (α, β, γ) -order and (α, β, γ) -lower order of an entire function f in terms of maximum modulus in the following way:

Definition 2. [1] *The (α, β, γ) -order denoted by $\rho_{(\alpha, \beta, \gamma)}[f]$ and (α, β, γ) -lower order denoted by $\lambda_{(\alpha, \beta, \gamma)}[f]$ of an entire function f are defined as:*

$$\rho_{(\alpha, \beta, \gamma)}[f] = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log^{[2]}(M(r, f)))}{\beta(\log(\gamma(r)))} \text{ and } \lambda_{(\alpha, \beta, \gamma)}[f] = \underline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log^{[2]}(M(r, f)))}{\beta(\log(\gamma(r)))}.$$

Remark 1. *If we take $\alpha(r) = \beta(r) = \gamma(r) = r$, then Definition 2 coincides with Definition 1.*

Definition 2 is alternatively written by Biswas et al. [4] using maximum term, which is as follows:

Definition 3. [4] *The (α, β, γ) -order $\rho_{(\alpha, \beta, \gamma)}[f]$ and (α, β, γ) -lower order $\lambda_{(\alpha, \beta, \gamma)}[f]$ of an entire function f are defined as:*

$$\rho_{(\alpha, \beta, \gamma)}[f] = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log^{[2]}(\mu(r, f)))}{\beta(\log(\gamma(r)))} \text{ and } \lambda_{(\alpha, \beta, \gamma)}[f] = \underline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log^{[2]}(\mu(r, f)))}{\beta(\log(\gamma(r)))}.$$

Usually, the growth investigation of entire functions are done through their maximum moduli function in comparison with those of exponential functions. But to evaluate the growth rates of any entire function with respect to a new entire function, the notions of relative growth indicators (see e.g. [2, 3]) are needed. Recently, Mulyavai et al. [7] discussed on relative growth of Hadamard compositions of entire Dirichlet series. Now in order to make some progresses in the study of relative order, one may introduce the definitions of relative (α, β, γ) -order and relative (α, β, γ) -lower order of an entire function with respect to another entire function in the following way:

Definition 4. *The relative (α, β, γ) -order denoted by $\rho_{(\alpha, \beta, \gamma)}[f]_h$ and relative (α, β, γ) -lower order denoted by $\lambda_{(\alpha, \beta, \gamma)}[f]_h$ of an entire function f are defined as:*

$$\rho_{(\alpha, \beta, \gamma)}[f]_h = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log M_h^{-1}(M_f(r)))}{\beta(\log(\gamma(r)))} \text{ and } \lambda_{(\alpha, \beta, \gamma)}[f]_h = \underline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log M_h^{-1}(M_f(r)))}{\beta(\log(\gamma(r)))}.$$

Remark 2. Clearly if $h(z) = \exp z$, then Definition 4 reduces to Definition 2. If we take $\alpha(r) = \beta(r) = \gamma(r) = r$ and $h(z) = \exp z$, then Definition 4 reduces to Definition 1.

In terms of maximum terms of entire functions, Definition 4 can be reformulated as:

Definition 5. The relative (α, β, γ) -order $\rho_{(\alpha, \beta, \gamma)}[f]_h$ and relative (α, β, γ) -lower order $\lambda_{(\alpha, \beta, \gamma)}[f]_h$ of an entire function f are defined as:

$$\rho_{(\alpha, \beta, \gamma)}[f]_h = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} \text{ and } \lambda_{(\alpha, \beta, \gamma)}[f]_h = \underline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))}.$$

In this paper, we study some growth properties of the composite entire functions on the basis of relative (α, β, γ) -order and relative (α, β, γ) -lower order relating to maximum term.

2. Lemma

In this section, we present some lemmas which will be needed in the sequel.

Lemma 1. [2] Suppose f is an entire function and $A > 1$, $0 < B < A$. Then for all sufficiently large r ,

$$M_f(Ar) \geq BM_f(r).$$

Lemma 2. Definition 4 and Definition 5 are equivalent.

Proof. Let h be an entire function. Then from the left-hand side of the inequalities $\mu_h(r) \leq M_h(r) \leq \frac{R}{R-r} \mu_h(R)$ {cf. [10]}, where $0 \leq r < R$, we obtain

$$M_h^{-1}(r) \leq \mu_h^{-1}(r)$$

and in the right-hand side putting $R = ar$ for any $a > 1$, we get that

$$\mu_h^{-1}(r) \leq aM_h^{-1}\left(\frac{ar}{(a-1)}\right).$$

Since $M_h^{-1}(r)$ and $\mu_h^{-1}(r)$ are increasing functions of r , for any $a > 1$ it follows from the above, Lemma 1 and the inequalities $\mu_f(r) \leq M_f(r) \leq \frac{a}{a-1} \mu_f(ar)$ {cf. [10]} that

$$M_h^{-1}\left(M_f\left(\frac{(a-1)r}{(2a-1)a}\right)\right) \leq \mu_h^{-1}(\mu_f(r)) \quad (1)$$

and

$$\mu_h^{-1}(\mu_f(r)) \leq aM_h^{-1}\left(M_f\left(\frac{(2a-1)}{(a-1)}r\right)\right). \quad (2)$$

Now from (1), we get

$$\begin{aligned}
& \alpha \left(\log M_h^{-1} \left(M_f \left(\frac{(a-1)r}{(2a-1)a} \right) \right) \right) \leq \alpha(\log \mu_h^{-1}(\mu_f(r))), \\
& \text{i.e., } \frac{\alpha \left(\log M_h^{-1} \left(M_f \left(\frac{(a-1)r}{(2a-1)a} \right) \right) \right)}{\beta(\log(\gamma(r)))} \leq \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))}, \\
& \text{i.e., } \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha \left(\log M_h^{-1} \left(M_f \left(\frac{(a-1)r}{(2a-1)a} \right) \right) \right)}{\beta \left(\log \left(\gamma \left(\frac{(a-1)r}{(2a-1)a} \right) \right) \right)} \cdot \lim_{r \rightarrow +\infty} \frac{\beta \left(\log \left(\gamma \left(\frac{(a-1)r}{(2a-1)a} \right) \right) \right)}{\beta(\log(\gamma(r)))} \\
& \leq \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))}, \\
& \text{i.e., } \rho_{(\alpha, \beta, \gamma)}[f]_h \cdot 1 \leq \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))}, \\
& \text{i.e., } \rho_{(\alpha, \beta, \gamma)}[f]_h \leq \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))}. \tag{3}
\end{aligned}$$

Similarly, using (2) we can prove that

$$\text{i.e., } \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} \leq \rho_{(\alpha, \beta, \gamma)}[f]_h. \tag{4}$$

From (3) and (4), we get the result.

Lemma 3. *If the integral $\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{t+1}} dr$ ($r_0 > 0$) is convergent for $0 < t < +\infty$, then*

$$\lim_{r \rightarrow +\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^t} = 0.$$

Proof. Since the integral $\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{t+1}} dr$ is convergent for $0 < t < +\infty$, for any given $\varepsilon (> 0)$ there exists a number $n_0 \in \mathbb{N}$, such that

$$\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{t+1}} dr < \varepsilon \text{ for } r_0 > n_0.$$

This means that for $r_0 > n_0$,

$$\int_{r_0}^{r_0+r} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{t+1}} dr < \varepsilon.$$

As $e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}$ is an increasing function of r , so

$$\int_{r_0}^{r_0+e^{\beta(\log(\gamma(r_0)))}} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{t+1}} dr \geq \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r_0)))]}}{[e^{\beta(\log(\gamma(r_0)))}]^{t+1}} \cdot e^{\beta(\log(\gamma(r_0)))}.$$

This means that

$$\int_{r_0}^{r_0+e^{\beta(\log(\gamma(r_0)))}} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{t+1}} dr \geq \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r_0)))]}}{[e^{\beta(\log(\gamma(r_0)))}]^t} \text{ for } r_0 > n_0,$$

$$\text{or } \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r_0)))]}}{[e^{\beta(\log(\gamma(r_0)))}]^t} < \varepsilon \text{ for } r_0 > n_0,$$

from which it is clear that

$$\lim_{r \rightarrow +\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^t} = 0.$$

This proves the lemma.

3. Main Results

In this section, the main results of the paper are presented.

Theorem 1. *The growth indicator $\rho_{(\alpha,\beta,\gamma)}[f]_h$ is alternatively defined as: The integral $\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{t+1}} dr$ ($r_0 > 0$) diverges when $t < \rho_{(\alpha,\beta,\gamma)}[f]_h$ and converges when $t > \rho_{(\alpha,\beta,\gamma)}[f]_h$.*

Proof. Case 1. Let $\rho_{(\alpha,\beta,\gamma)}[f]_h = +\infty$.

Since $\rho_{(\alpha,\beta,\gamma)}[f]_h = \infty$, by Definition 5 for arbitrary positive K , we have a sequence of real numbers r tending to infinity so that

$$\alpha(\log \mu_h^{-1}(\mu_f(r))) > K \cdot \beta(\log(\gamma(r))),$$

$$\text{i.e., } e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]} > [e^{\beta(\log(\gamma(r)))}]^K. \quad (5)$$

Suppose that the integral $\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}{[e^{\beta(\log(\gamma(r)))}]^{K+1}} dr$ ($r_0 > 0$) is convergent. Then by Lemma 3,

$$\overline{\lim}_{r \rightarrow +\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}{[e^{\beta(\log(\gamma(r)))}]^K} = 0.$$

So for all sufficiently large values of r ,

$$e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]} < [e^{\beta(\log(\gamma(r)))}]^K. \quad (6)$$

Now from (5) and (6) we reach at a contradiction.

Hence $\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}{[e^{\beta(\log(\gamma(r)))}]^{K+1}} dr$ ($r_0 > 0$) is divergent whenever K is finite.

Next, suppose that $\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}{[e^{\beta(\log(\gamma(r)))}]^{K+1}} dr$ ($r_0 > 0$) is divergent. Then for any chosen positive number K and for arbitrarily chosen positive number ε , we have a sequence of real numbers r tending to infinity, so that

$$e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]} > [e^{\beta(\log(\gamma(r)))}]^{K-\varepsilon},$$

$$i.e., \alpha(\log \mu_h^{-1}(\mu_f(r))) > (K - \varepsilon) \cdot \beta(\log(\gamma(r))).$$

This gives

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} \geq K - \varepsilon.$$

As $K > 0$ is arbitrary, it implies that

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} = \infty.$$

Thus Definition 5 follows.

Case 2. Let $0 \leq \rho_{(\alpha, \beta, \gamma)}[f]_h < +\infty$.

Subcase (I). $0 < \rho_{(\alpha, \beta, \gamma)}[f]_h < +\infty$.

If $0 < \rho_{(\alpha, \beta, \gamma)}[f]_h < \infty$, then for any arbitrary positive ε and for all sufficiently large values of r , we have

$$\frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} < \rho_{(\alpha, \beta, \gamma)}[f]_h + \varepsilon,$$

$$\begin{aligned}
& i.e., e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]} < [e^{\beta(\log(\gamma(r)))}]^{(\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon)}, \\
& i.e., \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^t} < \frac{[e^{\beta(\log(\gamma(r)))}]^{(\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon)}}{[e^{\beta(\log(\gamma(r)))}]^t}, \\
& i.e., \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^t} < \frac{1}{[e^{\beta(\log(\gamma(r)))}]^{t - (\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon)}}.
\end{aligned}$$

Therefore $\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{t+1}} dr$ ($r_0 > 0$) is divergent when $t < \rho_{(\alpha,\beta,\gamma)}[f]_h$ and convergent when $t > \rho_{(\alpha,\beta,\gamma)}[f]_h$.

Subcase (II). When $\rho_{(\alpha,\beta,\gamma)}[f]_h = 0$, then Definition 5 gives for all sufficiently large values of r that

$$\frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} \leq \varepsilon.$$

Then, with a similar arguments used in the previous sub-case, we get that

$\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{t+1}} dr$ ($r_0 > 0$) is convergent when $t > 0$ and divergent when $t < 0$.

If the integral $\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon + 1}} dr$ converge, then by Lemma 3 and for arbitrary positive ε , we get

$$\overline{\lim}_{r \rightarrow +\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon}} = 0,$$

i.e, for all sufficiently large values of r ,

$$\begin{aligned}
& \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]}}{[e^{\beta(\log(\gamma(r)))}]^{\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon}} < \varepsilon_0, \\
& i.e., e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]} < \varepsilon_0 \cdot [e^{\beta(\log(\gamma(r)))}]^{\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon}, \\
& i.e., \alpha(\log \mu_h^{-1}(\mu_f(r))) < \log \varepsilon_0 + (\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon) \cdot \beta(\log(\gamma(r))), \\
& i.e., \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} \leq \frac{\log \varepsilon_0}{\beta(\log(\gamma(r)))} + \rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon, \\
& i.e., \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} \leq \rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon.
\end{aligned}$$

Letting ε approach zero, we get

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} \leq \rho_{(\alpha,\beta,\gamma)}[f]_h. \quad (7)$$

If the integral $\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]} }{[e^{\beta(\log(\gamma(r)))}]^{\rho_{(\alpha, \beta, \gamma)}[f]_h - \varepsilon + 1}} dr$ is divergent, we have a sequence of values of r tending to infinity for which

$$\begin{aligned} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]} }{[e^{\beta(\log(\gamma(r)))}]^{\rho_{(\alpha, \beta, \gamma)}[f]_h - \varepsilon + 1}} &> \frac{1}{[e^{\beta(\log(\gamma(r)))}]^{1+\varepsilon}}, \\ i.e., e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]} &> (e^{\beta(\log(\gamma(r)))})^{\rho_{(\alpha, \beta, \gamma)}[f]_h - 2\varepsilon}, \\ i.e., \alpha(\log \mu_h^{-1}(\mu_f(r))) &> (\rho_{(\alpha, \beta, \gamma)}[f]_h - 2\varepsilon) \cdot \beta(\log(\gamma(r))), \\ i.e., \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} &> (\rho_{(\alpha, \beta, \gamma)}[f]_h - 2\varepsilon). \end{aligned}$$

As ε approaches zero, we get

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} \geq \rho_{(\alpha, \beta, \gamma)}[f]_h. \quad (8)$$

Thus from (7) and (8) it follows that

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_f(r)))}{\beta(\log(\gamma(r)))} = \rho_{(\alpha, \beta, \gamma)}[f]_h.$$

This is by the Definition 5.

Hence by Case 1 and Case 2, we reach at the conclusion.

The following Theorem 2 can be proved using a similar argument to that used for Theorem 1.

Theorem 2. *The growth indicator $\lambda_{(\alpha, \beta, \gamma)}[f]_h$ is alternatively defined as: The integral $\int_{r_0}^{\infty} \frac{e^{[\alpha(\log \mu_h^{-1}(\mu_f(r)))]} }{[e^{\beta(\log(\gamma(r)))}]^{t+1}} dr$ ($r_0 > 0$) diverges when $t < \lambda_{(\alpha, \beta, \gamma)}[f]_h$ and converges when $t > \lambda_{(\alpha, \beta, \gamma)}[f]_h$.*

Example 1. Let us consider the functions α, β, γ defined on $(-\infty, +\infty)$ such that

$$\alpha(x) = \beta(x) = \gamma(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

So α, β and γ are continuous, non-negative, increasing and unbounded functions. It is easily seen that $\alpha \in L_1, \beta \in L_2$ and $\gamma \in L_3$. Now let us consider the entire functions f and h defined on $\mathbb{C}$

$$f(z) = \exp(z^k) = \sum_{n=0}^{\infty} \frac{z^{nk}}{n!} \text{ and } h(z) = \exp(z) \text{ where } k \geq 1.$$

By definition of the maximum term

$$\mu_f(r) = \mu(r, f) = \max_{n \geq 0} \frac{r^{nk}}{n!}.$$

Using Stirling's approximation formula

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n,$$

one can show that the maximum term occurs when $n \sim \exp(r^k)$. Hence

$$\mu_f(r) \sim \exp(r^k) \text{ as } r \rightarrow \infty.$$

$$\text{As } r \rightarrow \infty, \mu_h(r) \sim \exp(r) \Rightarrow \mu_h^{-1}(s) \sim \log s.$$

$$\Rightarrow \alpha(\log(\mu_h^{-1}(\mu_f(r)))) \sim k \log r \text{ and } \beta(\log(\gamma(r))) = \log r.$$

$$\text{Thus } \rho_{(\alpha, \beta, \gamma)}[f]_h = \lambda_{(\alpha, \beta, \gamma)}[f]_h = k < \infty.$$

The integral in the Theorem 1 and Theorem 2 becomes

$$\int_{r_0}^{\infty} \frac{\exp[\alpha(\log(\mu_h^{-1}(\mu_f(r))))]}{[\exp(\beta(\log(\gamma(r))))]^{t+1}} dr = \int_{r_0}^{\infty} \frac{dr}{r^{t+1-k}}.$$

From the p-test of improper integral we can say that the above integral is divergent when $t < k$ and convergent when $t > k$. Therefore the example satisfy the Theorem 1 and Theorem 2.

Theorem 3. Let f, g and h be three entire functions such that $0 < \lambda_{(\alpha, \beta, \gamma)}[f]_h \leq \rho_{(\alpha, \beta, \gamma)}[f]_h < +\infty$ and $\lambda_{(\alpha, \beta, \gamma)}[f \circ g]_h = +\infty$. Then

$$\lim_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha(\log \mu_h^{-1}(\mu_f(r)))} = +\infty.$$

Proof. Suppose Theorem 3 does not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of r tending to infinity

$$\alpha(\log \mu_h^{-1}(\mu_{f \circ g}(r))) \leq \Delta \cdot \alpha(\log \mu_h^{-1}(\mu_f(r))). \quad (9)$$

Again from the definition of $\rho_{(\alpha, \beta, \gamma)}[f]_h$ given by Definition 5, it follows for all sufficiently large values of r that

$$\alpha(\log \mu_h^{-1}(\mu_f(r))) \leq (\rho_{(\alpha, \beta, \gamma)}[f]_h + \epsilon) \cdot \beta(\log(\gamma(r))). \quad (10)$$

From (9) and (10), for a sequence of values of r tending to $+\infty$, we have

$$\alpha(\log \mu_h^{-1}(\mu_{f \circ g}(r))) \leq \Delta(\rho_{(\alpha, \beta, \gamma)}[f]_h + \epsilon) \beta(\log(\gamma(r))),$$

$$i.e., \frac{\alpha(\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\beta(\log(\gamma(r)))} \leq \Delta(\rho_{(\alpha, \beta, \gamma)}[f]_h + \epsilon),$$

$$i.e., \lim_{r \rightarrow +\infty} \frac{\alpha(\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\beta(\log(\gamma(r)))} < +\infty.$$

Again from Definition 5, it is clear that

$$\lambda_{(\alpha, \beta, \gamma)}[f \circ g]_h < +\infty.$$

This is a contradiction.

Thus the theorem follows.

Example 2. Let us consider the functions α, β, γ defined on $(-\infty, +\infty)$ such that

$$\begin{aligned} \alpha(x) &= \gamma(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0, \end{cases} \quad \text{and} \\ \beta(x) &= \begin{cases} 0 & \text{if } x < 1 \\ \log x & \text{if } x \geq 1. \end{cases} \end{aligned}$$

So α, β and γ are continuous, non-negative, increasing and unbounded functions. It is easily seen that $\alpha \in L_1, \beta \in L_2$ and $\gamma \in L_3$. Now let us consider the entire functions f, g and h defined on $\mathbb{C}$ such that

$$f(z) = \exp(\log z)^k = \sum_{n=0}^{\infty} \frac{(\log z)^{nk}}{n!},$$

$$h(z) = \exp(z) \text{ and } g(z) = \exp(z^m) \text{ where } m, k \geq 1.$$

By definition of the maximum term

$$\mu_f(r) = \mu(r, f) = \max_{n \geq 0} \frac{(\log r)^{nk}}{n!}.$$

Using Stirling's approximation formula

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n,$$

one can show that the maximum term occurs when $n \sim \exp(\log r)^k$. Hence

$$\mu_f(r) \sim \exp(\log r)^k \text{ as } r \rightarrow \infty.$$

$$\text{As } r \rightarrow \infty, \mu_h(r) \sim \exp(r) \Rightarrow \mu_h^{-1}(s) \sim \log s.$$

$$\Rightarrow \alpha(\log(\mu_h^{-1}(\mu_f(r)))) \sim k \log(\log r) \text{ and } \beta(\log(\gamma(r))) = \log(\log r).$$

Thus

$$\rho_{(\alpha, \beta, \gamma)}[f]_h = \lambda_{(\alpha, \beta, \gamma)}[f]_h = k < \infty.$$

Now $(f \circ g)(z) = \exp(z^{mk})$

$$\text{implies } \mu_{f \circ g}(r) \sim \exp(r^{mk}),$$

$$\text{So, } \alpha(\log(\mu_h^{-1}(\mu_{f \circ g}(r)))) \sim mk \log r \text{ and } \beta(\log(\gamma(r))) = \log(\log r).$$

Which implies

$$\lambda_{(\alpha, \beta, \gamma)}[f \circ g]_h = \lim_{r \rightarrow +\infty} \frac{\alpha(\log(\mu_h^{-1}(\mu_{f \circ g}(r))))}{\beta(\log(\gamma(r)))} = \lim_{r \rightarrow +\infty} \frac{mk \log r}{\log(\log r)} = +\infty.$$

Therefore, the problem satisfy all the hypothesis of the Theorem 3. Now

$$\lim_{r \rightarrow \infty} \frac{\alpha(\log(\mu_h^{-1}(\mu_{f \circ g}(r))))}{\alpha(\log(\mu_h^{-1}(\mu_f(r))))} = \lim_{r \rightarrow \infty} \frac{mk \log r}{k \log(\log r)} = +\infty.$$

Hence the example satisfy the Theorem 3.

Remark 3. If we take $0 < \lambda_{(\alpha, \beta, \gamma)}[g]_h \leq \rho_{(\alpha, \beta, \gamma)}[g]_h < +\infty$ instead of $0 < \lambda_{(\alpha, \beta, \gamma)}[f]_h \leq \rho_{(\alpha, \beta, \gamma)}[f]_h < +\infty$ and other conditions remain same, the conclusion of Theorem 3 remains true with $\alpha(\log \mu_h^{-1}(\mu_g(r)))$ in place of $\alpha(\log \mu_h^{-1}(\mu_f(r)))$ in the denominator.

Remark 4. Theorem 3 and Remark 3 are also valid with “limit superior” instead of “limit” if $\lambda_{(\alpha, \beta, \gamma)}[f \circ g]_h = +\infty$ is replaced by $\rho_{(\alpha, \beta, \gamma)}[f \circ g]_h = +\infty$ and the other conditions remain the same.

Theorem 4. Let f, g and h be three entire functions such that $0 < \lambda_{(\alpha, \beta, \gamma)}[f \circ g]_h \leq$

$\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h < +\infty$ and $0 < \lambda_{(\alpha, \beta, \gamma)}[f]_h \leq \rho_{(\alpha, \beta, \gamma)}[f]_h < +\infty$. Then

$$\begin{aligned} \frac{\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\rho_{(\alpha_2, \beta, \gamma)}[f]_h} &\leq \lim_{r \rightarrow +\infty} \frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \\ &\leq \min \left\{ \frac{\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta, \gamma)}[f]_h}, \frac{\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\rho_{(\alpha_2, \beta, \gamma)}[f]_h} \right\} \\ &\leq \max \left\{ \frac{\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta, \gamma)}[f]_h}, \frac{\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\rho_{(\alpha_2, \beta, \gamma)}[f]_h} \right\} \\ &\leq \lim_{r \rightarrow +\infty} \frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \leq \frac{\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta, \gamma)}[f]_h}. \end{aligned}$$

Proof. Using Definition 5, we have from the definitions of $\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h$, $\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h$, $\lambda_{(\alpha_2, \beta, \gamma)}[f]_h$ and $\rho_{(\alpha_2, \beta, \gamma)}[f]_h$, for arbitrary positive ε and for all sufficiently large values of r that

$$\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r))) \geq (\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h - \varepsilon) \beta(\log(\gamma(r))), \quad (11)$$

$$\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r))) \leq (\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h + \varepsilon) \beta(\log(\gamma(r))), \quad (12)$$

$$\alpha_2 (\log \mu_h^{-1}(\mu_f(r))) \geq (\lambda_{(\alpha_2, \beta, \gamma)}[f]_h - \varepsilon) \beta(\log(\gamma(r))) \quad (13)$$

$$\text{and } \alpha_2 (\log \mu_h^{-1}(\mu_f(r))) \leq (\rho_{(\alpha_2, \beta, \gamma)}[f]_h + \varepsilon) \beta(\log(\gamma(r))). \quad (14)$$

Again for a sequence of values of r tending to infinity,

$$\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r))) \leq (\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h + \varepsilon) \beta(\log(\gamma(r))), \quad (15)$$

$$\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r))) \geq (\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h - \varepsilon) \beta(\log(\gamma(r))), \quad (16)$$

$$\alpha_2 (\log \mu_h^{-1}(\mu_f(r))) \leq (\lambda_{(\alpha_2, \beta, \gamma)}[f]_h + \varepsilon) \beta(\log(\gamma(r))), \quad (17)$$

$$\text{and } \alpha_2 (\log \mu_h^{-1}(\mu_f(r))) \geq (\rho_{(\alpha_2, \beta, \gamma)}[f]_h - \varepsilon) \beta(\log(\gamma(r))). \quad (18)$$

Now from (11) and (14) it follows for all sufficiently large values of r that

$$\frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \geq \frac{\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h - \varepsilon}{\rho_{(\alpha_2, \beta, \gamma)}[f]_h + \varepsilon}.$$

As ε tends to zero, we obtain that

$$\lim_{r \rightarrow +\infty} \frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \geq \frac{\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\rho_{(\alpha_2, \beta, \gamma)}[f]_h}. \quad (19)$$

Combining (13) and (15), we have for a sequence of values of r tending to infinity that

$$\frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \leq \frac{\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h + \varepsilon}{\lambda_{(\alpha_2, \beta, \gamma)}[f]_h - \varepsilon}.$$

Since ε tends to zero, it follows that

$$\lim_{r \rightarrow +\infty} \frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \leq \frac{\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta, \gamma)}[f]_h}. \quad (20)$$

Again from (11) and (17), for a sequence of values of r tending to infinity, we get

$$\frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \geq \frac{\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h - \varepsilon}{\lambda_{(\alpha_2, \beta, \gamma)}[f]_h + \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, we get from above that

$$\lim_{r \rightarrow +\infty} \frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \geq \frac{\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta, \gamma)}[f]_h}. \quad (21)$$

Also, it follows from (12) and (13), for all sufficiently large values of r that

$$\frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \leq \frac{\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h + \varepsilon}{\lambda_{(\alpha_2, \beta, \gamma)}[f]_h - \varepsilon}.$$

Since ε approach zero, we obtain that

$$\lim_{r \rightarrow +\infty} \frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \leq \frac{\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta, \gamma)}[f]_h}. \quad (22)$$

Now from (12) and (18), it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \leq \frac{\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h + \varepsilon}{\rho_{(\alpha_2, \beta, \gamma)}[f]_h - \varepsilon}.$$

As ε is arbitrary, we obtain that

$$\lim_{r \rightarrow +\infty} \frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \leq \frac{\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\rho_{(\alpha_2, \beta, \gamma)}[f]_h}. \quad (23)$$

Combining (14) and (16), we get for a sequence of values of r tending to infinity that

$$\frac{\alpha_1 (\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2 (\log \mu_h^{-1}(\mu_f(r)))} \geq \frac{\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h - \varepsilon}{\rho_{(\alpha_2, \beta, \gamma)}[f]_h + \varepsilon}.$$

Since ε is arbitrary, it follows that

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\alpha_1(\log \mu_h^{-1}(\mu_{f \circ g}(r)))}{\alpha_2(\log \mu_h^{-1}(\mu_f(r)))} \geq \frac{\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\rho_{(\alpha_2, \beta, \gamma)}[f]_h}. \quad (24)$$

Thus the theorem follows from (19), (20), (21), (22), (23) and (24).

Example 3. Let us consider the functions $\alpha, \alpha_1, \alpha_2, \beta, \gamma$ defined on $(-\infty, +\infty)$ such that

$$\alpha(x) = \alpha_1(x) = \alpha_2(x) = \beta(x) = \gamma(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0, \end{cases}$$

So $\alpha, \alpha_1, \alpha_2, \beta$ and γ are continuous, non-negative, increasing and unbounded functions. It is easily seen that $\alpha, \alpha_1, \alpha_2 \in L_1, \beta \in L_2$ and $\gamma \in L_3$. Now let us consider the entire functions f, g and h defined on $\mathbb{C}$ such that

$$f(z) = h(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ and } g(z) = z^m \text{ where } m \geq 1.$$

Hence

$$\mu_f(r) \sim \exp(r) \text{ as } r \rightarrow \infty.$$

$$\text{As } r \rightarrow \infty, \mu_h(r) \sim \exp(r) \Rightarrow \mu_h^{-1}(s) \sim \log s.$$

$$\Rightarrow \alpha(\log(\mu_h^{-1}(\mu_f(r)))) \sim \log r \text{ and } \beta(\log(\gamma(r))) = \log r.$$

Therefore,

$$\rho_{(\alpha, \beta, \gamma)}[f]_h = \lambda_{(\alpha, \beta, \gamma)}[f]_h = 1.$$

Hence,

$$0 < \lambda_{(\alpha, \beta, \gamma)}[f]_h \leq \rho_{(\alpha, \beta, \gamma)}[f]_h < +\infty$$

Now $(f \circ g)(z) = \exp(z^m)$,

As $r \rightarrow \infty$

$$\mu_{f \circ g}(r) \sim \exp(r^m).$$

$$\text{Again, } \mu_h(r) \sim \exp(r) \Rightarrow \mu_h^{-1}(s) \sim \log s.$$

$$\Rightarrow \alpha_1(\log(\mu_h^{-1}(\mu_{f \circ g}(r)))) \sim m \log r,$$

Thus,

$$\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h = \rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h = m.$$

Hence,

$$0 < \lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h \leq \rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h < +\infty.$$

Therefore the problem satisfy all the hypothesis of the Theorem 4. Now

$$\frac{\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\rho_{(\alpha_2, \beta, \gamma)}[f \circ g]_h} = \frac{m}{1} = m,$$

$$\lim_{r \rightarrow +\infty} \frac{\alpha_1(\log(\mu_h^{-1}(\mu_{f \circ g}(r))))}{\alpha_2(\log(\mu_h^{-1}(\mu_f(r))))} = \lim_{r \rightarrow +\infty} \frac{m \log r}{\log r} = m,$$

$$\min \left\{ \frac{\lambda_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta, \gamma)}[f]_h}, \frac{\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\rho_{(\alpha_2, \beta, \gamma)}[f]_h} \right\} = \min \left\{ \frac{m}{1}, \frac{m}{1} \right\} = m,$$

and

$$\frac{\rho_{(\alpha_1, \beta, \gamma)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta, \gamma)}[f]_h} = \frac{m}{1} = m.$$

Therefore the example satisfy conclusion of the Theorem 4.

Remark 5. If we take $0 < \lambda_{(\alpha_3, \beta, \gamma)}[g]_h \leq \rho_{(\alpha_3, \beta, \gamma)}[g]_h < +\infty$ instead of $0 < \lambda_{(\alpha_2, \beta, \gamma)}[f]_h \leq \rho_{(\alpha_2, \beta, \gamma)}[f]_h < +\infty$ and other conditions remain same, the conclusion of Theorem 4 remains true with $\lambda_{(\alpha_3, \beta, \gamma)}[g]_h$, $\rho_{(\alpha_3, \beta, \gamma)}[g]_h$ and $\alpha_3(\log \mu_h^{-1}(\mu_g(r)))$ in place of $\lambda_{(\alpha_2, \beta, \gamma)}[f]_h$, $\rho_{(\alpha_2, \beta, \gamma)}[f]_h$ and $\alpha_2(\log \mu_h^{-1}(\mu_f(r)))$ respectively in the denominators.

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References

- [1] Belaïdi, B. and Biswas, T., Study of complex oscillation of solutions of a second order linear differential equation with entire coefficients of (α, β, γ) -order, WSEAS Trans. Math., 21 (2022), 361-370, <https://doi.org/10.37394/23206.2022.21.43>.
- [2] Bernal, L., Orden relative de crecimiento de funciones enteras, Collect. Math., 39 (1988), 209-229.
- [3] Bernal-González, L., Crecimiento relativo de funciones enteras. Aportaciones al estudio de las funciones enteras con índice exponencial finito, Doctoral Thesis, Universidad de Sevilla, Spain, 1984.
- [4] Biswas, T., Biswas, C. and Bhattacharyya, S., Maximum term oriented growth analysis of composite entire functions from the view point of (α, β, γ) -order, J. Fract. Calc. Appl., 15(2), (12) (2024), 1-8, <https://doi.org/10.21608/jfca.2024.280977.1093>.

- [5] Chyzhykov, I. and Semochko, N., Fast growing entire solutions of linear differential equations, *Math. Bull. Shevchenko Sci. Soc.*, 13 (2016), 1-16.
- [6] Heittokangas, J., Wang, J., Wen, Z. T. and Yu, H., Meromorphic functions of finite φ -order and linear q -difference equations, *J. Difference Equ. Appl.*, 27(9) (2021), 1280-1309, <https://doi.org/10.1080/10236198.2021.1982919>.
- [7] Mulyavai, O., Sheremeta, M. and Trukhan, Y., Relative growth of Hadamard compositions of entire Dirichlet series, *Visnyk of the Lviv Univ. Series Mech. Math.*, 95 (2023), 83-93, <https://doi.org/10.30970/vmm.2023.95.083-093>.
- [8] Sato, D., On the rate of growth of entire functions of fast growth, *Bull. Amer. Math. Soc.*, 69 (1963), 411-414.
- [9] Sheremeta, M. N., Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion, *Izv. Vyssh. Uchebn. Zaved. Mat.* 2 (1967) 100-108 (in Russian).
- [10] Singh, A. P. and Baloria, M. S., On the maximum modulus and maximum term of composition of entire functions, *Indian J. Pure Appl. Math.*, 22(12) (1991), 1019-1026.
- [11] Valiron, G., *Lectures on the general theory of integral functions*, Chelsea Publishing Company, New York, 1949.
- [12] Yang, L., *Value distribution theory*, Springer-Verlag, Berlin, 1993.

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