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THE PROPERTIES OF FUZZY GRAPHIC MATROIDS

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Abstract: The minor and duality of a fuzzy graphic matroid are defined and also explored their properties. The charcteristics of fuzzy graphic matroid like augumentation, base exchange property, uniformity, submodularity, weak absorption, strong absorption, weak elimination and induced circuit properties are discussed with examples.

Keywords and Phrases: Fuzzy matroid, fuzzy graphic matroid, bases, circuit, minor, dual, induced circuit.

2020 Mathematics Subject Classification: 05C50, 05C72, 05C70.

1. Introduction

A graph is a structural representation of a set in which some pairs of objects are related [1]. In 18^{th} century, the basic ideas of graph theory introduced by Leonhard Euler [2]. In 1847 G.R. Kirchoff developed the tree theory in networks for their application.

Fuzzy graph theory is a branch of graph theory that incorporates fuzzy concepts allowing degrees of membership for vertices and edges. In 1965, Zadeh [17] introduced the concept of fuzzy sets laying the foundation for fuzzy logic and its applications in various fields. Zadeh's work paved the way for defining fuzzy graphs where edges have fuzzy weights or memberships. In 1975, Rosenfeld [6] developed the notion of fuzzy graphs and the concept of fuzzy adjacency matrices [3]. This work established a formal framework for analyzing fuzzy graphs.

A matroid is a combinatorial structure that provides a broader conceptual framework encompassing the idea of linear independence. It serves as a unifying bridge between graph theory and linear algebra [14]. In 1935, Whitney introduced matroids as combinatorial abstraction of linear independence and provided two axioms for independence [15]. Also, defined any structure adhering to these axioms to be matroids. The pioneering works in matroid theory done by Birkhoff in 1935 and MacLane in 1936. In 1950 W. T. Tutte provided a ground breaking characterization of matroids in terms of their independent sets, circuits and bases [13]. The theory of graphic matroids introduced by Whitney and further studied by Tutte in 1954 and others [13]. In the 1970s and 1980s, there was a significant expansion of matroid theory with the development of several fundamental concepts and results [14].

Graphic matroids are a class of matroids that arise from graphs [10]. Specifically, given a graph G and its graphic matroid is constructed by considering the set of edges as the ground set. A subset of edges is independent if it does not contain any cycle of the graph. This construction establishes a correspondence between graph theory and matroid theory illustrating how concepts from graph cycles and connectivity can be captured and analyzed using the framework of matroids. The theory of matroid minors introduced by Tutte in 1965 which provides a powerful tool for analyzing the structure of matroids and establishing connections between different classes of matroids.

In fuzzy matroids the elements can belong to sets with varying degrees of membership. In 1988, Goetschal and Voxman introduced the notion of fuzzy matroids [7]. Fuzzy matroids deals with graphical and algebrical structures related to the membership grades of a fuzzy graph. A new class of matroids from fuzzy graphs was constructed by O. K. Sabana and K. Sameena in 2019 [8],[9]. From a fuzzy graph G, subgraphs can be formed by deleting or contracting edges. The resulting fuzzy graphs are the minors of G.

Fuzzy graphic matroids are a generalization of graphic matroids, incorporating concepts from fuzzy graph theory. This framework extends traditional matroid theory by integrating the flexibility of fuzzy logic and making it useful for applications involving uncertainty and imprecision in network structures [10]. In this paper, the properties of fuzzy graphic matroids, minor and duality of fuzzy graphic matroid are discussed.

This paper is organised as follows: Section 2 deals with the basic definition. The definition of fuzzy graphic matroid is given and also discussed the properties of fuzzy graphic matroids in section 3. The dual and minor of fuzzy graphic matroids are discussed with examples in Section 4. Section 5 concludes the paper.

2. Preliminaries

Definition 2.1. [11] *Let $G = (\sigma^*, \mu^*)$ is a pair of functions with σ^* and μ^* where $\sigma^* : V \longrightarrow [0,1]$ is a fuzzy vertex set of G and $\mu^* : V \times V \longrightarrow [0,1]$ is a fuzzy edge set of G where $\mu_{ij} \leq min(\mu_i, \mu_j)$ for every $\mu_i,\ \mu_j\ \in V$. Then $G = (\sigma^*, \mu^*)$ is called as a fuzzy graph with n vertices and m edges.*

Definition 2.2. [12] *A fuzzy subset of a non-empty set is a mapping $\sigma : X \to [0,1]$ which assigns to each element in X a degree of membership.*

Definition 2.3. [5] *Let I be a non-empty family of a finite subsets of E and the pair $\mathcal{M} = (E, I)$ is called a matroid if it satisfies the following properties:*

(i) $\phi \in I$

(ii) If $A \in I, B \subset A$, *then* $B \in I$

(iii) If $A, B \in I; |B| \geq |A|$, *then there exists an* $e \in B - A$ *such that* $A \cup \{e\} \in I$

The collection of basis uniquely determines the matroid. A *basis (or base)* of a matroid refers to the maximal independent set within $\mathcal{M}$, while a *circuit* of the matroid corresponds to the minimal dependent set in $\mathcal{M}$. The size of the maximal independent set within the set E is defined to be the *rank of a matroid* and is represented as $r(I)$ [5].

Let $\mathbb{I}$ be a fuzzy subset on $\mathbb{E}$ is a function $\mathbb{I} : X \longrightarrow [0,1]$. The family of fuzzy set on $\mathbb{E}$ is given by $F(X)$. If $P, Q \in \mathfrak{F}(\mathbb{E})$, then

(i) $supp(P) = \{x \in \mathbb{E} \mid P(x) > 0\}$, a crisp set

(ii) $m(P) = min\{P(x) \mid x \in supp(P)\}$

(iii) $P \cup Q = max\{P(x), Q(x)\}, x \in \mathbb{E}$

(iv) $P \cap Q = min\{P(x), Q(x)\}, x \in \mathbb{E}$.

Note: For the sake of convenience, different notations are used here.

Definition 2.4. [7] [9] *Let $\mathbb{E}$ be a finite set and let $\mathbb{I} \in \mathfrak{F}(F_E)$ satisfying the following conditions:*

(i) $\phi \in \mathbb{I}$

(ii) $P \subset Q$ *and* $Q \in P$, *where* $P \subset Q, P(x) \leq Q(x)$ *forall* $x \in \mathbb{E}$

(iii) If $P, Q \in \mathbb{I}$ *with* $|supp\ (P)| \leq |supp\ (Q)|$, *there exists* $R \in \mathbb{I}$ *such that*

(a) $P \subset R \subseteq P \cup Q$

(b) $m(R) = min\,\{m(P), m(Q)\}$.

Then $F_M = (\mathbb{E}, \mathbb{I})$ *is a fuzzy matroid.*

Definition 2.5. [16] *A graphic matroid is derived from graph* G*. It is defined on the egde set of the graph* G *and it is denoted by* $M(G)$*.*

(i) A basis or a base of $M(G)$ *is the spanning trees of* G*.*

(ii) A circuit of $M(G)$ *is the simple cycles in a graph* G*.*

Definition 2.6. [4] *The rank function of a graphic matroid* $M(G)$ *defined on a graph* $G = (V, E)$ *is given by*

$$r(M(G)) = n - K(V, F)$$

where n *is the number of verticec and* $K(V, F)$ *is the number of connected components.*

3. Properties of Fuzzy Graphic Matroids

Definition 3.1. *A fuzzy matroid is said to be fuzzy graphic matroid when it can be isomorphically mapped to* $FM(G)$ *for a given fuzzy graph* G*.*

Definition 3.2. *The spanning trees of a fuzzy graph* G *is defined to be the basis or base of a fuzzy graphic matroid. It is denoted by* $\mathbb{B}(FGM)$*.*

Definition 3.3. *The minimal dependent edges of a fuzzy graph* G *that forms a simple cycle is defined to be the circuit of a fuzzy graphic matroid.* $\mathbb{C}(FGM)$*.*

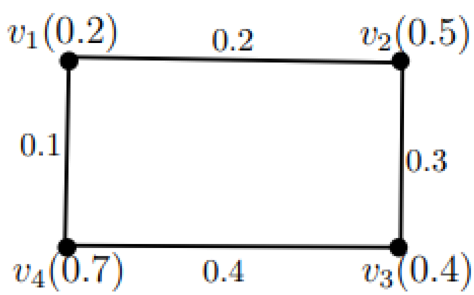

Figure 3.1 A fuzzy graphic matroid derived from a fuzzy graph G

From figure 3.1, Let $\mathbb{E} = \{0.2/\mu_1, 0.3/\mu_2, 0.4/\mu_2, 0.1/\mu_4\}$,
$\mathbb{I} = \{\{\phi\}, \{0.2/\mu_1\}, \{0.3/\mu_2\}, \{0.4/\mu_3\}, \{0.1/\mu_4\}, \{0.2/\mu_1, 0.3/\mu_2\}, \{0.3/\mu_2, 0.4/\mu_3\}, \{0.4/\mu_3, 0.1/\mu_4\}, \{0.1/\mu_4, 0.2/\mu_1\}, \{0.2/\mu_1, 0.3/\mu_2, 0.4/\mu_3\}, \{0.3/\mu_2, 0.4/\mu_3, 0.1/\mu_4\}, \{0.4/\mu_3, 0.1/\mu_4, 0.2/\mu_1\}$
P and Q are the subset of edges of G. $P \cup Q$ and $P \cap Q \in \mathbb{I}$. The cardinality of P and Q is 0.7 and 0.9. Therefore $|P| \leq |Q|$. The rank function of FGM is 0.9.

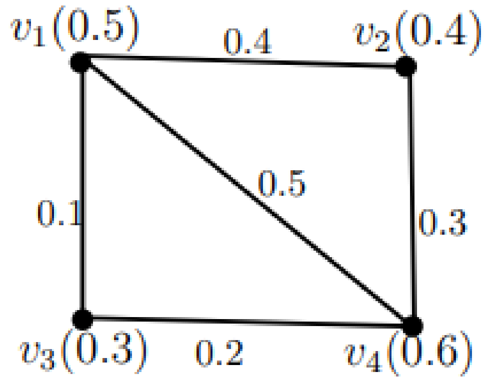

Figure 3.2 A fuzzy graph G

Consider a fuzzy graph G. Let $\mathbb{E}$ be the set of edges of G and $\mathbb{I}$ be its independent set of edges in G. P and Q are the subset of edges in $\mathbb{I}$. $\mathbb{B}, \mathbb{C}$ and $\mathbb{R}$ are bases, circuits and rank of a fuzzy graphic matroid. The following are the properties which are satisfied by a fuzzy graphic matroid.

Augmentation [I]. If $P, Q \in \mathbb{I}$, with $|Q| > |P|$ then $P + \{x\} \in \mathbb{I}$ for some $x \in Q - P$.

Uniformity [U]. $P \in \mathbb{I}$, then the maximal fuzzy subsets of $P \in \mathbb{I}$ have the same size.

Base Exchange [B]. If $\mathbb{B}_1, \mathbb{B}_2 \in \mathbb{B}$ and $x \in \mathbb{B}_1 \backslash \mathbb{B}_2$ then there exist $y \in \mathbb{B}_2 \backslash \mathbb{B}_1$ such that $(\mathbb{B}_1 - \{x\}) + \{y\} \in \mathbb{B}$.

Submodularity [R]. $\mathbb{R}(P \cap Q) + \mathbb{R}(P \cup Q) \leq \mathbb{R}(P) + \mathbb{R}(Q)$ whenever $P, Q \in \mathbb{E}$.

Weak absorption [A]. $\mathbb{R}(P + \{x\}) = \mathbb{R}(P + \{y\})$ implies $\mathbb{R}(P + \{x\} + \{y\}) = \mathbb{R}(\mathbb{I})$ whenever $P \subseteq \mathbb{E}$ and $x, y \in \mathbb{E}$.

Strong absorption [A']. If $P, Q \in \mathbb{E}$ and $\mathbb{R}(P + \{x\}) = \mathbb{R}(P)$ for all $x \in \mathbb{B}$, then $\mathbb{R}(P \cup Q) = \mathbb{R}(P)$.

Weak Elimination [C]. For distinct circuits $\mathbb{C}_1, \mathbb{C}_1 \in \mathbb{C}$ and $\{x\} \in \mathbb{C}_1 \cap \mathbb{C}_2$, there is another member of $\mathbb{C}$ contained in $(\mathbb{C}_1 \cup \mathbb{C}_2 - \{x\})$.

4. Minor and Dual of Fuzzy Graphic Matroids

Definition 4.1. *The dual of fuzzy graphic matroid FGM is FGM^* whose bases are the complements of the bases FGM.*

- *The sub-bases $\mathfrak{S}$ of FGM are the fuzzy set of edges containing a base.*
- *The hypobase $\mathfrak{H}$ of FGM are the maximal fuzzy set of edges containing no base.*
- *$\mathbb{B}^*$, $\mathbb{C}^*$ are the cobases and cocircuits of Fuzzy Graphic Matroid.*

Definition 4.2. *A minor of a fuzzy graphic matroid FGM is formed by two operations:*

- *The restriction that deletes an edge μ_{ij} of a fuzzy graph without changing the independence or rank of the remaining subset of the edge set of G.*
- *The contraction that deletes an edge μ_{ij} of a fuzzy graph by decreasing the rank of every edge set it belongs to.*

Lemma 4.3. *The dual of the dual of a fuzzy graphic matroid FGM is a fuzzy graphic matroid.*

$$(FGM^*)^* \quad = FGM \tag{1}$$

Proof. Let FGM be a fuzzy graphic matroid derived from a fuzzy graph G. Let F_E be the ground set of FGM then the dual FGM^* has the same ground set but the independent edges are different. A set of edges of a fuzzy graph G is independent if and only if its complement forms a circuit in G. similarly, the complement of a set of edges in FGM which forms a cycle corresponds to forest in FGM^*. Hence the dual of the dual of fuzzy graphic matroid is fuzzy graphic matroid.

Theorem 4.4. *Let FGM_1 and FGM_2 be two fuzzy graphic matroids with disjoint ground set of edges then the dual of $FGM_1 \oplus FGM_2$ is equal to $FGM_1^* \oplus FGM_2^*$.*
Proof. If G_1 and G_2 is a fuzzy graph and $\mathbb{E}_i$ for all $i = 1, 2$ be the edge set of G_1 and G_2. Let FGM_1 and FGM_2 be the fuzzy graphic matroids derived from G_1 and G_2. The families of bases of $FGM_1^* \oplus FGM_2^*$ and $(FGM_1 + FGM_2)^*$ coincides.

By the base exchange property of fuzzy graphic matroid

$$\begin{aligned}
\mathbb{B}((FGM_1 + FGM_2)^*) &= \{\mathbb{B}^* \mid \mathbb{E}\backslash\mathbb{B}^* \in \mathbb{B}(FGM_1 + FGM_2)\} && (2)\\
&= \{\mathbb{E}\backslash\mathbb{B} \mid \mathbb{B} \in \mathbb{B}(FGM_1 + FGM_2)\} && (3)\\
&= \{\mathbb{E}\backslash(\mathbb{B}_1 \cup \mathbb{B}_2) \mid \mathbb{B}_1 \in \mathbb{B}(FGM_1), \mathbb{B}_2 \in \mathbb{B}(FGM_2)\} && (4)\\
&= \{(\mathbb{E}_1\backslash\mathbb{B}_1) \cup (\mathbb{E}_2\backslash\mathbb{B}_2) \mid \mathbb{B}_1 \in \mathbb{B}(FGM_1), \mathbb{B}_2 \in \mathbb{B}(FGM_2)\} && (5)\\
&= \{\mathbb{B}_1^* \cup \mathbb{B}_2^* \mid \mathbb{B}_1^* \in \mathbb{B}_1^*(FGM_1^*), \mathbb{B}_2^* \in \mathbb{B}_2^*(FGM_2^*)\} && (6)\\
&= \mathbb{B}(FGM_1^* + FGM_2^*) && (7)
\end{aligned}$$

Theorem 4.5. *A fuzzy graphic matroid FGM_1 is a minor of a fuzzy graphic matroid FGM_2 if and only if FGM_{1^*} is a minor of FGM_{2^*}.*

Proof. Let FGM_1 and $FGM_2 2$ be a fuzzy graphic matroid isomorphically mapped from a fuzzy graphs G_1 and G_2 respectively. The dual of FGM_1^* and FGM_2^* are defined with same edge set but the independent edges are different. If F_A is a set of independent edges in FGM_1^* if and only if complement of F_A are the edge set of G forming a circuit.

Case(i). If FGM_1 is a minor of FGM_2 then FGM_1^* is a minor of FGM_2^*
Assume FGM_1 is a minor of FGM_2, then FGM_1 is obtained from FGM_2 by contraction or deletion of edges in G_2. Since FGM_1^* is a dual of FGM_1 that corresponds to contraction of edges in the circuit of G_1 which is equivalent to the contraction of edges in G_2 that forms the dual FGM_2^*. Therefore FGM_1^* is a minor of FGM_2^*.

Case (ii). If FGM_1^* is a minor of FGM_2^* then FGM_1 is a minor of FGM_2.
Assume FGM_1^* is a minor of FGM_2^*. From case (i) it is clear that FGM_1^* and FGM_2^* are obtained by the contraction or deletion of edges in the circuits of G_1 and G_2 respectively. Therefore FGM_1 is a minor of a fuzzy graphic matroid FGM_2.

Theorem 4.6. *If P and Q be two disjoint fuzzy subsets of $\mathbb{E}$ of a fuzzy graphic matroid FGM. The following holds:*

(i) $(\mathbb{E}\backslash P)\backslash Q = \mathbb{E}\backslash(P \cup Q) = (\mathbb{E}\backslash Q)\backslash P$

(ii) $(\mathbb{E}/P)/Q = \mathbb{E}/(P \cup Q) = (\mathbb{E}/Q)/P$

(iii) $(\mathbb{E}/P)\backslash Q = (\mathbb{E}\backslash Q)/P$

Proof. Proof of (i) Let G be a fuzzy graph $\mathbb{E}$ is the edge set of G. If P and Q are the subset of edges of G. If P and Q is disjoint subsets of G then $(\mathbb{E}\backslash P)\backslash Q$,$\mathbb{E}\backslash(P \cup Q)$ and $(\mathbb{E}\backslash Q)\backslash P$ are equal by the definition of minor of fuzzy graphic

matroid.
Proof of (ii) is obtained by using the property of dual of a fuzzy graphic matroid.
Proof of (iii) consider the rank function of fuzzy graphic matroid. If $\mathbb{S}$ be a subset of $\mathbb{E}$ then $\mathbb{S} \subseteq \mathbb{E} - (P \cup Q)$ then the rank function of $(\mathbb{E}/P)\backslash Q$ and $(\mathbb{E}\backslash Q)/P$ are equal.

$$\mathbb{R}_{(\mathbb{E}/P)\backslash Q}(\mathbb{S}) = \mathbb{R}_{(\mathbb{E}/P)}(\mathbb{S}) \tag{8}$$
$$= \mathbb{R}_{\mathbb{E}}(\mathbb{S} \cup P) - \mathbb{R}_{(\mathbb{E})}(P) \tag{9}$$
$$= \mathbb{R}_{(\mathbb{E}\backslash Q)}(\mathbb{S} \cup Q) - \mathbb{R}_{(\mathbb{E}\backslash Q)}(P) \tag{10}$$
$$= \mathbb{R}_{(\mathbb{E}\backslash Q)/P}(\mathbb{S}). \tag{11}$$

Lemma 4.7. *Let $G = (\sigma, \mu)$ be a fuzzy graph and P be a subset of edges in G then $P \subseteq \mathbb{E}$. The following holds*

(i) $(\mathbb{E}\backslash P)^* = \mathbb{E}^*\backslash P$

(ii) $(\mathbb{E}/P)^* = \mathbb{E}^*/P$

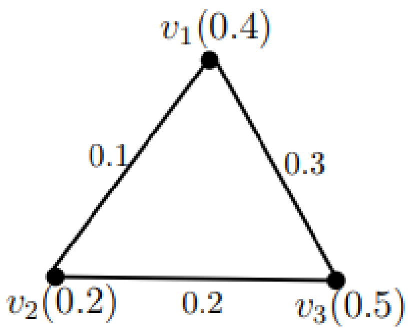

Figure 4.1 Duality and Minor of a fuzzy graphic matroid

Example 4.8. In Fig 4.1, $\mathbb{E} = \{0.3/\mu_1, 0.2/\mu_2, 0.1/\mu_3\}$, $\mathbb{I} = \{\phi, \{0.3/\mu_1\}, \{0.2/\mu_2\}, \{0.1/\mu_3\}, \{0.3/\mu_1, 0.2/\mu_2\}, \{0.2/\mu_2, 0.1/\mu_3\}, \{0.3/\mu_1, 0.1/\mu_3\}$ are the independent set of edges of G. $P = \{0.2/\mu_2, 0.1/\mu_3\}$ and $Q = \{0.3/\mu_1, 0.2/\mu_2\}$ are the subset of edges of G. From the definition of duality and minor of FGM the conditions (i) and (ii) is obvious.

Theorem 4.9. *For a fuzzy graphic matroid FGM, an edge $\mu(x) \in \mathbb{E}$ is a loop or a co-loop if and only if $\mathbb{E}\backslash\{x\} = \mathbb{E}/\{x\}$.*
Proof. If x is loop in a fuzzy graph G then $\mathbb{R}(x) = 0$ or $F_r(x) + \mathbb{R}((FGM) - \{x\}) =$

$\mathbb{R}(FGM)$ so by duality of the fuzzy graphic matroids $FGM\backslash\{x\} = FGM/\{x\}$.
If $\mathbb{R}^*(x) = 0$ then $\mathbb{E}$ is a co-loop of the fuzzy graphic matroid.

$$\mathbb{R}^*(x) = |x| - \mathbb{R}(FGM) + \mathbb{R}(FGM) - \{x\} \tag{12}$$

$$= 0 \tag{13}$$

$$\mathbb{R}(x) + \mathbb{R}(FGM) - \{x\} = \mathbb{R}(FGM) \tag{14}$$

If x is a co-loop then $\mathbb{R}(x) = \sum(\mu(x)), \mu(x) \in \mathbb{I}$.
Assume that x is not a loop then $F_r(x) = \sum(\mu(x))$ and $\mathbb{R}(x) + \mathbb{R}((FGM) - \{x\} + 1) = \mathbb{R}(FGM)$, which means that $(FGM) - \{x\}$ is not a spanning tree if it does not contain a basis. The set is not co-independent, and x is a co-loop.

5. Conclusion

The properties of fuzzy graphic matroids are explored and discussed the duality and minor of fuzzy graphic matroids. The authors further proposed to work on operations on fuzzy graphic matroids and its applications.

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