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RESULTS ON BASIC HYPERGEOMETRIC SERIES AND CONTINUED FRACTIONS

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Abstract: In this paper, making use of certain known identities, we have established some result involving q-series and continued fractions.

Keywords and Phrases: q-series, identities, continued fraction.

2020 Mathematics Subject Classification: 33D15.

1. Introduction, Notations and Definition

The q-rising factorial for complex numbers a and q with $\mid q \mid< 1$ is defined as:

$$(a;q)_0 = 1$$

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}),\ n \in \mathcal{N}$$

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r)$$

Ramanujan's Notebooks, especially second 'Lost' Notebook, contain a large number of q-series identities and fascinating results on continued fractions. Through out the paper, some interesting results involving q-series and continued fractions have been established by making use of certain known identities. We need some established results of the paper [3, 4, 5, 6, 7] to obtain certain continued fractions representations.

2. Main Results

The Rogers-Ramanujan continued fraction with an additional parameter 'a' is given by

(i)

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2}a^n}{(q;q)_n}} = \frac{1}{1+}\frac{aq}{1+}\frac{aq^2}{1+}\frac{aq^3}{1+...} \tag{2.1}$$

Proof of (2.1).

We have, left hand side of (2.1) is

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2}a^n}{(q;q)_n}} = \frac{1}{1+\frac{\sum_{n=0}^{\infty}\frac{q^{n^2}a^n}{(q;q)_n}-\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}}} = \frac{1}{1+\frac{\sum_{n=0}^{\infty}\frac{q^{n^2}a^n(1-q^n)}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}}}$$

$$= \frac{1}{1+\frac{\sum_{n=1}^{\infty}\frac{q^{n^2}a^n}{(q;q)_{n-1}}}{\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}}} = \frac{1}{1+\frac{aq\sum_{n=0}^{\infty}\frac{q^{n^2+2n}a^n}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}}}$$

$$= \frac{1}{1+\frac{aq}{\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+2n}a^n}{(q;q)_n}}}} = \frac{1}{1+\frac{aq}{1+\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}-\sum_{n=0}^{\infty}\frac{q^{n^2+2n}a^n}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+2n}a^n}{(q;q)_n}}}}$$

$$= \frac{1}{1+\frac{aq}{1+\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n(1-q^n)}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+2n}a^n}{(q;q)_n}}}} = \frac{1}{1+\frac{aq}{1+\frac{\sum_{n=1}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_{n-1}}}{\sum_{n=0}^{\infty}\frac{q^{n^2+2n}a^n}{(q;q)_n}}}}$$

$$= \cfrac{1}{1+\cfrac{aq}{1+\cfrac{aq^2}{\cfrac{\sum_{n=0}^{\infty}\frac{q^{n^2+2n}a^n}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+3n}a^n}{(q;q)_n}}}}} = \cfrac{1}{1+\cfrac{aq}{1+\cfrac{aq^2}{1+\cfrac{\sum_{n=0}^{\infty}\frac{q^{n^2+2n}a^n}{(q;q)_n}-\sum_{n=0}^{\infty}\frac{q^{n^2+3n}a^n}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+3n}a^n}{(q;q)_n}}}}}$$

Iterating this process, we find the right hand side of (2.1).
Taking a=1 in (2.1), we get

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}} = \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\ldots} \tag{2.2}$$

which is celebrated continued fraction due to Rogers-Ramanujan.
For a=q, (2.1) yields

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}} = \frac{1}{1+}\frac{q^2}{1+}\frac{q^3}{1+\ldots} \tag{2.3}$$

For a=-1, (2.1) yields

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(q;q)_n}} = \frac{1}{1-}\frac{q}{1-}\frac{q^2}{1-\ldots} \tag{2.4}$$

For $a = q^{-1}$, (2.1) gives

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2-n}}{(q;q)_n}} = \frac{1}{1+}\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\ldots} \tag{2.5}$$

Comparing (2.2) and (2.5), we get

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n^2-n}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}} = 1 + \frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}} \tag{2.6}$$

(ii) An identity due to Ramanujan is

$$(-bq;q)_\infty \sum_{n=0}^{\infty}\frac{(-\lambda/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n} = (-aq;q)_\infty \sum_{n=0}^{\infty}\frac{(-\lambda/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n} \tag{2.7}$$

[Andrews, G. E. and Berndt B. C. [1], 6.2.9, p.146]

Putting $\frac{a}{q}$ for a and $\frac{\lambda}{q}$ for λ in (2.7), we get

$$(-bq;q)_\infty \sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n-1)/2}}{(q;q)_n(-bq;q)_n} = (-a;q)_\infty \sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{bq};q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-a;q)_n} \tag{2.8}$$

Taking ratio of (2.7) and (2.8), we have

$$\frac{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n-1)/2}}{(q;q)_n(-bq;q)_n}} = \frac{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{b};q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}}{(1+a)\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-a;q)_n}} \tag{2.9}$$

The left side of (2.9) is

$$\frac{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n-1)/2}}{(q;q)_n(-bq;q)_n}} = \frac{1}{1+\frac{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n-1)/2}}{(q;q)_n(-bq;q)_n} - \sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}}$$

$$= \cfrac{1}{1+\cfrac{(1+\frac{\lambda}{a})a/(1+bq)}{1+\cfrac{\sum_{n=0}^{\infty}\frac{a^n q^{n(n-1)/2}}{(q;q)_n}\left(\frac{(\frac{-\lambda}{a};q)_n}{(-bq;q)_n}-\frac{(\frac{-\lambda q}{a};q)_n}{(-bq^2;q)_n}\right)}{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda q}{a};q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq^2;q)_n}}}}$$

$$= \cfrac{1}{1+\cfrac{(1+\frac{\lambda}{a})a/(1+bq)}{1-\cfrac{aq(bq-\frac{\lambda}{a})/(1+bq)(1+bq^2)}{\cfrac{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda q}{a};q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq^2;q)_n}}{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda q}{a};q)_n a^n q^{n(n+3)/2}}{(q;q)_n(-bq^3;q)_n}}}}}$$

Iterating this process, we finally get

$$\frac{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n-1)/2}}{(q;q)_n(-bq;q)_n}} = \frac{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{b};q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}}{(1+a)\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-a;q)_n}}$$

$$= \frac{1}{1+}\cfrac{(\lambda+a)/(1+bq)}{1+\cfrac{q(\lambda-abq)/(1+bq)(1+bq^2)}{1+...}}$$

$$= \frac{1}{1+}\frac{(\lambda+a)(1+bq)}{1+}\frac{q(\lambda-abq)/(1+bq)(1+bq^2)}{1+}\frac{q(a+\lambda q)/(1+bq^2)(1+bq^3)}{1+...}$$

$$= \frac{1}{1+}\frac{(a+\lambda)}{(1+bq)+}\frac{q(\lambda-abq)}{(1+bq^2)+}\frac{q(a+\lambda q)}{(1+bq^3)+}\frac{q^3(\lambda-abq^2)}{(1+bq^3)+...} \tag{2.10}$$

As $b \to 0$, (2.10) yields

$$\frac{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n+1)/2}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{a};q)_n a^n q^{n(n-1)/2}}{(q;q)_n}} = \frac{\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2}}{(q;q)_n(-aq;q)_n}}{(1+a)\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2-n}}{(q;q)_n(-a;q)_n}}$$

$$= \frac{1}{1+}\frac{(a+\lambda)}{1+}\frac{\lambda q}{1+}\frac{q(a+\lambda q)}{1+}\frac{\lambda q^3}{1+...} \tag{2.11}$$

Taking a = 0 in (2.11), we obtain

$$\frac{\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2-n}}{(q;q)_n}} = \frac{1}{1+}\frac{\lambda}{1+}\frac{\lambda q}{1+}\frac{\lambda q^2}{1+...} \qquad ((2.12)$$

For $\lambda = 1$, (2.12) is same as (2.5) and for $\lambda =$ q, (2.12) is same as (2.2).
Taking a = 0 in (2.10), we get

$$\frac{\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2}}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2-n}}{(q;q)_n(-bq;q)_n}} = \frac{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{b};q)_n b^n q^{n(n+1)/2}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{bq};q)_n b^n q^{n(n+1)/2}}{(q;q)_n}}$$

$$= \frac{1}{1+}\frac{\lambda}{(1+bq)+}\frac{\lambda q}{(1+bq^2)+}\frac{\lambda q^2}{(1+bq^3)+}\frac{\lambda q^3}{(1+bq^4)+...} \tag{2.13}$$

Taking b = 1 in (2.13), we get

$$\frac{\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2-n}}{(q^2;q^2)_n}} = \frac{\sum_{n=0}^{\infty}\frac{(-\lambda;q)_n q^{n(n+1)/2}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{(\frac{-\lambda}{q};q)_n q^{n(n+1)/2}}{(q;q)_n}}$$

$$= \frac{1}{1+}\frac{\lambda}{(1+q)+}\frac{\lambda q}{(1+q^2)+}\frac{\lambda q^2}{(1+q^3)+}\frac{\lambda q^3}{(1+q^4)+...} \tag{2.14}$$

Taking b = -1 in (2.13), we get

$$\frac{\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2}}{{(q;q)^2}_n}}{\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2-n}}{{(q;q)^2}_n}} = \frac{\sum_{n=0}^{\infty}\frac{(-1)^n(\lambda;q)_n q^{n(n+1)/2}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\lambda}{q};q)_n q^{n(n+1)/2}}{(q;q)_n}}$$

$$= \frac{1}{1+}\frac{\lambda}{(1-q)+}\frac{\lambda q}{(1-q^2)+}\frac{\lambda q^2}{(1-q^3)+}\frac{\lambda q^3}{(1-q^4)+...} \tag{2.15}$$

Taking λ= q in (2.13), we get

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)^2{}_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)^2{}_n}} = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}$$

$$= \frac{1}{1+}\frac{q}{(1-q)+}\frac{q^2}{(1-q^2)+}\frac{q^3}{(1-q^3)+}\frac{q^4}{(1-q^4)+...} \tag{2.16}$$

(iii) Following are two identities due to [Slater, L. J., 3; (34)(36), p.155]

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \frac{1}{(q^3,q^4q^5;q^8)_\infty} \tag{2.17}$$

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \frac{1}{(q,q^4,q^7;q^8)_\infty} \tag{2.18}$$

Taking the ratio of the left sides (2.17) and (2.18), we have

$$\frac{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}} = \frac{1}{1+\frac{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2(1-q^{2n})}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}}$$

$$= \frac{1}{1+\frac{q(1+q)}{\frac{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}}} = \frac{1}{1+\frac{q(1+q)}{1+\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^2;q^2)_n}[(-q;q^2)_n-(-q^3;q^2)]}{\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}}}$$

$$= \cfrac{1}{1+\cfrac{q(1+q)}{1+\cfrac{q^4}{\cfrac{\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+4n}}{(q^2;q^2)_n}}}}}$$

Iterating this process, we have

$$\frac{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}} = \frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}$$

$$= \frac{1}{1+}\frac{q(1+q)}{1+}\frac{q^4}{1+}\frac{q^3(1+q^3)}{1+}\frac{q^8}{1+...} \tag{2.19}$$

which is a known result [Andrews G. E. and Berndt B. C., 1; (6.2.38) p.154]
(iv) Let us consider the following summation formula:

$$\phi_1\left[\begin{array}{c} a,b;q;-q/b \\ aq/b \end{array}\right] = \frac{(-q;q)_\infty(\frac{aq^2}{b^2};aq;q^2)_\infty}{(\frac{aq}{b};\frac{-q}{b};q)_\infty}, \quad \text{where} \quad |\frac{q}{b}|<1 \tag{2.20}$$

[Gasper G. and Raheman M., 2; App.II (II.9), p.236]

Taking b$\rightarrow\infty$ in (2.20), we obtain

$$\sum_{n=0}^{\infty}\frac{(a;q)_n q^{n(n+1)/2}}{(q;q)_n} = (-q;q)_\infty(aq;q^2)_\infty \qquad ((2.21)$$

Taking q^2 for q and then putting a = -q and a = $-\frac{1}{q}$ in (2.21), we get

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n+1)/2}}{(q^2;q^2)_n} = \frac{1}{(q^2,q^3,q^7;q^8)_\infty} \tag{2.22}$$

$$\sum_{n=0}^{\infty}\frac{(-\frac{1}{q};q^2)_n q^{n(n+1)/2}}{(q^2;q^2)_n} = \frac{1}{(q,q^5,q^6;q^8)_\infty} \tag{2.23}$$

Taking the ratio of (2.22) and (2.23) and proceeding as in the previous cases, we easily get

$$\frac{\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty} \frac{(-\frac{1}{q};q^2)_n q^{n^2+n}}{(q^2;q^2)_n}} = \frac{(q,q^5,q^6;q^8)_\infty}{(q^2,q^3,q^7;q^8)_\infty}$$

$$= \frac{1}{1+}\frac{q}{1+}\frac{q^2(1+q)}{1+}\frac{q^4}{1+}\frac{q^4(1+q^3)}{1+...} \tag{2.24}$$

References

[1] Andrews G. E. and Berndt B. C., Ramanujan's Lost Notebook, Part I, Springer-New York, 2005.

[2] Gasper G. and Raheman M., Basic Hypergeometric Series, Cambridge University Press, 1991.

[3] Mehta Trishla and Yadav Vijay, On Hypergeometric Series and Continued Fractions, South East Asian J. of Mathematics and Mathematical Sciences, Vol. 11, No. 1 (2015), 79-88.

[4] Pant G. S. and Pande V. P., On Continued Fractions and Lambert Series, J. of Ramanujan Society of Math. and Math. Sc., Vol.6, No.1 (2017), 79-90.

[5] Pant G. S. and Pande V. P., Partition Generating Functions and Continued Fractions, J. of Ramanujan Society of Mathematics and Mathematical Sciences, Vol. 7, No. 1 (2019), 101-106

[6] Pant G. S. and Pande V. P. and Pathak Manoj Kumar, Basic Bilateral Hypergeometric Function ${}_2\Psi_2$ and Continued Fractions, South East Asian J. of Mathematics and Mathematical Sciences, Vol. 19, No. 1 (2023), 81-90.

[7] Singh Amit Kumar,Yadav Vijay, A note on Partition and Continued Fractions, J. of Ramanujan Society of Math. and Math. Sc., Vol. 1, No.1 (2012), 85-90.

[8] Slater, L. J., Further identities of Roger-Ramanujan Type, Proc. London Mathematical Society, 54 (2) (1952).