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A NOTE ON A NEW AND GENERAL SUMMATION FORMULA FOR THE GENERALIZED HYPERGEOMETRIC SERIES ${}_4F_3(1)$

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Abstract: The main purpose of this note is to demonstrate how one can obtain a new and general summation formula for the generalized hypergeometric series ${}_4F_3(1)$. This is achieved by employing two generalizations of the Gauss's second summation theorems obtained earlier by Rakha and Rathie in an identity due to Bailey. Several results obtained earlier by Ali et al. follow special cases of our main findings.

Keywords and Phrases: Generalized hypergeometric function, Gauss summation formula, Generalization, Bailey identity.

2020 Mathematics Subject Classification: 33C20.

1. Introduction

The generalized hypergeometric function with p numerator and q denominator parameters is defined by [3, 5]

$$ {}_pF_q\left[\begin{array}{ccc} \alpha_1, & \ldots, & \alpha_p \\ & & ;\ z \\ \beta_1, & \ldots, & \beta_q \end{array}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!} \tag{1.1}$$

where, as usual, (α) is the well-known Pochhammer symbol (or the shifted or the raised factorial, since $(1)_n = n!$) defined for any complex number $\alpha(\neq 0)$ by

$$(\alpha)_n = \begin{cases} 1, & (n = 0) \\ \alpha(\alpha+1)\ldots(\alpha+n-1), & (n \in \mathbb{N}) \end{cases} \tag{1.2}$$

In terms of gamma function $(\alpha)_n$ is defined by

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, (\alpha \neq 0) \tag{1.3}$$

For convergence conditions of ${}_pF_q$ (including absolute convergence) and its properties, we refer [3, 5].
For $p = 2$ and $q = 1$, we have the following well-known Gauss's hypergeometric function defined by

$$ {}_2F_1\left[\begin{array}{cc} a, b & \\ & ;\ z \\ c & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n}{(c)_n} \frac{z^n}{n!} \tag{1.4}$$

In the theory of hypergeometric series, classical summation theorems such as those of Gauss, second Gauss, Kummer and Bailey play a key role. Applications of these classical summation theorems are well-known. For this, we refer an interesting research paper by Bailey [2].
In our present investigation, we are interested in the following Gauss summation theorem [4] viz.

$$ {}_2F_1\left[\begin{array}{c} a, b \\ \frac{1}{2}(1+a+b) \end{array};\frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} = \Delta_0 \tag{1.5}$$

In 1996, Lavoie, Grondin and Rathie [3] generalized Gauss second summation theorem (1.5) and obtained the explicit expressions of

$$ {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+i+1) \end{matrix} ; \frac{1}{2}\right] $$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.
For $i = 0$, we recover Gauss second summation theorem(1.5).
In 2011, Rakha and Rathie [6] generalized Gauss's second summation theorem (1.5) and discovered the following two very general summation formulas viz,

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix} \alpha, \beta \\ \frac{1}{2}(\alpha+\beta+i+1) \end{matrix} ; \frac{1}{2}\right] \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}+\frac{i}{2}\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{i}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{i}{2}+\frac{1}{2}\right)} \times \\ &\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}i+\frac{i}{2}r+\frac{1}{2}\right)} \\ &= \Delta_i \end{aligned} \tag{1.6} $$

and

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix} \alpha, \beta \\ \frac{1}{2}(\alpha+\beta-i+1) \end{matrix} ; \frac{1}{2}\right] \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{i}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)} \times \sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}i+\frac{i}{2}r+\frac{1}{2}\right)} \\ &= \Delta_{-i} \end{aligned} \tag{1.7} $$

each for $i = 0, 1, 2, ...$

Remark

For $i = 0$, (1.6) or (1.7) reduces to (1.5). Further for $i = 0, 1, 2, 3, 4, 5$, the results (1.6) and (1.7) are recorded in Lavoie et al. [4].
In addition to this, we shall require the following identity due to Bailey [2] viz.

$$ \begin{aligned} &{}_4F_3\left[\begin{matrix} \alpha, \beta, \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\beta, \gamma, 1+\alpha+\beta-\gamma \end{matrix} ; 4z(1-z)\right] \\ &= {}_2F_1\left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} ; z\right] \times {}_2F_1\left[\begin{matrix} \alpha, \beta \\ 1+\alpha-\beta-\gamma \end{matrix} ; z\right] \end{aligned} \tag{1.8} $$

The main purpose of this note is to demonstrate how one can obtain a new and general summation formula for the generalized hypergeometric series ${}_4F_3(1)$. This is achieved by employing two generalizations (1.6) and (1.7) of the Gauss's second summation theorem obtained earlier by Rakha and Rathie [6] in an identity (1.8) due to Bailey [3]. Several results obtained earlier by Ali et al. [1] follow special cases of our main findings.

2. Main Summation Formula

In this section, we shall establish the following new and general summation formula for the series ${}_4F_3$ asserted in the following theorem.

Theorem 2.1. *For $i = 0, 1, 2, \ldots$, the following general summation formula holds true.*

$${}_4F_3\left[\begin{matrix}\alpha, \beta, \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\beta, \frac{1}{2}(\alpha+\beta+i+1), \frac{1}{2}(1+\alpha+\beta-i+1)\end{matrix}; 1\right] = \Delta_i \Delta_{-i} \tag{2.1}$$

where Δ_i and Δ_{-i} are the same as given in (1.6) and (1.7) respectively.

Proof. The derivation of the result (2.1) asserted in the theorem is quite straight forward. Thus in order to derive the result (2.1), we proceed as follows:
In the Bailey's identity (1.8), if we set $z = \frac{1}{2}$ and $\gamma = \frac{1}{2}(\alpha+\beta+i+1)$, then for $i = 0, 1, 2, \ldots$, it takes the following form.

$$\begin{aligned} &{}_4F_3\left[\begin{matrix}\alpha, \beta, \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\beta, \frac{1}{2}(\alpha+\beta+i+1), \frac{1}{2}(\alpha+\beta-i+1)\end{matrix}; 1\right] \\ &= {}_2F_1\left[\begin{matrix}\alpha, \beta \\ \frac{1}{2}(\alpha+\beta+1+i)\end{matrix}; \frac{1}{2}\right] \times {}_2F_1\left[\begin{matrix}\alpha, \beta \\ \frac{1}{2}(\alpha+\beta+1-i)\end{matrix}; \frac{1}{2}\right] \end{aligned} \tag{2.2}$$

We now observe that the first and the second ${}_2F_1$ appearing on the right hand side of (2.2) can respectively be evaluated with the help of the known results (1.6) and (1.7) due to Rakha and Rathie [6] and we at once get the desired result (2.1). This completes the proof of our main result (2.1) asserted in the theorem.

3. Special cases

In this section, we shall mention several known results obtained very recently by Ali et al [1].
In (2.1) if we set $i = 0, 1, 2, 3, 4, 5$, we respectively get

$${}_3F_2\left[\begin{matrix}\alpha, \beta, \frac{1}{2}(\alpha+\beta), \\ \frac{1}{2}(\alpha+\beta+1), \alpha+\beta\end{matrix}; 1\right] = \Delta_0^2 \tag{3.1}$$

where Δ_0 is given by (1.5) or can be obtained from (1.6) or (1.7) by putting $i = 0$.

$$ {}_3F_2\left[\begin{matrix} \alpha, \beta, \frac{1}{2}(\alpha+\beta+1), \\ \frac{1}{2}(\alpha+\beta+2), \alpha+\beta \end{matrix}; 1\right] = \Delta_1\Delta_{-1} \tag{3.2} $$

where Δ_1 and Δ_{-1} can be obtained respectively from (1.6) or (1.7) by putting $i = 1$.

$$ {}_4F_3\left[\begin{matrix} \alpha, \beta, \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\beta, \frac{1}{2}(\alpha+\beta+3), \frac{1}{2}(\alpha+\beta-1) \end{matrix}; 1\right] = \Delta_2\Delta_{-2} \tag{3.3} $$

where Δ_2 and Δ_{-2} can be obtained respectively from (1.6) or (1.7) by putting $i = 2$.

$$ {}_4F_3\left[\begin{matrix} \alpha, \beta, \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\beta, \frac{1}{2}(\alpha+\beta+4), \frac{1}{2}(\alpha+\beta-2) \end{matrix}; 1\right] = \Delta_3\Delta_{-3} \tag{3.4} $$

where Δ_3 and Δ_{-3} can be obtained respectively from (1.6) or (1.7) by putting $i = 3$.

$$ {}_4F_3\left[\begin{matrix} \alpha, \beta, \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\beta, \frac{1}{2}(\alpha+\beta+5), \frac{1}{2}(\alpha+\beta-3) \end{matrix}; 1\right] = \Delta_4\Delta_{-4} \tag{3.5} $$

where Δ_4 and Δ_{-4} can be obtained respectively from (1.6) or (1.7) by putting $i = 4$.

$$ {}_4F_3\left[\begin{matrix} \alpha, \beta, \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\beta, \frac{1}{2}(\alpha+\beta+6), \frac{1}{2}(\alpha+\beta-4) \end{matrix}; 1\right] = \Delta_5\Delta_{-5} \tag{3.6} $$

where Δ_5 and Δ_{-5} can be obtained respectively from (1.6) or (1.7) by putting $i = 5$.

The results (3.1) to (3.6) have recently been obtained by Ali et al. [1].

Similarly other results can be obtained.

4. Concluding remark

In this note, we have demonstrated how one can obtain a new and general summation formula for the generalized hypergeometric series ${}_4F_3$ with unit argument. This is achieved by employing the generalizations of the Gauss's second summation theorem obtained earlier by Rakha and Rathie in an identity due to Bailey. Several results obtained earlier by Ali et al [1] follow special cases of our main findings. The results given in this paper are simple interesting, easily established and may be potential useful in applied mathematics, mathematical physics and engineering mathematics.

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