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A NOTE ON TWO INTEGRALS INVOLVING PRODUCT OF TWO GENERALIZED HYPERGEOMETRIC FUNCTIONS

Madhav Prasad Poudel, Sunil D. Singh* and Arjun K. Rathie**

School of Engineering,
Pokhara University, Kaski, Pokhara, NEPAL

E-mail : pdmadav@gmail.com

*Department of Mathematics
The Institute of Science, Dr. Homi Bhabha University,
Madan Cama Road Mumbai, INDIA

E-mail : sunilsingh@iscm.ac.in

**Department of Mathematics
Vedant College of Engineering & Technology
(Rajasthan Technical University)
Tulsi, Jakhamund, Bundi, Rajasthan, INDIA

E-mail : arjunkumarrathie@gmail.com

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Abstract: In this note, two interesting integrals involving the product of two generalized hypergeometric functions have been evaluated in terms of gamma function. The results are derived with the help of a known integral involving hypergeometric function recorded in the paper of Lavoie and Trottlier. We also give several interesting special cases of our main findings.

Keywords and Phrases: Hypergeometric function, Generalized hypergeometric function, Watson summation theorem.

2020 Mathematics Subject Classification: 33C20.

1. Introduction and Results Required

To proceed with our work, let us start with the quotation of Sylvester [11]: "*It seems to be expected of every pilgrim up the slope of the mathematical Parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock.*"

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters defined as follows [1, 2, 7, 8];

$$ {}_pF_q\left[\begin{array}{ccccc} \alpha_1, & \ldots, & \alpha_p & & \\ & & & ; & x \\ \beta_1, & \ldots, & \beta_q & & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{x^n}{n!} \tag{1.1} $$

where $(\alpha)_n$ is the well-known Pochhammer symbol defined (for $\alpha \in \mathbb{C}$) by

$$ \begin{aligned} (\alpha)_n :&= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha} \qquad (\alpha \in \mathbb{C}\backslash\mathbb{Z}_0^-) \\ &= \begin{cases} \alpha(\alpha+1)...(\alpha+n-1) & ; n \in \mathbb{N} \\ 1 & ; n = 0 \end{cases} \end{aligned} \tag{1.2} $$

where $\Gamma(\alpha)$ is the familiar Gamma function. Here an empty product is to be interpreted as unity and we assume that the variable x, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ takes complex values, provided that no zeros appear in denominator of (1.1), that is,

$$ (\beta_j \in \ \mathbb{C}\backslash\mathbb{Z}_0^-; j = 1, \ldots, q) $$

Here we let $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, integers and positive integers respectively and moreover let

$$ \mathbb{N}_0 := \mathbb{N} \cup 0 \ \text{ and } \ \mathbb{Z}_0^- := \mathbb{Z}\backslash\mathbb{N} $$

See [1, 2, 7, 8] for more information on ${}_pF_q$ including its convergence, absolute convergence, various special and limiting cases.

It is interesting to mention here that whenever the hypergeometric function ${}_pF_q$ and its special case ${}_2F_1$ with some special arguments 1 or $\frac{1}{2}$ or -1 can be summed to be expressed in terms of gamma function, the results can be important from the application point of view. Thus the classical summation theorems such as those of Gauss, Gauss's second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz's for the series ${}_3F_2$ and others play an important role. Applications of the above mentioned classical summation theorems are well-known

now.
For our purpose, we would like to mention the following classical Watson's summation theorem [2]:

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{a+b+1}{2}, & 2c & \end{matrix}\; ;1\right]
= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)} \tag{1.3}
$$

provided $Re(2c-a-b) > -1$.
From (1.3), It is not difficult to evaluate the following two integrals involving hypergeometric functions which are due to Rakha et al [9]

$$
\int_0^1 x^{c-1}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-1}\left(1-\frac{x}{4}\right)^{c-1}
\times {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix}\; ;\; \frac{9}{4}x\left(1-\frac{x}{3}\right)^2\right]dx
= \left(\frac{2}{3}\right)^{2c}\Omega \tag{1.4}
$$

provided that $Re(c) > 0$ and $Re(2c-a-b) > -1$.

$$
\int_0^1 x^{c-1}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-1}\left(1-\frac{x}{4}\right)^{c-1}
\times {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix}\; ;\; \left(1-\frac{x}{4}\right)(1-x)^2\right]dx
= \left(\frac{2}{3}\right)^{2c}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma^2(c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)}
= \left(\frac{2}{3}\right)^{2c}\Omega \tag{1.5}
$$

provided that $Re(c) > 0$ and $Re(2c-a-b) > -1$.The value of Ω is the same as given in (1.4)

These results have been obtained by the following well-known results of Lavoie and Trottier [5].

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}dx=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)(\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.6}$$

provided that $Re(\alpha)>0$ and $Re(\beta)>0$

In this paper, we will evaluate two interesting integrals involving product of two generalized hypergeometric function in terms of gamma function. The integrals are evaluated with the help of known integrals (1.4) and (1.5). Several very interesting special cases have also been given from our main finding.

2. Main Integral Formula

In this section, we evaluate two integrals involving product of two generalized hypergeometric functions asserted in the following theorem.

Theorem 2.1. *For $Re(c)>0$ and $Re(2c-a-b)>-1$, the following results hold.*

$$\begin{aligned}&\int_0^1 x^{c-1}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-1}\left(1-\frac{x}{4}\right)^{c-1}\\&\quad\times{}_2F_1\left[\begin{matrix}a,b\\ \frac{1}{2}(a+b+1)\end{matrix};\ \frac{9}{4}x\left(1-\frac{x}{3}\right)^2\right]\\&\quad\times\ {}_2F_2\left[\begin{matrix}c-\frac{1}{2}a+\frac{1}{2}, & c-\frac{1}{2}b+\frac{1}{2}\\ c, & c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\end{matrix};\ 9x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx\\&\quad=\left(\frac{2}{3}\right)^{2c}\frac{e\sqrt{\pi}\Gamma^2(c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\\&\quad=e\left(\frac{2}{3}\right)^{2c}\Omega=\Omega_1\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&\int_0^1 x^{c-1}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-1}\left(1-\frac{x}{4}\right)^{c-1}\\&\quad\times{}_2F_1\left[\begin{matrix}a,b\\ \frac{1}{2}(a+b+1)\end{matrix};\ \left(1-\frac{x}{4}\right)(1-x)^2\right]\end{aligned}$$

$$\times\ {}_2F_2\left[\begin{matrix} c-\frac{a}{2}+\frac{1}{2}, & c-\frac{b}{2}+\frac{1}{2} \\ c, & c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\end{matrix}\ ;\ 9x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$= \Omega_1 \tag{2.2}$$

The value of Ω_1 *is same as given in (2.1).*

Proof. In order to evaluate the integral (2.1), we proceed as follows;
Denoting the left hand side of (2.1) by I, we have

$$I = \int_0^1 x^{c-1}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-1}\left(1-\frac{x}{4}\right)^{c-1}$$

$$\times\ {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1)\end{matrix}\ ;\ \frac{9}{4}x\left(1-\frac{x}{3}\right)^2\right]$$

$$\times\ {}_2F_2\left[\begin{matrix} c-\frac{1}{2}a+\frac{1}{2}, & c-\frac{1}{2}b+\frac{1}{2} \\ c, & c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\end{matrix}\ ;\ 9x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx$$

Express ${}_2F_2$ as a series, interchanging the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series involved in the process, we have

$$I = \sum_{n=0}^{\infty}\frac{\left(c-\frac{1}{2}a+\frac{1}{2}\right)_n\left(c-\frac{1}{2}b+\frac{1}{2}\right)_n}{(c)_n\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)_n n!}\left(\frac{3}{2}\right)^{2n}$$
$$\times\int_0^1 x^{c+n-1}(1-x)^{2c+2n-1}\left(1-\frac{x}{3}\right)^{2c+2n-1}\left(1-\frac{x}{4}\right)^{c+n-1}$$
$$\times\ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+1) & \end{matrix}\ ;\ \frac{9}{4}x\left(1-\frac{x}{3}\right)^2\right]dx$$

Evaluating the integral with the help of the result (1.4) and simplifying it, we have

$$I = \left(\frac{2}{3}\right)^{2c}\frac{\Gamma(\frac{1}{2})\Gamma^2(c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\sum_{n=0}^{\infty}\frac{1}{n!}$$

Finally observing that $\sum\limits_{n=0}^{\infty} \frac{1}{n!} = e$, we easily arrive in a right hand side of the result (2.1). This completes the proof of (2.1) asserted in the theorem.
In the same manner, we can establish the results (2.2) with the help of known results (1.5).

3. Special cases

In this section, we mention a few very interesting special cases of the main integral (2.1) and (2.2)in the following forms.

(a) In (2.1) if we let $b = -2n$ and replace a by $a + 2n$, for $n \in \mathbb{N}_0$, then we get the following result:

$$\int_0^1 x^{c-1}(1-x)^{2c-1} \left(1-\frac{x}{3}\right)^{2c-1} \left(1-\frac{x}{4}\right)^{c-1}$$

$$\times\ {}_2F_1\left[\begin{matrix} -2n, & a+2n \\ & \\ \frac{1}{2}(a+1) & \end{matrix} ;\ \frac{9}{4}x\left(1-\frac{x}{3}\right)^2\right]$$
$$\times\ {}_2F_2\left[\begin{matrix} c+n+\frac{1}{2}, & c-\frac{1}{2}a+\frac{1}{2}-n \\ & \\ c, & c-\frac{1}{2}a+\frac{1}{2} \end{matrix} ;\ 9x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right] dx$$
$$= \left(\frac{2}{3}\right)^{2c} \frac{e\Gamma^2(c)}{\Gamma(2c)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}+\frac{1}{2}a-c\right)_n}{\left(c+\frac{1}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}\right)_n}$$
$$= \Omega_2 \tag{3.1}$$

(b) In (2.1), If we let $b = -2n-1$ and replace a by $a+2n+1$, where $n \in \mathbb{N}_0$, then we get the following interesting result.

$$\int_0^1 x^{c-1}(1-x)^{2c-1} \left(1-\frac{x}{3}\right)^{2c-1} \left(1-\frac{x}{4}\right)^{c-1}$$

$$\times\ {}_2F_1\left[\begin{matrix} -2n-1, & a+2n+1 \\ & \\ \frac{1}{2}(a+1) & \end{matrix} ;\ \frac{9}{4}x\left(1-\frac{x}{3}\right)^2\right]$$

$$\times \ {}_2F_2\left[\begin{array}{c} c+n+1, \quad c-\frac{1}{2}a+\frac{1}{2}-n \\ \\ c, \qquad\qquad c-\frac{1}{2}a+\frac{1}{2} \end{array} ; \ 9x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx$$
$$= 0 \tag{3.2}$$

(c) In (2.1), if we take $a = b = \frac{1}{2}$ and making use of the known result [7, p.473, Equ.(75)]

$${}_2F_1\left[\begin{array}{c} \frac{1}{2}, \quad \frac{1}{2} \\ \\ 1 \end{array} ; \ x\right] = \frac{2}{\pi}K(\sqrt{x}) \tag{3.3}$$

where $K(k)$ is the complete Elliptic function of the first kind defined by.

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2sin^2t}} \tag{3.4}$$

then we get the following interesting result:

$$\int_0^1 x^{c-1}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-1}\left(1-\frac{x}{4}\right)^{c-1} K\left(\frac{3}{2}\sqrt{x}\left(1-\frac{x}{3}\right)\right)$$
$$\times \ {}_2F_2\left[\begin{array}{c} c+\frac{1}{4}, \quad c+\frac{1}{4} \\ \\ c, \qquad c \end{array} ; \ 9x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx$$
$$= \left(\frac{2}{3}\right)^{2c}\frac{e\pi^{\frac{3}{2}}}{2}\frac{\Gamma^3(c)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma(2c)\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(c+\frac{1}{4}\right)}$$
$$= \Omega_3 \tag{3.5}$$

provided $Re(c) > 0$.

(d) In (2.1), if we take $a = b = 1$ and making use of the known result [7, p.476, Equ.(147)]

$${}_2F_1\left[\begin{array}{c} 1, \quad 1 \\ \\ \frac{3}{2} \end{array} ; \ x\right] = \frac{sin^{-1}(\sqrt{x})}{\sqrt{x(1-x)}} \tag{3.6}$$

then, we get the following interesting result:

$$\int_0^1 x^{c-\frac{3}{2}}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-2}\left(1-\frac{x}{4}\right)^{c-1} sin^{-1}\left(\frac{3}{2}\sqrt{x}(1-x)\right)$$
$$\times_1 F_1\left[\begin{matrix} c \\ \\ c-\frac{1}{2}\end{matrix} ;\ 9x(1-x)^2\left(1-\tfrac{x}{3}\right)^2\left(1-\tfrac{x}{4}\right)\right]dx$$
$$=\left(\frac{2}{3}\right)^{2c}\frac{e\pi}{2}\frac{\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma(2c)}$$
$$=\Omega_4 \tag{3.7}$$

provided $Re(c) > \frac{1}{2}$.

(e) In (2.1), If we take $b=-a$ and using result [7, p. 459, Equ.(83)]

$${}_2F_1\left[\begin{matrix} a, & -a \\ & \\ \frac{1}{2} & \end{matrix} ;\ x\right] = cos(2asin^{-1}(\sqrt{x}), \tag{3.8}$$

then we get the following interesting result.

$$\int_0^1 x^{c-1}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-1}\left(1-\frac{x}{4}\right)^{c-1} cos(2asin^{-1}\left(\frac{3}{2}\sqrt{x}(1-x)\right)$$
$${}_2F_2\left[\begin{matrix} c-\frac{1}{2}a+\frac{1}{2}, & c+\frac{1}{2}a+\frac{1}{2} \\ & \\ c, & c+\frac{1}{2}\end{matrix} ;\ 9x(1-x)^2\left(1-\tfrac{x}{3}\right)^2\left(1-\tfrac{x}{4}\right)\right]dx$$
$$=\left(\frac{2}{3}\right)^{2c}\frac{\pi e\Gamma^2(c)\Gamma^2\left(c+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{1}{2}-\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(c+\frac{1}{2}-\frac{1}{2}a\right)\Gamma\left(c+\frac{1}{2}+\frac{1}{2}a\right)}$$
$$=\Omega_5 \tag{3.9}$$

(f) In (2.2), If we let $b=-2n$ and replace a by $a+2n$, where $n\in\mathbb{N}_0$, then we get the following result.

$$\int_0^1 x^{c-1}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-1}\left(1-\frac{x}{4}\right)^{c-1}$$
$$\times\ {}_2F_1\left[\begin{matrix} -2n, & a+2n \\ & \\ & \frac{1}{2}(a+1)\end{matrix} ;\ (1-x)^2\left(1-\tfrac{x}{4}\right)\right]$$

$$\times\ {}_2F_2\left[\begin{array}{c} c+n+\frac{1}{2},\quad c-\frac{a}{2}+\frac{1}{2}-n \\ \\ c,\qquad c-\frac{1}{2}a+\frac{1}{2} \end{array};\ 9x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx$$
$$=\Omega_2 \tag{3.10}$$

where Ω_2 is same as defined in (3.1).

(g) In (2.2), If we let $b=-2n-1$ and replace a by $a+2n+1$, where $n\in\mathbb{N}_0$,then we get the following interesting result:

$$\int_0^1 x^{c-1}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-1}\left(1-\frac{x}{4}\right)^{c-1}$$
$$\times\ {}_2F_1\left[\begin{array}{c} -2n-1,\quad a+2n+1 \\ \\ \frac{1}{2}(a+1) \end{array};\ (1-x)^2\left(1-\frac{x}{4}\right)\right]$$
$$\times\ {}_2F_2\left[\begin{array}{c} c+n+1,\quad c-\frac{1}{2}a+\frac{1}{2}-n \\ \\ c,\qquad c-\frac{1}{2}a+\frac{1}{2} \end{array};\ 9x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx$$
$$=0 \tag{3.11}$$

where Ω_2 is same as defined in (3.1)

(h) In (2.2), If we let $a=b=\frac{1}{2}$ and making use of the known result (3.3) then we get the following interesting result.

$$\int_0^1 x^{c-1}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-1}\left(1-\frac{x}{4}\right)^{c-1}K\left((1-x)\sqrt{1-\frac{x}{4}}\right)$$
$$\times\ {}_2F_2\left[\begin{array}{c} c+\frac{1}{4},\quad c+\frac{1}{4} \\ \\ c,\qquad c \end{array};\ 9x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx$$
$$=\Omega_3 \tag{3.12}$$

where Ω_3 is same as obtained in (3.5).

(i) In (2.2), If we let $a=b=1$ and making use of the known result (3.6), then we get the following interesting result.

$$\int_0^1 x^{c-\frac{3}{2}}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-2}\left(1-\frac{x}{4}\right)^{c-1} sin^{-1}\left((1-x)\sqrt{1-\frac{x}{4}}\right)$$
$$\times\ {}_1F_1\left[\begin{array}{c} c \\ \\ c-\frac{1}{2}\end{array};\ 9x(1-x)^2\left(1-\tfrac{x}{3}\right)^2\left(1-\tfrac{x}{4}\right)\right]dx$$
$$=\Omega_4 \qquad (3.13)$$

where Ω_4 is same as obtained in (3.7).

(j) In (2.2) if we take $b=-a$, and making use of the known result (3.8), then we get the following interesting result:

$$\int_0^1 x^{c-1}(1-x)^{2c-1}\left(1-\frac{x}{3}\right)^{2c-1}\left(1-\frac{x}{4}\right)^{c-1} cos\left(2asin^{-1}(1-x)\sqrt{1-\frac{x}{4}}\right)$$
$$\times\ {}_2F_2\left[\begin{array}{cc} c-\frac{a}{2}+\frac{1}{2}, & c+\frac{a}{2}+\frac{1}{2} \\ \\ c, & c+\frac{1}{2}\end{array};\ 9x(1-x)^2\left(1-\tfrac{x}{3}\right)^2\left(1-\tfrac{x}{4}\right)\right]dx$$
$$=\Omega_5 \qquad (3.14)$$

where Ω_5 is same as obtained in (3.9)

Similarly other results can be obtained.

Remarks

For integrals (single and double) of this type, see [3, 4, 6].

4. Concluding Remarks

In this paper, we have evaluated two interesting integrals involving the product of two generalized hypergeometric functions in terms of Gamma function. It should be remarked here that whenever an integral is evaluated in terms of gamma function, the result may be important in application point of view.Thus our results established in this paper may be potentially useful.

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