J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 11, No. 2 (2024), pp. 103-116

DOI: 10.56827/JRSMMS.2024.1102.7 **ISSN (Online): 2582-5461**

ISSN (Print): 2319-1023

UNVEILING THE POWER OF CAUCHY'S RESIDUE THEOREM FOR EVALUATING THE INTEGRATION OF DIFFERENT TYPES OF COMPLEX FUNCTIONS

Ganesh Prasad Adhikari

Central Department of Education,
Tribhuvan University,
Kirtipur, Kathmandu, NEPAL

E-mail : gpadhikarin@gmail.com

(**Received:** *Jan. 28, 2024* **Accepted:** *Jun. 08, 2024* **Published:** *Jun. 30, 2024)*

Abstract: Cauchy's residue theorem gives a relatively general form for a complex integral along a simple closed contour. With the help of Cauchy's residue theorem, an appropriate closed contour can be chosen to calculate some abnormal definite integrals that might be very complicated and difficult to solve by conventional methods. This study focuses on four distinct types of definite integrals: integrals involving sine and cosine functions, polynomial functions, exponential functions, and logarithmic functions. The contours chosen are a sector of a circle that involves one or several isolated singularities of the function. The residue at the isolated singularities of the function is then calculated. The value of the residues is substituted in the formula deducted from Cauchy's residue theorem. The integral along the simple closed contour can be expressed in two parts, one along the real axis and the other along the circle. This study demonstrates that Cauchy's Residue Theorem is superior to conventional real analysis methods for evaluating the integrals of different types of complex functions.

Keywords and Phrases: Residue, definite integral, improper integral, proper integral.

2020 Mathematics Subject Classification: 97I80.

1. Introduction

Over the years, numerous mathematicians have contributed to the continuous development of the residue theorem. The residue theorem was developed by several notable mathematicians, including Weierstrass, Cauchy, Laurent, Riemann, and Mittag-Leffler. Cauchy first presented it in the early 19^{th} century as a component of his work on complex analysis. However, German mathematician Bernhard Riemann did not fully develop and properly prove the residue theorem until the late 19th century. Riemann's proof of the theorem was based on the concept of Laurent series expansions of complex functions, which enable the expression of a function as the sum of positive and negative powers of z [4]

In complex analysis, the residue theorem is an effective tool that makes it possible to assess specific contour integrals. The residue theorem has easy-to-understand the integral of complex functions. The value of the integral of $f(z)$ around **C** is equal to $2\pi i$ times the sum of the residues of $f(z)$ at each of the singular points inside **C** if $f(z)$ is a function that is analytic everywhere inside and on a closed contour**C** with the exception of isolated singularities at $z_1, z_2, z_3, \cdots, z_k$ [8].

The Residue Theorem simplifies evaluating integrals by extending them to the complex plane and has many applications in mathematics, physics, and engineering [1]. Real integrals can be transformed into contour integrals on the real-axis in the complex plane, using a one-to-one correspondence between real-valued and complex-valued functions. In this study a closed contour is created by defining a contour on the real axis and a half-circle in the upper or halfplane [3]. In view of the background information mentioned above, no study has been conducted using a specific, simple method to find the integral of complex functions. Moreover, mathematicians are still actively researching the retention number theorem, which is a basic discovery in the field with numerous significant applications and extensions of the residue theorem for complex function integrals. Therefore, this study focuses on how to evaluating proper integrals involving sine and cosines functions, rational function of the form $\int\limits_{-\infty}^{\infty} f(z)dz$, rational function of the form $\int\limits_{-\infty}^{\infty} f(x)sinmxdx$ or $\int\limits_{-\infty}^{\infty} f(x)cosmxdx$ and improper integral involving logarithm function using the Residue Theorems.

Theorem 1.1. (**Cauchy Residue Theorem**) *Let $f(z)$ be an analytic inside and on a simple closed contour $\boldsymbol{C}$ expect at finite number of singularities z_1, z_2, z_3, $\cdots$, z_n inside $\boldsymbol{C}$ at which the residues are β_1, β_2, β_3, $\cdots$,β_n respectively of $f(z)$, then*

$\int\limits_C f(z)dz = 2\pi i \sum\limits_{k=1}^{n} \beta_k$ [2,3]

Proof. Letus suppose that $c_1,\ c_2,\ c_3,\ \cdots,\ c_n$ be the small circles with center at $z_1,\ z_2,\ z_3,\ \cdots,\ z_n$ respectively and radii so small such that they lie entirely within positively oriented simple closed contour **C** and having no common parts, then by extension of Cauchy Gorsat's Theorem we have

$$\int\limits_C f(z)dz - \sum_{k=1}^{n} \int\limits_{c_k} f(z)dz = 0$$

$$\Rightarrow \int\limits_C f(z)dz = \sum_{k=1}^{n} \int\limits_{c_k} f(z)dz \tag{1}$$

Also, by the definition of residue, we have

$$\int\limits_{c_k} f(z)dz = 2\pi i \underset{z=z_k}{Res} f(z), \quad for \quad k = 1, 2, 3, \cdots, n. \tag{2}$$

Now, from equations (1) and (2), we have

$$\int\limits_C f(z)dz = \sum_{k=1}^{n} \int\limits_{c_k} f(z)dz = 2\pi i \underset{z=z_k}{Res} f(z)$$

If we suppose that $\underset{z=z_k}{Res} f(z) = \beta_k$ for $k = 1, 2, 3, \cdots, n.$, then we have

$\int\limits_C f(z)dz = 2\pi i \sum\limits_{k=1}^{n} \beta_k$ for $k = 1, 2, 3, \cdots, n.$

2. Applications of Cauchy's Residue Theorem

The process of evaluating integrals would be as simple as finding the residue(s), adding them up, and multiplying by $2\pi i$ If the Residue Theorem's criteria are satisfied [3]. Although the Residue Theorem can only be used to evaluate specific integrals of rational functions, it can be extended to a wider range of situations. The Residue Theorems acknowledged in this study will only be used to demonstrate the strength and applicability of the Residue Theorems in integral evaluation. Every Residue Theorem can be used to evaluate a particular kind of integral that is commonly encountered. The applications are introduced with a concrete example of their application, which is followed by a demonstration of the related theorems[2].

Example 2.1. Find the integral of $f(z) = \frac{1}{(z^2+1)^3}$ with $\mathbf{C} : |z| = 2$ [2].

Solution. Let $f(z) = \frac{1}{(z^2+1)^3}$, Then the singular points of $f(z)$ are $z = i$ and $z = -i$ where both singular points of order three lie inside the given contour $\mathbf{C} : |z| = 2$. Now, the residue of $f(z)$ at $z = i$ is
$\beta_1 = \lim\limits_{z \to i} \frac{1}{(3-1)!} \frac{d^2}{dz^2} \left[(z-i)^3 f(z)\right] = \lim\limits_{z \to i} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z-i)^3 \frac{1}{(z^2+1)^3}\right] = \lim\limits_{z \to i} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z+i)^{-3}\right] = \frac{3}{16i}$. Similarly, the residue of $f(z)$ at $z = -i$ is $\beta_2 = \frac{-3}{16i}$.
Therefore, $\int\limits_C \frac{1}{(z^2+1)^3} = 2\pi i \left(\beta_1 + \beta_2\right) = 0$

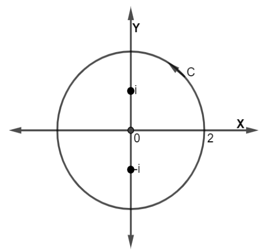

Figure 1

2.1. Proper Integrals Involving sine and cosines Functions

The following rules give an easy way to find the proper integrals of a function involving $sin\theta$ and $cos\theta$ functions $\left(\int\limits_0^{2\pi} F\left(sin\theta, cos\theta\right) d\theta\right)$[2, 3, 5].

1. Put $z = e^{i\theta}$, $(0 \le \theta \le 2\pi)$, then $d\theta = \frac{dz}{iz}$
2. Put $sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ in the given functions of integral.
3. Write the relation $\int\limits_0^{2\pi} F\left(sin\theta, cos\theta\right) d\theta = \int_C F\left(\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right) = \int\limits_C f(z)$, where C is the unit circle.
4. Calculate the residues of $f(z)$ inside the unit circle C by finding poles.
5. Finally,we use Cauchy Residue Theorem such that $\int_C f(z)dz = 2\pi i \sum\limits_{k=1}^{n} \underset{z=z_k}{Res} f(z)$.

Example 2.2. Find the integral of $\int\limits_0^{2\pi} \frac{d\theta}{3-2cos\theta+sin\theta}$ [5].

Solution. Since,

$$I = \int_0^{2\pi} \frac{d\theta}{3 - 2cos\theta + sin\theta} \tag{3}$$

Let $z = e^{i\theta}$, where θ varies from 0 to 2π, then z traces the unit circle with center origin in positive direction. Also, $d\theta = \frac{dz}{iz}$
Again, put $sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ in equation (3), then we get

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{3 - 2cos\theta + sin\theta} &= \int_C \frac{1}{3 - \left(z + \frac{1}{z}\right) + \frac{1}{2i\left(z - \frac{1}{z}\right)}} \frac{dz}{iz} \\
&= \int_C \frac{\frac{dz}{iz}}{\frac{6iz + 2iz^2 - 2i + z^2 - 1}{2iz}} = \int_C \frac{2dz}{z^2(1-2i) + 6iz - (1+2i)} \\
&= \int_C \frac{2(1+2i)\,dz}{z^2(1-2i)(1+2i) + 6iz(1+2i) - (1+2i)^2} \\
&= \int_C \frac{\frac{2}{5}(1+2i)\,dz}{z^2 + \frac{6}{5}i(1+2i)z - \frac{1}{5}(1+2i)^2} = \int_C f(z)dz
\end{aligned}$$

where, $f(z) = z^2 + \frac{6}{5}i(1+2i)z - \frac{1}{5}(1+2i)^2$.

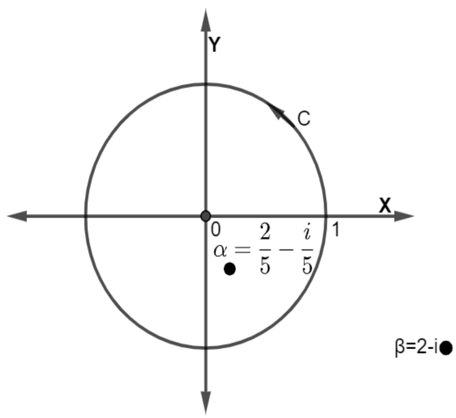

Figure 2

Now, the pole of $\phi(z)$ can be obtain by $z^2+\frac{6}{5}i\left(1+2i\right)z-\frac{1}{5}\left(1+2i\right)^2=0$. Thus, $z=\frac{\frac{-6i}{5}(1+2i)\pm\frac{4i}{5}(1+2i)}{2}$. Hence, the poles of $f(z)$ are Let $\alpha=\frac{2}{5}-\frac{i}{5}$ and $\beta=2-i$. Since, both poles are simple where α lies within contour C whereas β does not lie within C. $(see\ in\ figure\, 2)$
Here, the residue of $f(z)$ at $z=\alpha$ is

$$\underset{z=\alpha}{Res}\, f(z)=\lim_{z\to\alpha}\left(z-\alpha\right)f(z)=\lim_{z\to\alpha}\left(z-\alpha\right)\frac{\frac{2}{5}\left(1+2i\right)}{\left(z-\alpha\right)\left(z-\beta\right)}=\frac{\frac{2}{5}\left(1+2i\right)}{\left(\alpha-\beta\right)}$$
$$=\frac{\left(1+2i\right)}{2\left(-2+i\right)}$$

Now, by Cauchy Residue Theorem, we have

$$\int_0^{2\pi}\frac{d\theta}{3-2cos\theta+sin\theta}=\int_C f(z)dz=2\pi i\times Sum\ Residue\, of\ f(z)=2\pi i\frac{\left(1+2i\right)}{2\left(-2+i\right)}=\pi$$

Hence,

$$\int_0^{2\pi}\frac{d\theta}{3-2cos\theta+sin\theta}=\pi$$

2.2. Evaluation of Integral of the form $\int_{-\infty}^{\infty} f(x)dx$, where $f(z)$ is Rational Function Having no Poles in Real Axis

The following rules give an easy way to find the improper integrals of the form $\int_{-\infty}^{\infty} f(x)dx$ having no poles in real axis [2]:

1. First of all, we consider the integral $\int_C f(z)dz$, where C is called counter consisting upper half plane of semi circle Γ of large radius and real axis from $-R$ to R including all poles of $f(z)$.

2. Calculate all the poles of $f(z)$ and select the poles which lie inside C.

3. Apply the Cauchy Residue Theorem: $\int_C f(z)dz=2\pi i\sum R^+$, where $\sum R^+$ is the sum of residue in the upper half plane. $\Rightarrow \int_{-R}^{R} f(x)dx+\int_\Gamma f(z)dz=2\pi i\sum R^+$

4. We make $\int\limits_{\Gamma} f(z)dz \to 0$ as $R \to \infty$, then $\lim\limits_{R\to\infty} \int\limits_{-R}^{R} f(x)dx = 2\pi i \sum R^{+}$.

5. Finally, we get $\int\limits_{-\infty}^{\infty} f(x)dx = 2\pi i \sum R^{+}$.

Example 2.3. Find the integral of $\int\limits_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$ [2].

Solution. Let us suppose that $\int\limits_{C} f(z)dz = \int\limits_{C} \frac{dz}{(1+z^2)^2}$, where C is the closed contour consisting of upper half large semi-circle Γ together with real axis from $x = -R$ to $x = R$, then $\int\limits_{C} f(z) = \int\limits_{-R}^{R} f(x)dx + \int\limits_{\Gamma} f(z)dz$

Now, the pole of $f(z)$ can be obtain by $(1+z^2)^2 = 0 \Rightarrow z = \pm i$. Hence, $f(z)$ has poles at at $z = \pm i$ of order two, but only the pole $z = i$ of $f(z)$ lies within C, so that

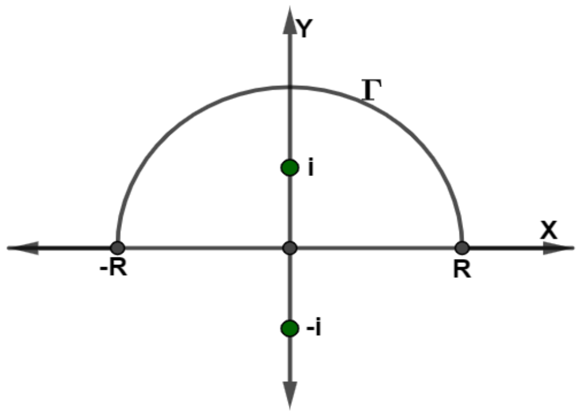

Figure 3

$$\underset{z=i}{Res} f(z) = \lim_{z\to i} \frac{1}{(2-1)!} \frac{d}{dz}\left[(z-i)^2 \frac{1}{(1+z^2)^2}\right] = \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{1}{(z-i)^2 (z+i)^2}\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right] = \lim_{z\to i} -2\,(2+i)^{-3} = \frac{1}{4i}$$

Now, by Cauchy Residue Theorem we have

$$\int_C f(z)dz = 2\pi i \sum R^+.$$

$$\Rightarrow \int_C f(z) = \int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz = \frac{\pi}{2} \tag{4}$$

Here, $\left|\int_\Gamma f(z)dz\right| \leq \int_\Gamma |f(z)dz| = \int_\Gamma |f(z)| = \int_\Gamma |f(z)||dz| = \int_\Gamma |\frac{1}{(1+z^2)^2}||dz|$

Put, $Z = Re^{i\theta} \Rightarrow dz = Rie^{i\theta}d\theta$, $\Rightarrow |dz| = |Rie^{i\theta}d\theta| = Rd\theta$, and $|z| = R$, then $|\int_\Gamma f(z)dz| \leq \int_0^\pi \frac{Rd\theta}{(1+R^2)^2} \leq \int_0^\pi \frac{R}{(R^2-1)} \to 0$ as $R \to \infty$.
Hence, the equation (4) becomes

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \frac{\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$$

2.3. The Integrals of the Form $\int_{-\infty}^{\infty} f(x)sinmxdx$ or $\int_{-\infty}^{\infty} f(x)cosmxdx$, where $f(x)$ is Rational Fraction

Now, we consider $\int_C f(z)e^{imz}dz$, where C is called counter consisting upper half plane of semi circle Γ of large radius and real axis from $-R$ to R including all poles of $f(z)$ and $e^{imx} = cos\, mx + i\, sin\, mx$. All of the procedure is similar to the form of $\int_{-\infty}^{\infty} f(x)dx$.

Theorem 2.1. (**Jodern Lemma**). *Suppose that $f(z)$ is analytic at all points in upper half plane $y \geq 0$. Let Γ denote any semi-circle $|z| = R$, and for all points z on Γ there is a positive constant M_R such that $|f(z)| \leq M_R$ and $M_R \to 0$ as $R \to \infty$, then $\lim_{R\to 0} \int_\Gamma f(z)e^{imz}dz = 0$, where $m > 0$* [2, 3].

Proof. Let us suppose that $f(z)$ has no singularity with on Γ for sufficiently large value of radius R. Let $z = Re^{i\theta}$, then $dz = Rie^{i\theta}d\theta$. $\Rightarrow |dz| = Rd\theta$, $(0 \leq \theta \leq \pi)$.

Here,

$$\left|\int\limits_{\Gamma} f(z)e^{imz}dz\right| \leq \int\limits_{\Gamma} |f(z)||e^{imz}||dz| \leq M_R \int\limits_{0}^{\pi} |e^{imz}|Rd\theta = RM_R \int\limits_{0}^{\pi} |e^{imRe^{i\theta}}|d\theta$$

$$= RM_R \int\limits_{0}^{\pi} |e^{imR(cos\theta+isin\theta)}|d\theta = RM_R \int\limits_{0}^{\pi} |e^{imRcos\theta+i^2mRsin\theta}|d\theta$$

$$= RM_R \int\limits_{0}^{\pi} |e^{imRcos\theta-mRsin\theta}|d\theta = RM_R \int\limits_{0}^{\pi} |e^{imRcos\theta}e^{-mRsin\theta}|d\theta$$

$$= RM_R \int\limits_{0}^{\pi} |cos\, mRcos\theta + isin\, mRcos\theta||e^{-mRsin\theta}|d\theta$$

$$\leq RM_R \int\limits_{0}^{\pi} |e^{-mRsin\theta}|d\theta < RM_R\frac{\pi}{mR} \,[By\ Jordan\ Inequality]$$

$$= \frac{\pi M_R}{m} \to 0 as R \to \infty$$

Thus, $\left|\int\limits_{\Gamma} f(z)e^{imz}dz\right| \to 0$ as $R \to \infty$. Hence, $\lim\limits_{R\to\infty} \int\limits_{\Gamma} f(z)e^{imz}dz = 0$, where $m > 0$.

Example 2.4. Find the integral of $\int\limits_{-\infty}^{\infty} \frac{sinmx}{x^2+4}dx$ [2, 3, 5].

Solution. Let us suppose that $\int\limits_{C} f(z)dz = \int\limits_{C} \frac{e^{imz}}{(z^2+4)}dz = \int\limits_{-\infty}^{\infty} \frac{sinmz}{z^2+4}dz$, where C is the closed contour consisting upper half semi circle Γ of large radius R with real axis from $-R$ to R, then we have

$$\int\limits_{C} f(z)dz = \int\limits_{-R}^{R} f(x)dx + \int\limits_{\Gamma} f(z)dz \tag{5}$$

and $f(z) = \frac{sin\, mx}{x^2+4}$
Now, the poles of $f(z)$ can be obtained by $z^2 + 4 = 0$. $\Rightarrow z = \pm 2i$, which are simple poles, but only the pole $z = 2i$ of $f(z)$ lies within the contour C, so that, the Residue of $f(z)$ is

$$\underset{z=2i}{Res} f(z) = \lim_{z\to 2i}[(z-2i)\, f(z)] = \lim_{z\to 2i}\left[(z-2i)\,\frac{e^{imz}}{(z^2+4)}\right]$$
$$= \lim_{z\to 2i}\left[(z-2i)\,\frac{e^{imz}}{(z-2i)\,(z+2i)}\right] = \lim_{z\to 2i}\left[\frac{e^{imz}}{z+2i}\right] = \frac{e^{-2m}}{4i}$$

Thus, $\underset{z=2i}{Res} f(z) = \frac{e^{-2m}}{4i}$

Now, by Cauchy Residue Theorem, we have

$$\int_C f(z)dz = 2\pi i\cdot\frac{e^{-2m}}{4i} = \frac{\pi e^{-2m}}{2} \Rightarrow \int_C f(z)dz = \frac{\pi e^{-2m}}{2}$$
$$\Rightarrow \int_{-R}^{R} f(x)dx + \int_{\Gamma} f(z)dz = \frac{\pi e^{-2m}}{2}$$
$$\Rightarrow \lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_{\Gamma} f(z)dz = \frac{\pi e^{-2m}}{2} \tag{6}$$

Since, by Jordan Lemma, we have $\lim_{R\to\infty}\int_{\Gamma} f(z)dz = 0$, so that

$$\int_{-\infty}^{\infty}\frac{e^{imx}}{(x^2+4)}dx = \frac{\pi e^{-2m}}{2} \Rightarrow \int_{-\infty}^{\infty}\frac{cos\, mx}{(x^2+4)}dx + i\int_{-\infty}^{\infty}\frac{sin\, mx}{(x^2+4)}dx = \frac{\pi e^{-2m}}{2} + i0 \tag{7}$$

Now, comparing real and imaginary parts of equation (7), then we have

$$\int_{-\infty}^{\infty}\frac{sin\, mx}{(x^2+4)}dx = 0$$

2.4. Improper Integrals Involving Logarithm Function

The following rules give an easy way to find improper integrals of a function involving logarithm functions [5, 7].

Put x=z. Consider the integral $\int_C f(z)dz$, where $f(z) = \frac{(log z)^2}{z^2+1}$, and the contour C consists of positive oriented semi-circle Γ given by $|z| = R$ described upper part of the real axis. The segment AB on$x$-axis from $-R$ to $-r$, and the segment ED on$x$-axis from r to R, take $R\to\infty$, and $r\to 0$. In addition, the pole at $z = 0$ is surrounded by the negatively orientated semi-circle Γ_1. Here, the line of integration, $z = 0$, is a pole on the x-axis. Since we cannot integrate through a singularity, we have adjusted the contour by indenting the path at $z = 0$, and as a result, we have taken the semi-circle Γ_1. Thus, the contour $C = \Gamma + AB + \Gamma_1 + ED$.

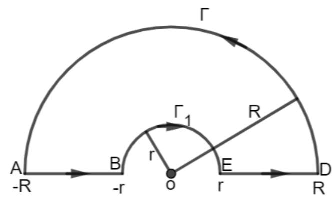

Figure 4

Example 2.5. Evaluate $\int\limits_{-\infty}^{\infty} \frac{(logx)^2}{x^2+1} dx$ [5, 6].

Solution. Let us suppose that $\int\limits_C f(z)$, where $f(z) = \frac{(logz)^2}{z^2+1}$, and C is the contour of the figure 4. Now, the poles of $f(z)$ can be obtained by $z^2 + 1 = 0$. $\Rightarrow z = \pm i$, which are simple poles, but only the pole $z = i$ of $f(z)$ lies within the contour C, so the Residue of $f(z)$ is

$$\begin{aligned}
\underset{z=i}{Res} f(z) &= \lim_{z \to i} \left[(z-i) f(z) \right] = \lim_{z \to i} \left[(z-i) \frac{(logz)^2}{z^2+1} \right] \\
&= \lim_{z \to i} \left[(z-i) \frac{(logz)^2}{(z-i)(z+i)} \right] = \lim_{z \to i} \left[\frac{(logz)^2}{z+i} \right] \\
&= \frac{(logi)^2}{2i} = \frac{\left[log(0+i) \right]^2}{2i} = \frac{1}{2i} \left[\frac{1}{2} log\left(0^2 + 1^2\right) + itan^{-1}\left(\frac{1}{0}\right) \right]^2 \\
&= \frac{1}{2i} \left[\frac{1}{2} log(1) + itan^{-1}(\infty) \right]^2 = \frac{1}{2i} \left[0 + i\frac{\pi}{2} \right]^2 = \frac{i\pi^2}{8}
\end{aligned}$$

$$\Rightarrow \underset{z=i}{Res} f(z) = \frac{i\pi^2}{8}$$

Now, by Cauchy Residue Theorem, we have

$$\int\limits_C f(z) dz = 2\pi i \cdot \frac{i\pi^2}{8} = \frac{-\pi^3}{4}$$

$$\Rightarrow \int\limits_C f(z) dz = \frac{-\pi^3}{4} \Rightarrow \int\limits_{-R}^{-r} f(z) dz + \int\limits_{\Gamma_1} f(z) dz + \int\limits_r^R f(z) dz + \int\limits_{\Gamma} f(z) dz = \frac{-\pi^3}{4}$$

$$\Rightarrow \int_{-R}^{-r} \frac{(log(z))^2}{z^2+1}dz + \int_{\Gamma_1} \frac{(logz)^2}{z^2+1}dz + \int_{r}^{R} \frac{(logz)^2}{z^2+1}dx + \int_{\Gamma} \frac{(logz)^2}{z^2+1}dz = \frac{-\pi^3}{4} \quad (8)$$

Now, the variable $z = -x$ in segment AB at x-axis, then we have $log(z) = log(-x) = logx + log(-1) = logx + log(e^{i\pi})$, and $dz = -dx$. Therefore,

$$\int_{-R}^{-r} \frac{(log(z))^2}{z^2+1}dz = -\int_{-R}^{-r} \frac{(log(-x))^2}{x^2+1}dx = \int_{r}^{R} \frac{[logx+\pi i]^2}{x^2+1}dx \quad (9)$$

Again, the variable $z = x$ in segment ED at x-axis, then we have

$$\int_{r}^{R} \frac{(log(z))^2}{z^2+1}dz = \int_{r}^{R} \frac{(log(x))^2}{x^2+1}dx \quad (10)$$

Thus, from equations (8), (9), and (10), we have

$$\int_{r}^{R} \frac{(logx+\pi i)^2}{x^2+1}dx + \int_{\Gamma_1} \frac{(logz)^2}{z^2+1}dz + \int_{r}^{R} \frac{(logx)^2}{x^2+1}dx + \int_{\Gamma} \frac{(logz)^2}{z^2+1}dz = \frac{-\pi^3}{4}$$

$$\Rightarrow \lim_{\substack{R\to\infty,\\ r\to 0}} \int_{r}^{R} \frac{(logx+\pi i)^2}{x^2+1}dx + \lim_{\substack{R\to\infty,\\ r\to 0}} \int_{\Gamma_1} \frac{(logz)^2}{z^2+1}dz + \lim_{\substack{R\to\infty,\\ r\to 0}} \int_{r}^{R} \frac{(logx)^2}{x^2+1}dx \quad (11)$$

$$+ \lim_{\substack{R\to\infty,\\ r\to 0}} \int_{\Gamma} \frac{(logz)^2}{z^2+1}dz = \frac{-\pi^3}{4}$$

$$\Rightarrow \int_{0}^{\infty} \frac{(logx+\pi i)^2}{x^2+1}dx + \int_{0}^{\infty} \frac{(logx)^2}{x^2+1}dx = \frac{-\pi^3}{4}$$

$$\Rightarrow \int_{0}^{\infty} \frac{[(logx)^2+2\pi i logx-\pi^2]^2}{x^2+1}dx + \int_{0}^{\infty} \frac{(logx)^2}{x^2+1}dx = \frac{-\pi^3}{4}$$

$$\Rightarrow \int_{0}^{\infty} \frac{(logx)^2}{x^2+1}dx + 2\pi i \int_{0}^{\infty} \frac{logx}{x^2+1}dx - \pi^2 \int_{0}^{\infty} \frac{dx}{x^2+1} + \int_{0}^{\infty} \frac{(logx)^2}{x^2+1}dx = \frac{-\pi^3}{4}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{(logx)^2}{x^2+1}dx + 2\pi i \int_{0}^{\infty} \frac{logx}{x^2+1}dx - \pi^2 \int_{0}^{\infty} \frac{dx}{x^2+1} = \frac{-\pi^3}{4}$$

$$\Rightarrow 2\int_0^\infty \frac{(logx)^2}{x^2+1}dx + 2\pi i\int_0^\infty \frac{logx}{x^2+1}dx - \pi^2\left(tan^{-1}x\right)_0^\infty = \frac{-\pi^3}{4}$$

$$\Rightarrow 2\int_0^\infty \frac{(logx)^2}{x^2+1}dx + 2\pi i\int_0^\infty \frac{logx}{x^2+1}dx - \pi^2\left(\frac{\pi}{2}\right) = \frac{-\pi^3}{4}$$

$$\Rightarrow \int_0^\infty \frac{(logx)^2}{x^2+1}dx + 2\pi i\int_0^\infty \frac{logx}{x^2+1}dx = \frac{\pi^3}{8} + 0i \tag{12}$$

Comparing real and imaginary parts of the equation (12), then we have

$$\int_0^\infty \frac{(logx)^2}{x^2+1}dx = \frac{\pi^3}{8} \tag{13}$$

Moreover, we have

$$\int_{-\infty}^\infty \frac{(logx)^2}{x^2+1}dx = 2\int_0^\infty \frac{(logx)^2}{x^2+1}dx \tag{14}$$

Hence, from equations (13), and (14), we have

$$\int_{-\infty}^\infty \frac{(logx)^2}{x^2+1}dx = \frac{\pi^3}{4}$$

3. Conclusion

In conclusion, this study has demonstrated how effectively Cauchy's Residue Theorem supports complex integrals along simple closed contours. By carefully selecting simple closed contours, specifically upper semi-circle containing isolated singularities, this study has effectively addressed the problem of evaluating anomalous definite integrals, which are frequently complex and difficult to solve using traditional methods.

The study focuses on four primary categories of definite integrals: sine and cosines functions, rational function of the form $\int_{-\infty}^{\infty} f(z)dz$, rational function of the form $\int_{-\infty}^{\infty} f(x)sinmxdx$ or $\int_{-\infty}^{\infty} f(x)cosmxdx$ and improper integral involving logarithm function. The residues at the singularities could be computed since the selected contours effectively enclosed them. After then, the integrals were simplified and assessed using these residues, which are essential parts in the application of Cauchy's Residue Theorem.

Furthermore, this study proved that using Cauchy's residual theorem has a clear advantage over traditional real analysis techniques. The method's efficiency stemmed from its ability to represent the integral along a simple closed contour as the sum of integrals along the real axis and the circle. This not only simplified the evaluation process but also provided an in-depth understanding of the behavior of the integral.

In essence, the results of this study demonstrate how effective Cauchy's Residue Theorem is at addressing a wide range of definite integrals involving various functions. This study's methodology brings up new possibilities for solving difficult mathematical problems that would otherwise be difficult or probably unsolvable when utilizing conventional real analysis methods.

References

[1] Ablowitz M. J. and Fokas A. S., Complex Variable, Cambridge University Press, 2008.

[2] Adhikari G. P., Complex Analysis, Dikshanta Prakashan, Kirtipur, Kathmandu, Nepal, first edition, 2021.

[3] Agarwal R. P. Perera K. and Pinelas S., Introduction to Complex Analysis, Springer Science Business Media, New York, first edition, 2011.

[4] Brown J. W. and Charchill R. V., Complex Variables and Applications. McGraw-Hill Higher Education, New York, eight edition, 2009.

[5] Chatarji P. N., Complex Analysis, Rajhans Agencies Educational Publishers, Dharma Alok, Ram Nagar, Meerut, India, ninth edition, 2013.

[6] Harris F. E., Mathematics for Physical Science and Engineering: Symbolic Computing Applications in Maple and Mathematica, Academic Press, first edition, 2014.

[7] Sharma J. N. and Swarup S., Function of a Complex Variable. Krishna Prakashan Media P.Ltd., Shivaji Road, Meerut, India, seventh edition, 2001.

[8] Tao T., Analysis I. Hindustan Book Agency, third edition, 2018.