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A NEW TYPE OF REGULARITY IN FUZZY MINIMAL SPACE

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Abstract: This paper deals with a new type of open-like set in fuzzy minimal space [2], viz., fuzzy m-α-preopen set taking fuzzy m-α-open set [3] as a basic tool. Afterwards, we introduce an idempotent operator, viz., fuzzy m-α-preclosure operator. With the help of this operator we introduce and study two new types of functions, viz., fuzzy (m, m_1)-α-precontinuous function and fuzzy (m, m_1)-α-preirresolute function. It is shown that fuzzy (m, m_1)-α-preirresolute function implies fuzzy (m, m_1)-α-precontinuous function, but reverse implication is not necessarily true, in general. Moreover, we introduce fuzzy m-α-preregular space in which the reverse implication holds.

Keywords and Phrases: Fuzzy m-open set, fuzzy m-α-preopen set, fuzzy (m, m_1)-α-precontinuous function, fuzzy (m, m_1)-α-preirresolute function, fuzzy m-α- preregular space.

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1. Introduction

In [8], L.A. Zadeh introduced fuzzy set as follows : A fuzzy set A is a mapping from a non-empty set X into the closed interval $[0, 1]$, i.e., $A \in I^X$. In 1968, C.L. Chang introduced fuzzy topology [5]. Afterwards, Alimohammady and Roohi introduced a more general version of fuzzy topology by introducing fuzzy minimal structure as follows : A family $\mathcal{M}$ of fuzzy sets in a non-empty set X is said to be a fuzzy minimal structure on X if $\alpha 1_X \in \mathcal{M}$ for every $\alpha \in [0, 1]$ [1]. However a more general version of it (in the sense of Chang) is introduced in [4, 6] as follows : A family $\mathcal{F}$ of fuzzy sets in a non-empty set X is a fuzzy minimal structure on X if $0_X \in \mathcal{F}$ and $1_X \in \mathcal{F}$. In this paper, we use the notion of fuzzy minimal structure in the sense of Chang. In [2], we introduced fuzzy minimal space (fuzzy m-space, for short) as follows : Let X be a non-empty set and $m \subset I^X$. Then (X, m) is called fuzzy m-space if $0_X \in m$ and $1_X \in m$. The members of m are called fuzzy m-open sets and the complement of a fuzzy m-open set is called fuzzy m-closed set [2].

2. Preliminary

Throughout this paper, (X, m) or simply by X we shall mean a fuzzy minimal space (fuzzy m-space, for short). The support [8] of a fuzzy set A, denoted by $suppA$ or A_0 and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [8] of a fuzzy set A in X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [8] while AqB means A is quasi-coincident (q-coincident, for short) [7] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy set A and a fuzzy point x_α in X, $x_\alpha \in A$ means $A(x) \geq \alpha$. A fuzzy set A in a fuzzy m-space (X, m) is called a fuzzy m-nbd of a fuzzy point x_α in X if there exists a fuzzy m-open set U in X such that $x_\alpha \in U \leq A$ [2]. If, in addition, A is fuzzy m-open, then A is called a fuzzy m-open nbd of x_α [2]. A fuzzy set A in a fuzzy m-space (X, m) is called a fuzzy m-q-nbd of a fuzzy point x_α in X if there exists a fuzzy m-open set U in X such that $x_\alpha q U \leq A$ [2]. If, in addition, A is fuzzy m-open, then A is called a fuzzy m-open q-nbd of x_α[2].

3. Fuzzy m-α-Preopen Set : Some Properties

Using fuzzy m-α-open set as a basic tool, here we introduce fuzzy m-α-preopen set, the class of which is strictly larger than that of fuzzy m-open as well as fuzzy m-α-open sets. Afterwards, we introduce fuzzy m-α-preclosure operator which is an idempotent operator.

We first recall some definitions from [2, 3] for ready references.

Definition 3.1. [2] *Let X be a non-empty set and $m \subset I^X$ an m-structure on X. For $A \in I^X$, the m-closure of A and m-interior of A are defined as follows :*

$$mclA = \bigwedge\{F : A \leq F, 1_X \setminus F \in m\}$$

$$mintA = \bigvee\{D : D \leq A, D \in m\}$$

It can be observed that a given fuzzy minimal structure on X, $A \in I^X$ does not imply that $mintA \in m$ and $mclA$ is fuzzy m-closed. But if m satisfies M-condition (i.e., m is closed under arbitrary union), then $mintA \in m$ and $mclA$ is fuzzy m-closed.

Proposition 3.2. [2] *Let X be a non-empty set and m, an m-structure on X. Then for any $A \in I^X$, a fuzzy point $x_\alpha \in mclA$ if and only if for any $U \in m$ with $x_\alpha qU$, UqA.*

Lemma 3.3. [2] *Let X be a non empty set and m, an m-structure on X. For $A, B \in I^X$, the following hold :*
(i) $A \leq B$ which implies that (a) $mintA \leq mintB$, (b) $mclA \leq mclB$.
(ii) (a) $mcl0_X = 0_X$, $mcl1_X = 1_X$, (b) $mint0_X = 0_X$, $mint1_X = 1_X$.
(iii) $mintA \leq A \leq mclA$.
(iv) (a) $mclA = A$ if $1_X \setminus A \in m$, (b) $mintA = A$, if $A \in m$.
(v) (a) $mcl(1_X \setminus A) = 1_X \setminus mintA$, (b) $mint(1_X \setminus A) = 1_X \setminus mclA$.
(vi) (a) $mcl(mclA) = mclA$, (b) $mint(mintA) = mintA$.
(vii) (a) $mcl(A \bigwedge B) \leq mclA \bigwedge mclB$, (b) $mint(A \bigvee B) \geq mintA \bigvee mintB$.

Definition 3.4. [3] *Let (X, m) be a fuzzy m-space and $A \in I^X$. Then A is called fuzzy m-α-open* [3] *if $A \leq mint(mcl(mintA))$.*

The complement of fuzzy m-α-open set is called fuzzy m-α-closed [3].
The union (intersection) of all fuzzy m-α-open (resp., fuzzy m-α-closed) sets contained in (resp., containing) a fuzzy set A is called fuzzy m-α-interior [3] (resp., fuzzy m-α-closure [3]) of A denoted by $m\alpha intA$ (resp., $m\alpha clA$).
The collection of all fuzzy m-α-open (resp., fuzzy m-α-closed) sets in a fuzzy m-space X is denoted by $Fm\alpha O(X)$ (resp., $Fm\alpha C(X)$).

Proposition 3.5. [3] *Let (X, m) be a fuzzy m-space and $A \in I^X$. Then a fuzzy point $x_\alpha \in m\alpha clA$ if and only if for every fuzzy m-α-open set U in X, $x_\alpha qU$, UqA, i.e., for every fuzzy m-α-open q-nbd U of x_α, UqA.*

Result 3.6. [3] *Let (X, m) be a fuzzy m-space and $A, B \in I^X$. Then the following statements hold :*
(i) $A \leq B$ which implies that (a) $m\alpha intA \leq m\alpha intB$, (b) $m\alpha clA \leq m\alpha clB$.
(ii) (a) $m\alpha cl0_X = 0_X$, $m\alpha cl1_X = 1_X$, (b) $m\alpha int0_X = 0_X$, $m\alpha int1_X = 1_X$.
(iii) $m\alpha intA \leq A \leq m\alpha clA$.
(iv) (a) $m\alpha clA = A$ if $A \in Fm\alpha C(X)$, (b) $m\alpha intA = A$, if $A \in Fm\alpha O(X)$.
(v) (a) $m\alpha cl(1_X \setminus A) = 1_X \setminus m\alpha intA$, (b) $m\alpha int(1_X \setminus A) = 1_X \setminus m\alpha clA$.
(vi) (a) $m\alpha cl(m\alpha clA) = m\alpha clA$, (b) $m\alpha int(m\alpha intA) = m\alpha intA$.

(vii) (a) $m\alpha cl(A \bigwedge B) \leq m\alpha clA \bigwedge m\alpha clB$, *(b)* $m\alpha int(A \bigvee B) \geq m\alpha intA \bigvee m\alpha intB$,
(viii) (a) $m\alpha cl(A \bigvee B) \geq m\alpha clA \bigvee m\alpha clB$, *(b)* $m\alpha cl(A \bigwedge B) \leq m\alpha intA \bigwedge m\alpha intB$.

Definition 3.7. *A fuzzy set A in a fuzzy m-space (X, m) is called fuzzy m-α-preopen if $A \leq m\alpha int(mclA)$. The complement of this set is called fuzzy m-α-preclosed set.*
The collection of fuzzy m-α-preopen (resp., fuzzy m-α-preclosed) sets in (X, m) is denoted by $Fm\alpha PO(X)$ (resp., $Fm\alpha PC(X)$).
The union (resp., intersection) of all fuzzy m-α-preopen (resp., fuzzy m-α-preclosed) sets contained in (containing) a fuzzy set A is called fuzzy m-α-preinterior (resp., fuzzy m-α-preclosure) of A, denoted by $m\alpha pintA$ (resp., $m\alpha pclA$).

Result 3.8. *Union of two fuzzy m-α-preopen sets in a fuzzy m-space X is also so.*
Proof. Let $A, B \in Fm\alpha PO(X)$. Then $A \leq m\alpha int(mclA), B \leq m\alpha int(mclB)$. Now $m\alpha int(mcl(A \bigvee B)) = m\alpha int(mclA \bigvee mclB) \geq m\alpha int(mclA) \bigvee m\alpha int\,(mclB) \geq A \bigvee B$.

Remark 3.9. *Intersection of two fuzzy m-α-preopen sets need not be so, follows from the next example.*

Example 3.10. Let $X = \{a, b\}$, $m = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, m) is a fuzzy m-space. Consider two fuzzy sets U, V defined by $U(a) = 0.4, U(b) = 0.5, V(a) = 0.6, V(b) = 0.4$. Then clearly $U, V \in Fm\alpha PO(X)$. Let $W = U \bigwedge V$. Then $W(a) = W(b) = 0.4$. Now $m\alpha int(mclW) \not\geq W \Rightarrow W \notin Fm\alpha PO(X)$.

Remark 3.11. *Fuzzy m-open and fuzzy m-α-open sets are fuzzy m-α-preopen, but not conversely follow from the next example.*

Example 3.12. Consider Example 3.10. Here $U \in Fm\alpha PO(X)$. But $U \notin m$, $U \notin Fm\alpha O(X)$.

Definition 3.13. *A fuzzy set A in a fuzzy m-space (X, m) is called fuzzy m-α-pre neighbourhood (fuzzy m-α-pre nbd, for short) of a fuzzy point x_α if there exists a fuzzy m-α-preopen set U in X such that $x_\alpha \in U \leq A$. If, in addition, A is fuzzy m-α-preopen, then A is called fuzzy m-α-preopen nbd of x_α.*

Definition 3.14. *A fuzzy set A in a fuzzy m-space (X, m) is called fuzzy m-α-pre quasi neighbourhood (fuzzy m-α-pre q-nbd, for short) of a fuzzy point x_α if there exists a fuzzy m-α-preopen set U in X such that $x_\alpha q U \leq A$. If, in addition, A is fuzzy m-α-preopen, then A is called fuzzy m-α-preopen q-nbd of x_α.*

Remark 3.15. *Since a fuzzy m-open set is fuzzy m-α-preopen, we can conclude*

that
(i) fuzzy m-nbd (resp., fuzzy m-open nbd) of a fuzzy point x_α is a fuzzy m-α-pre nbd (resp., fuzzy m-α-preopen nbd) of x_α,
(ii) fuzzy m-q-nbd (resp., fuzzy m-open q-nbd) of a fuzzy point x_α is a fuzzy m-α-pre q-nbd (resp., fuzzy m-α-preopen q-nbd) of x_α.
But the reverse implications are not necessarily true follow from the following example.

Example 3.16. Let $X = \{a, b\}$, $m = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, m) is a fuzzy m-space. Consider two fuzzy sets B, C defined by $B(a) = 0.4, B(b) = 0.5, C(a) = 0.6, C(b) = 0.4$. Then $B, C \in Fm\alpha O(X)$. Consider two fuzzy points $a_{0.3}$ and $a_{0.45}$. Now $a_{0.3} \in B \leq B \Rightarrow B$ is a fuzzy m-α-pre nbd as well as fuzzy m-α-preopen nbd of $a_{0.3}$. But there does not exist any fuzzy m-open set U in (X, m) such that $a_{0.3} \in U \leq B$. So B is not a fuzzy m-nbd and fuzzy m-open nbd of $a_{0.3}$.
Next $a_{0.45}qC \leq C \Rightarrow C$ is a fuzzy m-α-pre q-nbd as well as fuzzy m-α-preopen q-nbd of $a_{0.45}$. But there does not exist a fuzzy m-open set U in X with $a_{0.45}qU \leq C \Rightarrow C$ is not a fuzzy m-q-nbd and fuzzy m-open q-nbd of $a_{0.45}$.

Theorem 3.17. *For any fuzzy set A in a fuzzy m-space (X, m), $x_t \in m\alpha pclA$ if and only if every fuzzy m-α-preopen q-nbd U of x_t, UqA.*
Proof. Let $x_t \in m\alpha pclA$ for any fuzzy set A in a fuzzy m-space (X, m). Let $U \in Fm\alpha PO(X)$ with x_tqU. Then $U(x) + t > 1 \Rightarrow x_t \notin 1_X \setminus U \in Fm\alpha PC(X)$. Then by definition, $A \not\leq 1_X \setminus U \Rightarrow$ there exists $y \in X$ such that $A(y) > 1 - U(y) \Rightarrow A(y) + U(y) > 1 \Rightarrow UqA$.
Conversely, let the given condition hold. Let $U \in Fm\alpha PC(X)$ with $A \leq U$ ... (1). We have to show that $x_t \in U$, i.e., $U(x) \geq t$. If possible, let $U(x) < t$. Then $1 - U(x) > 1 - t \Rightarrow x_tq(1_X \setminus U)$ where $1_X \setminus U \in Fm\alpha PO(X)$. By hypothesis, $(1_X \setminus U)qA \Rightarrow$ there exists $y \in X$ such that $1 - U(y) + A(y) > 1 \Rightarrow A(y) > U(y)$, contradicts (1).

Theorem 3.18. *$m\alpha pcl(m\alpha pclA) = m\alpha pclA$ for any fuzzy set A in a fuzzy m-space (X, m).*
Proof. Let $A \in I^X$. Then $A \leq m\alpha pclA \Rightarrow m\alpha pclA \leq m\alpha pcl(m\alpha pclA)$ ... (1).
Conversely, let $x_t \in m\alpha pcl(m\alpha pclA)$. If possible, let $x_t \notin m\alpha pclA$. Then there exists $U \in Fm\alpha PO(X)$,

$$x_tqU, U \not{q} A ...(2)$$

But as $x_t \in m\alpha pcl(m\alpha pclA)$, $Uq(m\alpha pclA) \Rightarrow$ there exists $y \in X$ such that $U(y) + (m\alpha pclA)(y) > 1 \Rightarrow U(y) + s > 1$ where $s = (m\alpha pclA)(y)$. Then $y_s \in m\alpha pclA$

and $y_s q U$ where $U \in Fm\alpha PO(X)$. Then by definition, UqA, contradicts (2). So

$$m\alpha pcl(m\alpha pclA) \leq m\alpha pclA...(3)$$

Combining (1) and (3), we get the result.

4. Fuzzy (m, m_1)-α-Precontinuous Function: Some Characterizations

In this section a new type of function is introduced and studied, the class of which is strictly larger than that of fuzzy (m, m_1)-continuous function [3].

Definition 4.1. *A function $f : (X, m) \to (Y, m_1)$ is said to be fuzzy (m, m_1)-α-precontinuous if for each fuzzy point x_t in X and every fuzzy m_1-nbd V of $f(x_t)$ in Y, $mcl(f^{-1}(V))$ is a fuzzy m-α-nbd of x_t in X.*

Theorem 4.2. *For a function $f : (X, m) \to (Y, m_1)$ where m, m_1 satisfy M-condition, the following statements are equivalent :*
(a) f is fuzzy (m, m_1)-α-precontinuous,
(b) $f^{-1}(B) \leq m\alpha int(mcl(f^{-1}(B)))$, for all fuzzy m_1-open set B of Y,
(c) $f(m\alpha clA) \leq m_1cl(f(A))$, for all fuzzy m-open set A in X.

Proof. (a) $\Rightarrow$ (b). Let B be any fuzzy m_1-open set in Y and $x_t \in f^{-1}(B)$. Then $f(x_t) \in B \Rightarrow B$ is a fuzzy m_1-nbd of $f(x_t)$ in Y. By (a), $mcl(f^{-1}(B))$ is a fuzzy m-α-nbd of x_t in X. So $x_t \in m\alpha int(mcl(f^{-1}(B)))$. Since x_t is taken arbitrarily, $f^{-1}(B) \leq m\alpha int(mcl(f^{-1}(B)))$.
(b) $\Rightarrow$ (a). Let x_t be a fuzzy point in X and B be a fuzzy m_1-nbd of $f(x_t)$ in Y. Then $x_t \in f^{-1}(B) \leq m\alpha int(mcl(f^{-1}(B)))$ (by (b)) $\leq mcl(f^{-1}(B))$. So $mcl(f^{-1}(B))$ is a fuzzy m-α-nbd of x_t in X.
(b) $\Rightarrow$ (c). Let A be a fuzzy m-open set in X. Then $1_Y \setminus m_1cl(f(A))$ is a fuzzy m_1-open set in Y (as m_1 satisfies M-condition). By (b), $f^{-1}(1_Y \setminus m_1cl(f(A))) \leq m\alpha int(mcl(f^{-1}(1_Y \setminus m_1cl(f(A))))) = m\alpha int(mcl(1_X \setminus f^{-1}(m_1cl(f(A))))) \leq m\alpha int\ (mcl(1_X \setminus f^{-1}(f(A)))) \leq m\alpha int\ (mcl(1_X \setminus A)) = m\alpha int(1_X \setminus A) = 1_X \setminus m\alpha clA$. Then $m\alpha clA \leq 1_X \setminus f^{-1}(1_Y \setminus m_1cl(f(A))) = f^{-1}(m_1cl(f(A)))$. So $f(m\alpha clA) \leq m_1cl(f(A))$.
(c) $\Rightarrow$ (b). Let B be any fuzzy m_1-open set in Y. Then $mint(f^{-1}(1_Y \setminus B))$ is a fuzzy m-open set in X (as m satisfies M-condition). By (c), $f(m\alpha cl(mint(f^{-1}(1_Y \setminus B)))) \leq m_1cl(f(mint(f^{-1}(1_Y \setminus B)))) \leq m_1cl(f(f^{-1}(1_Y \setminus B))) \leq m_1cl(1_Y \setminus B) = 1_Y \setminus B$ (as m_1 satisfies M-condition) $\Rightarrow B \leq 1_Y \setminus f(m\alpha cl(mint(f^{-1}(1_Y \setminus B))))$. Then $f^{-1}(B) \leq f^{-1}(1_Y \setminus f(m\alpha cl(mint(f^{-1}(1_Y \setminus B))))) = 1_X \setminus f^{-1}(f(m\alpha cl(mint(f^{-1}(1_Y \setminus B))))) \leq 1_X \setminus m\alpha cl(mint(f^{-1}(1_Y \setminus B))) = 1_X \setminus m\alpha cl(mint(1_X \setminus f^{-1}(B))) = m\alpha int(mcl(f^{-1}(B)))$.

Note 4.3. It is clear from above theorem that the inverse image under fuzzy

(m, m_1)-α-precontinuous function of any fuzzy m_1-open set is fuzzy m-α-preopen.

Theorem 4.4. *For a function $f : (X, m) \to (Y, m_1)$ where m_1 satisfies M-condition, the following statements are equivalent :*
(a) f is fuzzy (m, m_1)-α-precontinuous,
(b) $f^{-1}(B) \leq m\alpha int(mcl(f^{-1}(B)))$, for all fuzzy m_1-open set B of Y,
(c) for each fuzzy point x_t in X and each fuzzy m_1-open nbd V of $f(x_t)$ in Y, there exists $U \in Fm\alpha PO(X)$ containing x_t such that $f(U) \leq V$,
(d) $f^{-1}(F) \in Fm\alpha PC(X)$, for all fuzzy m_1-closed sets F in Y,
(e) for each fuzzy point x_t in X, the inverse image under f of every fuzzy m_1-nbd of $f(x_t)$ is a fuzzy m-α-pre nbd of x_t in X,
(f) $f(m\alpha pclA) \leq m_1cl(f(A))$, for all fuzzy set A in X,
(g) $m\alpha pcl(f^{-1}(B)) \leq f^{-1}(m_1clB)$, for all fuzzy set B in Y,
(h) $f^{-1}(m_1intB) \leq m\alpha pint(f^{-1}(B))$, for all fuzzy set B in Y.

Proof. (a) $\Leftrightarrow$ (b). Follows from Theorem 4.2 (a) $\Leftrightarrow$ (b).
(b) $\Rightarrow$ (c). Let x_t be a fuzzy point in X and V be a fuzzy open m_1-nbd of $f(x_t)$ in Y. By (b), $f^{-1}(V) \leq m\alpha int(mcl(f^{-1}(V)))$ ... (1) . Now $f(x_t) \in V \Rightarrow x_t \in f^{-1}(V)$ ($= U$, say). Then $x_t \in U$ and by (1), $U(= f^{-1}(V)) \in Fm\alpha PO(X)$ and $f(U) = f(f^{-1}(V)) \leq V$.
(c) $\Rightarrow$ (b). Let V be a fuzzy m_1-open set in Y and let $x_t \in f^{-1}(V)$. Then $f(x_t) \in V \Rightarrow V$ is a fuzzy m_1-open nbd of $f(x_t)$ in Y. By (c), there exists $U \in Fm\alpha PO(X)$ containing x_t such that $f(U) \leq V$. Then $x_t \in U \leq f^{-1}(V)$. Now $U \leq m\alpha int(mclU)$. Then $U \leq m\alpha int(mclU) \leq m\alpha int(mcl(f^{-1}(V))) \Rightarrow x_t \in U \leq m\alpha int(mcl(f^{-1}(V)))$. Since x_t is taken arbitrarily, $f^{-1}(V) \leq m\alpha int(mcl(f^{-1}(V)))$.
(b) $\Leftrightarrow$ (d). Obvious.
(b) $\Rightarrow$ (e). Let W be a fuzzy m_1-nbd of $f(x_t)$ in Y. Then there exists a fuzzy m_1-open set V in Y such that $f(x_t) \in V \leq W \Rightarrow V$ is a fuzzy m_1-open nbd of $f(x_t)$ in Y. Then by (b), $f^{-1}(V) \in Fm\alpha PO(X)$ and $x_t \in f^{-1}(V) \leq f^{-1}(W) \Rightarrow f^{-1}(W)$ is a fuzzy m-α-pre nbd of x_t in X.
(e) $\Rightarrow$ (b). Let V be a fuzzy m_1-open set in Y and $x_t \in f^{-1}(V)$. Then $f(x_t) \in V$. Then V is a fuzzy m_1-open nbd of $f(x_t)$ in Y. By (e), there exists $U \in Fm\alpha PO(X)$ containing x_t such that $U \leq f^{-1}(V) \Rightarrow x_t \in U \leq m\alpha int(mclU) \leq m\alpha int(mcl(f^{-1}(V)))$. Since x_t is taken arbitrarily, $f^{-1}(V) \leq m\alpha int(mcl(f^{-1}(V)))$.
(d) $\Rightarrow$ (f). Let $A \in I^X$. Then $m_1cl(f(A))$ is a fuzzy m_1-closed set in Y (as m_1 satisfies M-condition). By (d), $f^{-1}(m_1cl(f(A))) \in Fm\alpha PC(X)$ containing A. Therefore, $m\alpha pclA \leq m\alpha pcl(f^{-1}(m_1cl(f(A)))) = f^{-1}(m_1cl(f(A))) \Rightarrow f(m\alpha pclA) \leq m_1cl(f(A))$.
(f) $\Rightarrow$ (d). Let B be a fuzzy m_1-closed set in Y. Then $f^{-1}(B) \in I^X$. By (f), $f(m\alpha pcl(f^{-1}(B))) \leq m_1cl(f(f^{-1}(B))) \leq m_1clB = B$ (as m_1 satisfies M-condition)

$\Rightarrow m\alpha pcl(f^{-1}(B)) \leq f^{-1}(B) \Rightarrow f^{-1}(B) \in Fm\alpha PC(X)$.
(f) $\Rightarrow$ (g). Let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. By (f), $f(m\alpha pcl(f^{-1}(B))) \leq m_1cl(f(f^{-1}(B))) \leq m_1clB \Rightarrow m\alpha pcl(f^{-1}(B)) \leq f^{-1}(m_1clB)$.
(g) $\Rightarrow$ (f). Let $A \in I^X$. Let $B = f(A)$. Then $B \in I^Y$. By (g), $m\alpha pcl(f^{-1}(f(A))) \leq f^{-1}(m_1cl(f(A))) \Rightarrow m\alpha pclA \leq f^{-1}(m_1cl(f(A))) \Rightarrow f(m\alpha pclA) \leq m_1cl(f(A))$.
(b) $\Rightarrow$ (h). Let $B \in I^Y$. Then m_1intB is a fuzzy m_1-open set in Y (as m_1 satisfies M-condition). By (b), $f^{-1}(m_1intB) \leq m\alpha int(mcl(f^{-1}(m_1intB))) \Rightarrow f^{-1}(m_1intB) \in Fm\alpha PO(X) \Rightarrow f^{-1}(m_1intB) = m\alpha pint(f^{-1}(m_1intB)) \leq m\alpha pint (f^{-1}(B))$.
(h) $\Rightarrow$ (b). Let A be any fuzzy m_1-open set in Y. Then $f^{-1}(A) = f^{-1}(m_1intA) \leq m\alpha pint(f^{-1}(A))$ (by (h)) $\Rightarrow f^{-1}(A) \in Fm\alpha PO(X)$.

Theorem 4.5. *A function $f : (X, m) \to (Y, m_1)$ is fuzzy (m, m_1)-α-precontinuous if and only if for each fuzzy point x_t in X and each fuzzy m_1-open q-nbd V of $f(x_t)$ in Y, there exists a fuzzy m-α-pre q-nbd W of x_t in X such that $f(W) \leq V$.*

Proof. Let f be fuzzy (m, m_1)-α-precontinuous function and x_t be a fuzzy point in X and V be a fuzzy m_1-open q-nbd of $f(x_t)$ in Y. Then $f(x_t)qV$. Let $f(x) = y$. Then $V(y) + t > 1 \Rightarrow V(y) > 1 - t \Rightarrow V(y) > \beta > 1 - t$, for some real number β. Then V is a fuzzy m_1-open nbd of y_β. By Theorem 4.4 (a)$\Rightarrow$(c), there exists $W \in Fm\alpha PO(X)$ containing x_β, i.e., $W(x) \geq \beta$ such that $f(W) \leq V$. Then $W(x) \geq \beta > 1 - t \Rightarrow x_tqW$ and $f(W) \leq V$.
Conversely, let the given condition hold and let V be a fuzzy m_1-open set in Y. Put $W = f^{-1}(V)$. If $W = 0_X$, then we are done. Suppose $W \neq 0_X$. Then for any $x \in W_0$, let $y = f(x)$. Then $W(x) = [f^{-1}(V)](x) = V(f(x)) = V(y)$. Let us choose $m \in \mathcal{N}$ where $\mathcal{N}$ is the set of all natural numbers such that $1/m \leq W(x)$. Put $\alpha_n = 1 + 1/n - W(x)$, for all $n \in \mathcal{N}$. Then for $n \in \mathcal{N}$ and $n \geq m$, $1/n \leq 1/m \Rightarrow 1+1/n \leq 1+1/m \Rightarrow \alpha_n = 1+1/n-W(x) \leq 1+1/m-W(x) \leq 1$. Again $\alpha_n > 0$, for all $n \in \mathcal{N} \Rightarrow 0 < \alpha_n \leq 1$ so that $V(y) + \alpha_n > 1 \Rightarrow y_{\alpha_n}qV \Rightarrow V$ is a fuzzy m_1-open q-nbd of y_{α_n}. By the given condition, there exists $U_n^x \in Fm\alpha PO(X)$ such that $x_{\alpha_n}qU_n^x$ and $f(U_n^x) \leq V$, for all $n \geq m$. Let $U^x = \bigvee\{U_n^x : n \in \mathcal{N}, n \geq m\}$. Then $U^x \in Fm\alpha PO(X)$ (by Result 3.8) and $f(U^x) \leq V$. Again $n \geq m \Rightarrow U_n^x(x) + \alpha_n > 1 \Rightarrow U_n^x(x) + 1 + 1/n - W(x) > 1 \Rightarrow U_n^x(x) > W(x) - 1/n \Rightarrow U_n^x(x) \geq W(x)$, for each $x \in W_0$. Then $W \leq U_n^x$, for all $n \geq m$ and for all $x \in W_0 \Rightarrow W \leq U^x$, for all $x \in W_0 \Rightarrow W \leq \bigvee_{x \in W_0} U^x = U$ (say) ... (1) and $f(U^x) \leq V$, for all $x \in W_0 \Rightarrow f(U) \leq V \Rightarrow U \leq f^{-1}(f(U)) \leq f^{-1}(V) = W$ ... (2). By (1) and (2), $U = W = f^{-1}(V) \Rightarrow f^{-1}(V) \in Fm\alpha PO(X)$. Hence by Theorem 4.2, f is fuzzy (m, m_1)-α-precontinuous function.

Remark 4.6. *Composition of two fuzzy (m, m_1)-α-precontinuous functions need not be so, follows from the following example.*

Example 4.7. Let $X = \{a, b\}$, $m_1 = \{0_X, 1_X, A\}$, $m_2 = \{0_X, 1_X\}$, $m_3 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.4$. Then (X, m_1), (X, m_2) and (X, m_3) are fuzzy m-spaces. Consider two identity functions $i_1 : (X, m_1) \to (X, m_2)$ and $i_2 : (X, m_2) \to (X, m_3)$. Clearly i_1 and i_2 are fuzzy (m_1, m_2)-α-precontinuous and fuzzy (m_2, m_3)-α-precontinuous functions. Now $B \in m_3$. $(i_2 \circ i_1)^{-1}(B) = B \not\leq m_1\alpha int(m_1 clB) = 0_X \Rightarrow B \notin Fm_1\alpha PO(X) \Rightarrow i_2 \circ i_1$ is not fuzzy (m_1, m_3)-α-precontinuous function.

Let us now recall the following definitions from [3] for ready references.

Definition 4.8. [3] *A function $f : (X, m) \to (Y, m_1)$ is called fuzzy (m, m_1)-continuous function if the inverse image of every fuzzy m_1-open set in Y is fuzzy m-open set in X.*

Note 4.9. It is clear from above definition that fuzzy (m, m_1)-continuous function is fuzzy (m, m_1)-α-precontinuous. But the converse is not necessarily true follows from the next example.

Example 4.10. Let $X = \{a, b\}$, $m_1 = \{0_X, 1_X\}$, $m_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5 = A(b)$. Then (X, m_1) and (X, m_2) are fuzzy m-spaces. Consider the identity function $i : (X, m_1) \to (X, m_2)$. Now $A \in m_2, i^{-1}(A) = A \notin m_1$. Clearly i is not fuzzy (m_1, m_2)-continuous function. Now every fuzzy set in (X, m_1) is fuzzy m_1-α-preopen in $(X, m_1) \Rightarrow i$ is fuzzy (m_1, m_2)-α-precontinuous function.

5. Fuzzy (m, m_1)-α-Preirresolute Function: Some Properties

In this section we introduce a new type of function, viz., fuzzy (m, m_1)-α-preirresolute function, the class of which is coarser than that of fuzzy (m, m_1)-α-precontinuous function.

Definition 5.1. *A function $f : (X, m) \to (Y, m_1)$ is called fuzzy (m, m_1)-α-preirresolute if the inverse image of every fuzzy m_1-α-preopen set in Y is fuzzy m-α-preopen in X.*

Theorem 5.2. *For a function $f : (X, m) \to (Y, m_1)$ where m_1 satisfies M-condition, the following statements are equivalent :*
(a) f is fuzzy (m, m_1)-α-preirresolute,
(b) for each fuzzy point x_t in X and each fuzzy m_1-α-preopen nbd V of $f(x_t)$ in Y, there exists a fuzzy m-α-preopen nbd U of x_t in X and $f(U) \leq V$,
(c) $f^{-1}(F) \in Fm\alpha PC(X)$, for all $F \in Fm_1\alpha PC(Y)$,
(d) for each fuzzy point x_t in X, the inverse image under f of every fuzzy m_1-α-

preopen nbd of $f(x_t)$ *is a fuzzy* m-α-*preopen nbd of* x_t *in* X,
(e) $f(m\alpha pclA) \leq m_1\alpha pcl(f(A))$, *for all* $A \in I^X$,
(f) $m\alpha pcl(f^{-1}(B)) \leq f^{-1}(m_1\alpha pclB)$, *for all* $B \in I^Y$,
(g) $f^{-1}(m_1\alpha pintB) \leq m\alpha pint(f^{-1}(B))$, *for all* $B \in I^Y$.
Proof. The proof is similar to that of Theorem 4.4 and hence is omitted.

Theorem 5.3. *A function* $f : (X, m) \to (Y, m_1)$ *is fuzzy* (m, m_1)-α-*preirresolute if and only if for each fuzzy point* x_t *in* X *and corresponding to any fuzzy* m_1-α-*preopen q-nbd* V *of* $f(x_t)$ *in* Y, *there exists a fuzzy* m-α-*preopen q-nbd* W *of* x_t *in* X *such that* $f(W) \leq V$.
Proof. The proof is similar to that of Theorem 4.5 and hence is omitted.

Remark 5.4. *Clearly composition of two fuzzy* (m, m_1)-α-*preirresolute functions is fuzzy* (m, m_1)-α-*preirresolute.*

Theorem 5.5. *If* $f : (X, m) \to (Y, m_1)$ *is fuzzy* (m, m_1)-α-*preirresolute and* $g : (Y, m_1) \to (Z, m_2)$ *is fuzzy* (m_1, m_2)-α-*precontinuous (resp. fuzzy* (m_1, m_2)-*continuous), then* $g \circ f : (X, m) \to (Z, m_2)$ *is fuzzy* (m, m_2)-α-*precontinuous.*
Proof. Obvious.

Remark 5.6. *Every fuzzy* (m, m_1)-α-*preirresolute function is fuzzy* (m, m_1)-α-*precontinuous, but the converse is not true, in general, follows from the following example.*

Example 5.7. Fuzzy (m, m_1)-α-precontinuous function $\not\Rightarrow$ fuzzy (m, m_1)-α - preirresolute function.
Let $X = \{a, b\}$, $m = \{0_X, 1_X, A\}$, $m_1 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, m) and (X, m_1) are fuzzy m-spaces. Consider the identity function $i : (X, m) \to (X, m_1)$. Clearly i is fuzzy (m, m_1)-α-precontinuous function. Now every fuzzy set in (X, m_1) is fuzzy m_1-α-preopen set in (X, m_1). Consider the fuzzy set B defined by $B(a) = B(b) = 0.4$. Then $B \in Fm_1\alpha PO(X)$. Now $i^{-1}(B) = B \not\leq m\alpha int(mclB) = 0_X \Rightarrow B \notin Fm\alpha PO(X) \Rightarrow i$ is not fuzzy (m, m_1)-α-preirresolute function.

6. Fuzzy m-α-Preregular Space

In this section we introduce fuzzy m-α-preregular space in which space fuzzy m-α-preopen set and fuzzy m-open set coincide.

Definition 6.1. *A fuzzy* m-*space* (X, m) *is said to be fuzzy* m-α-*preregular if for each fuzzy* m-α-*preclosed set* F *in* X *and each fuzzy point* x_t *in* X *with* $x_tq(1_X \setminus F)$, *there exist a fuzzy* m-*open set* U *in* X *and a fuzzy* m-α-*preopen set* V *in* X *such that* x_tqU, $F \leq V$ *and* $U \not{q} V$.

Theorem 6.2. *For a fuzzy m-space (X, m) where m satisfies M-condition, the following statements are equivalent:*
(a) X is fuzzy m-α-preregular,
(b) for each fuzzy point x_t in X and each fuzzy m-α-preopen set U in X with x_tqU, there exists a fuzzy m-open set V in X such that $x_tqV \leq m\alpha pclV \leq U$,
(c) for each fuzzy m-α-preclosed set F in X, $\bigcap\{mclV : F \leq V, V \in Fm\alpha PO(X)\} = F$,
(d) for each fuzzy set G in X and each fuzzy m-α-preopen set U in X such that GqU, there exists a fuzzy m-open set V in X such that GqV and $m\alpha pclV \leq U$.
Proof. (a)⇒(b). Let x_t be a fuzzy point in X and U, a fuzzy m-α-preopen set in X with x_tqU. Then $x_t \notin 1_X \setminus U \in Fm\alpha PC(X)$. By (a), there exist a fuzzy m-open set V and a fuzzy m-α-preopen set W in X such that x_tqV, $1_X \setminus U \leq W$, $V \not{q} W$. Then $x_tqV \leq 1_X \setminus W \leq U \Rightarrow x_tqV \leq m\alpha pclV \leq m\alpha pcl(1_X \setminus W) = 1_X \setminus W \leq U$.
(b)⇒(a). Let F be a fuzzy m-α-preclosed set in X and x_t be a fuzzy point in X with $x_t \notin F$. Then $x_tq(1_X \setminus F) \in Fm\alpha PO(X)$. By (b), there exists a fuzzy m-open set V in X such that $x_tqV \leq m\alpha pclV \leq 1_X \setminus F$. Put $U = 1_X \setminus m\alpha pclV$. Then $U \in Fm\alpha PO(X)$ and x_tqV, $F \leq U$ and $U \not{q} V$.
(b)⇒(c). Let F be fuzzy m-α-preclosed set in X. Then $F \leq \bigwedge\{mclV : F \leq V, V \in Fm\alpha PO(X)\}$.
Conversely, let $x_t \notin F \in Fm\alpha PC(X)$. Then $F(x) < t \Rightarrow x_tq(1_X \setminus F)$ where $1_X \setminus F \in Fm\alpha PO(X)$. By (b), there exists a fuzzy m-open set U in X such that $x_tqU \leq m\alpha pclU \leq 1_X \setminus F$. Put $V = 1_X \setminus m\alpha pclU$. Then $F \leq V$ and $U \not{q} V \Rightarrow x_t \notin mclV \Rightarrow \bigwedge\{mclV : F \leq V, V \in Fm\alpha PO(X)\} \leq F$.
(c)⇒(b). Let V be any fuzzy m-α-preopen set in X and x_t be any fuzzy point in X with x_tqV. Then $V(x) + t > 1 \Rightarrow x_t \notin (1_X \setminus V)$ where $1_X \setminus V \in Fm\alpha PC(X)$. By (c), there exists $G \in Fm\alpha PO(X)$ such that $1_X \setminus V \leq G$ and $x_t \notin mclG$. Then there exists a fuzzy m-open set U in X with x_tqU, $U \not{q} G \Rightarrow U \leq 1_X \setminus G \leq V$ $\Rightarrow x_tqU \leq m\alpha pclU \leq m\alpha pcl(1_X \setminus G) = 1_X \setminus G \leq V$.
(c)⇒(d). Let G be any fuzzy set in X and U be any fuzzy m-α-preopen set in X with GqU. Then there exists $x \in X$ such that $G(x) + U(x) > 1$. Let $G(x) = t$. Then $x_tqU \Rightarrow x_t \notin 1_X \setminus U$ where $1_X \setminus U \in Fm\alpha PC(X)$. By (c), there exists $W \in Fm\alpha PO(X)$ such that $1_X \setminus U \leq W$ and $x_t \notin mclW \Rightarrow (mclW)(x) < t \Rightarrow x_tq(1_X \setminus mclW)$. Let $V = 1_X \setminus mclW$. Then V is fuzzy m-open set in X (as m satisfies M-condition) and $V(x) + t > 1 \Rightarrow V(x) + G(x) > 1 \Rightarrow VqG$ and $m\alpha pclV = m\alpha pcl(1_X \setminus mclV) \leq m\alpha pcl(1_X \setminus W) = 1_X \setminus W \leq U$.
(d)⇒(b). Obvious.

Note 6.3. It is clear from Theorem 6.2 that in a fuzzy m-α-preregular space, every fuzzy m-α-preclosed set is fuzzy m-closed and hence every fuzzy m-α-preopen set is fuzzy m-open. As a result, in a fuzzy m-α-preregular space, the collection of all fuzzy m-closed (resp., fuzzy m-open) sets and fuzzy m-α-preclosed (resp., fuzzy m-α-preopen) sets coincide.

Theorem 6.4. *If $f : (X, m) \to (Y, m_1)$ is fuzzy (m, m_1)-α-precontinuous function where Y is fuzzy m-α-preregular space, then f is fuzzy (m, m_1)-α-preirresolute function.*

Proof. Let x_t be a fuzzy point in X and V be any fuzzy m_1-α-preopen q-nbd of $f(x_t)$ in Y where Y is fuzzy m-α-preregular space. By Theorem 6.2 (a)⇒(b), there exists a fuzzy m_1-open set W in Y such that $f(x_t)qW \leq m_1\alpha pclW \leq V$. Since f is fuzzy (m, m_1)-α-precontinuous function, by Theorem 4.5, there exists $U \in Fm\alpha PO(X)$ with x_tqU and $f(U) \leq W \leq V$. By Theorem 5.3, f is fuzzy (m, m_1)-α-preirresolute function.

Let us now recall following definitions from [3] for ready references.

Definition 6.5. [3] *A collection $\mathcal{U}$ of fuzzy sets in a fuzzy minimal space (X, m) is said to be a fuzzy cover of X if $\bigcup \mathcal{U} = 1_X$. If, in addition, every member of $\mathcal{U}$ is fuzzy m-open, then $\mathcal{U}$ is called a fuzzy m-open cover of X.*

Definition 6.6. [3] *A fuzzy cover $\mathcal{U}$ of a fuzzy minimal space (X, m) is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 = 1_X$.*

Definition 6.7. [3] *A fuzzy m-space (X, m) is said to be fuzzy almost m-compact if every fuzzy m-open cover $\mathcal{U}$ of X has a finite proximate subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{mclU : U \in \mathcal{U}_0\}$ is again a fuzzy cover of X.*

Theorem 6.8. *If $f : (X, m) \to (Y, m_1)$ is a fuzzy (m, m_1)-α-precontinuous, surjective function and X is fuzzy m-α-preregular and fuzzy almost m-compact space, then Y is fuzzy almost m_1-compact space, where m_1 satisfies M-condition.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy m_1-open cover of Y. Then as f is fuzzy (m, m_1)-α-precontinuous function, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy cover of X by fuzzy m-α-preopen and hence by fuzzy m-open sets of X as X is fuzzy m-α-preregular space (by Note 6.3). Since X is fuzzy almost m-compact, there are finitely many members $U_1, U_2, ..., U_n$ of $\mathcal{U}$ such that $\bigcup_{i=1}^{n} mcl(f^{-1}(U_i)) = 1_X$. Since X is fuzzy m-α-preregular, by Note 6.3, $mclA = m\alpha pclA$ for all $A \in I^X$. So $1_X = \bigcup_{i=1}^{n} m\alpha pcl(f^{-1}(U_i)) \Rightarrow 1_Y = f(\bigcup_{i=1}^{n} m\alpha pcl(f^{-1}(U_i))) = \bigcup_{i=1}^{n} f(m\alpha pcl(f^{-1}(U_i))) \leq$

$\bigcup_{i=1}^{n} m_1 cl(f(f^{-1}(U_i)))$ (by Theorem 4.4 (a)⇒(f)) $\leq \bigcup_{i=1}^{n} m_1 cl(U_i) \Rightarrow \bigcup_{i=1}^{n} m_1 cl(U_i) = 1_Y \Rightarrow Y$ is fuzzy almost m_1-compact space.

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