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SOME ASPECTS OF NON –LINEAR DYNAMICAL SYSTEMS CARRYING NEAR-RING STRUCTURE

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Abstract: In this paper, we study large classes of nonlinear systems that admit a transfer function completely described by their input-output behavior. Our objective is to identify and analyze the aforementioned classes, which exhibit unique characteristics related to separable systems. We aim to fit certain examples of automata/dynamical systems with new concepts. We observe that for a Discrete System or Automaton, the state set forms a group. Furthermore, there exists a natural near-ring for Separable Systems. Some substructures of this near-ring are generated by $id.$ and a map from state set Q to itself if the state set Q is an abelian structure. It is interesting to note that Separable Systems themselves form a near-ring with respect to parallel and series connections. We discuss certain results and provide examples to validate separable systems and the outcomes. This paper offers a theoretical and practical overview of dynamical systems in our daily lives.

Keywords and Phrases: Automata, Separable system, Transfer Function, Near-rings.

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1. Introduction

A near-ring is precisely what is required to adequately describe the endomorphism formation of various mathematical structures. The initial step towards studying near-rings was axiomatic research conducted by L. E. Dickson in 1905. He revealed the existence of "fields with only one distributive law." Progress in the advancement of near-rings has naturally followed. Researchers such as Zassenhaus and Wielandt, who studied "near-fields" and "abstract near-rings," as well as scholars like Frohlich, Blackett, and Betsch, significantly developed the theory. Authors such as Beidleman and Clay have also contributed to its growth. Extensive research is currently underway on near-rings and dynamical systems, where system theory holds another level of importance [9]. The most famous near-ringers, including G. F. Pliz, W. M. I. Holecombe, J. L. Casti, G. Hofer, R. Lidl, K. C. Chowdhury, H. K. Saikia, et al., have conducted considerable work (approximately 1970-2000) on various aspects of near-rings, encompassing chain conditions, automata, reachability, feedback systems, etc. [19], [13]. In the period 2005-2012, mathematicians such as O. Moreno, D. Bollman, M. Alicia, Guangwa Xu, Yi Ming, Sangfa Y, Y. Feng, Ming Cao, etc., conducted extensive work on linear automata, near-rings, finite fields, etc. [22], [23]. Moreover, recently, authors like T. Boykett, G. Wendt, K. Srinivas, S. Raju, G. Koppe, H. Toutoungi, etc., have related their works to radical in automata, near-rings, regular semi-near-rings, generative recurrent neural networks, etc. (2014-2019) [14].
Dynamical systems emerge across a broad spectrum of physical, biological, and social contexts. However, progress in fields such as control theory, signal and image processing, data mining, mobile networks, robotics, nano-computing, etc., has historically lagged due to the underutilization of automata's full potential, creating a gap in technological advancements compared to other spheres of life. To address this disparity, the research fields of dynamical systems, near-rings, and automata have played pivotal roles.

An automaton, or non-linear discrete dynamical system, is a system whose states are drawn from a defined set. It receives either a single or a series of inputs—such as energy, information, or materials—from its environment and adjusts its state accordingly. These adjustments occur without direct human intervention, making an automaton a self-operating machine. Synonymous with 'robot,' examples include automatic watches, packing machines, printing machines, ATMs, etc.

In recent decades, due to its widespread significance across diverse areas including Electrical Engineering, Linguistics, Philosophy, Biology, Mathematics, Computer Science, etc., researchers have increasingly turned their attention to this field

[1], [17]. Dynamical systems, near-rings, reachability, etc., are highly captivating topics. Near-ring theory finds extensive applications across various subject areas, including digital computing, sequential machines, graph theory, and combinatorics.

This paper offers readers a comprehensive understanding and development of near-ring theory, automata, transfer functions, series and parallel connections, reachability, feedback systems linked to graph theory, automata rings, automata theory in number theory, genetic structures, etc. [4], [5], [10], [20]. Recently, A. Permatasari, et al. [18], M. Dutta, et al. [6], [7] and I. Jones, et al. [12] did related works.

2. Preliminaries

In this section, we present some fundamental definitions in automata that are utilized in the following sections.

Definition 2.1. Automata
The abstract definition of automata given by Gunter F. Pilz [10] *is a quintuple which is described as follows : An Automaton is a system of the type* $\sum = (Q, A, B, F, G)$, *where* Q *is a set of states,* A *is a set of inputs,* B *is a set of outputs,* $F : Q \times A \to Q$ *and* $G : Q \times A \to B$ *are functions usually known as State transition function and Output function respectively. The definition of automation is usually known as "local description" of* $\sum$*. If instead of a single input, a series of input signals it reveals the "global description" of the system. This leads up to a consistent input sequence of the type* $(a_i \in \mathbb{Z})$*. Sometimes it may be called a Discrete, Dynamical, Time-invariant system.*

Definition 2.2. Ring of Formal Power Series
Let R *be a ring and* $R[[x]] = \{(a_i) = (a_0, a_1, a_2, \ldots\ldots\ldots\ldots) : a_i \in \mathbb{R}\}$ *and observe* [2] *that with usual operations described as* $(a_i) + (b_i) = (a_i + b_i), (a_i) \cdot (b_i) = (c_i)$ *where* $c_i = \sum_{j+k=i} a_j b_k$ *is called Ring of Formal Power Series.*

Note: An arbitrary element of the ring of formal power series of the type: $\sum_{i=0}^{\infty} a_i x^i$

Definition 2.3. Linear System
$\sum = (Q, A, B, F, G)$ *is called Linear* [10], [22], [23] *if* Q, A, B *are vector spaces over some field* K *and* F, G *are linear maps on the product space* $Q \times A$*. In this case* F *and* G *can be decomposed into linear functions* $\alpha : Q \to Q, \beta : A \to Q, \gamma : Q \to B$ *and* $\delta : A \to B$*, such that* $F(q, a) = \alpha(q) + \beta(a), G(q, a) = \gamma(q) + \delta(a)$ *hold for all* $(q, a) \in Q \times A$*.*

Definition 2.4. Separable System
A system $\sum$ *(as in definition above) is called Separable* [10], [19] *if* Q, A, B *are groups (written additively, but not necessarily abelian) and if there are maps* α :

$Q \to Q, \gamma : Q \to B$ *and two homomorphisms* $\beta : A \to Q, \delta : A \to B$ *such that* $F(q,a) = \alpha(q) + \beta(a), G(q,a) = \gamma(q) + \delta(a), \forall q \in Q, a \in A$. *We then denote* $\sum$ *by* $(Q, A, B, \alpha, \beta, \gamma, \delta)$ *or simply by* $(\alpha, \beta, \gamma, \delta)$. $\sum$ *is called zero-symmetric if* $\alpha(0) = \gamma(0) = 0$.

Definition 2.5. Near-Rings
An algebraic system $(N, +, \bullet)$ *consisting of a non empty set* N *and two binary operations* " $+$ " *and* " $\bullet$ " *is called a Near-rings* [19] *if* $(N, +)$ *is a group (not necessarily abelian),* $(N, \bullet)$ *is a semi group and* $\left(n' + n''\right) n = n'n + n''n$ *holds for all* $n, n', n'' \in N$, N *is said to be zero symmetric if* $n0 = 0$, $\forall n \in N$. *So a ring, is a Near-ring with* $(N, +)$ *abelian and* $n\left(n' + n''\right) = nn' + nn''$ *holds for all* $n, n', n'' \in N$.

Definition 2.6. *Let* X *be any set containing* 0 *(a special element).* $L(X)$ *be the set* [10] *of all sequence* $(x_i)_{i \in \mathbb{Z}}$ *where for some* $k \in \mathbb{Z}, x_i = 0, \forall i < k$, *i.e.* $L(X) = \{(....x_i......) : x_i \in X \text{ and } x_k = 0 \text{ for } i < k\} = \{(0, 0, ...x_k, x_{k+1}......) : x_{k+i}, 0 \in X\}$.

We now go through the routine work and observe the following.

Observation 1: If $(A, +)(A \equiv inputset)$ is a group then $(L(A), +)$ is a group w.r.t. " $+$ " such that $(a_i) + (b_i) = (a_i + b_i)$ [21].

Notation 1: $A^{\mathbb{Z}}$ is the set of all maps from $\mathbb{Z}$ to A i.e. $A^{\mathbb{Z}} = \{f : \mathbb{Z} \to A\}$.

Observation 2: $L(A)$ is isomorphic to a subgroup of $A^{\mathbb{Z}}$ and in the sense that $L(A)$ is a subgroup of $A^{\mathbb{Z}}$.
Now we see how these two maps F and G can be decomposed into four maps $\alpha, \beta, \gamma, \delta$ inheriting the linear character, though it appears that the linearity has not played any role. In other words, such types of decompositions are possible not only for linear systems; it is true even if the maps under discussion are not linear. This fact leads us to the notion that can be interpreted as a separable system.

3. Main Works

In this section, we present certain examples of automata/discrete dynamical systems with new concepts. Before delving into a detailed discussion of our objectives, we would like to elucidate our motivation through the use of some examples based on our observations.

Example 3.1. Compound interest problem as an example of dynamical system:

We can show that the compound interest problem is a case of a dynamical system. Consider the set Q the states as a collection P of possible capitals, A the set of inputs as the set of Y years, B the set of outputs as the set I of re-

spective interest, the transition function F as the amount $A : P \times Y \to P$ such that $A(p,n) = p\left(1 + \frac{r}{100}\right)^n, \forall p \in P, n \in Y, r \in I$, and output function G as the compound interest $B : P \times Y \to I$ such that $B(p,n) = A(p,n) - P$.
There are some aspects to study regarding this kind of problems in terms of dynamical systems.

Example 3.2. Consider $Q = A = B = (\mathbb{R}, +), F(q,a) = q^2 + \sin q + 3a$, $G(q,a) = e^q - 1 + \pi a$. Then $(Q, A, B, \alpha, \beta, \gamma, \delta)$ is a separable systems.
Since $Q = A = B = (\mathbb{R}, +)$ are groups (written additively, but not necessarily abelian) and there are two maps $\alpha : Q \to Q$ defined by $\alpha(q) = q^2 + \sin q$ and $\gamma : Q \to B$ defined by $\gamma(q) = e^q - 1$ and two homomorphisms $\beta : A \to Q$ defined by $\beta(a) = 3a$ and $\delta : A \to B$ by $\delta(a) = \pi a$ hold, $\forall q \in Q$ and $a \in A$.
The system is non-linear.
Since $F(r(q_1, a_1) + s(q_2, a_2)) = F((rq_1, ra_1) + (sq_2, sa_2))$
$= F(rq_1 + sq_2, ra_1 + sa_2) = (rq_1 + sq_2)^2 + \sin(rq_1 + sq_2) + 3(ra_1 + sa_2)$
$= r^2 q_1^2 + 2rq_1 sq_2 + s^2 q_2^2 + \sin(rq_1)\cos(sq_2) + \cos(rq_1)\sin(sq_2) + 3ra_1 + 3sa_2$.
$rF(q_1, a_1) + sF(q_2, a_2) = r(q_1^2 + \sin q_1 + 3a_1) + s(q_2^2 + \sin q_2 + 3a_2)$
$= rq_1^2 + r\sin q_1 + 3ra_1 + sq_2^2 + s\sin q_2 + 3sa_2$, where r, s are any two scalars.
Therefore $F(r(q_1, a_1) + s(q_2, a_2)) \neq rF(q_1, a_1) + sF(q_2, a_2), q_1, q_2 \in Q, a_1, a_2 \in A$.
This example shows the zero symmetric property, Since $\alpha(0) = 0^2 + \sin 0 = 0$, $\gamma(0) = e^0 - 1 = 0$.

Example 3.3. Consider $Q = B = (\mathbb{R}, +)$, $A = (\{f | f : \mathbb{R} \to \mathbb{R},\ f$ is a continuous function $\}, +)$,
$F : Q \times A \to Q$ defined by $F(q,a) = log(1 - q) + \int_0^k a(x)\,dx$, k is a fixed constant, $G : Q \times A \to B$ defined by $G(q,a) = Sin^{-1}q + \int_{-k}^k a(x)\,dx$. Then $(Q, A, B, \alpha, \beta, \gamma, \delta)$ is a separable systems. Since $Q = B = (\mathbb{R}, +), A = (\{f | f : \mathbb{R} \to \mathbb{R}, f$ is a continuous function$\}, +)$ are groups and there are two maps $\alpha : Q \to Q$ defined by $\alpha(q) = log(1 - q)$ and $\gamma : Q \to B$ defined by $\gamma(q) = Sin^{-1}q$ and two homomorphisms $\beta : A \to Q$ defined by $\beta(a) = \int_0^k a(x)\,dx, \forall a \in A$. Since $a_1, a_2 \in A$, then $\beta(a_1 + a_2) = \int_0^k (a_1 + a_2)(x)\,dx = \int_0^k (a_1(x) + a_2(x))\,dx = \int_0^k a_1(x)\,dx + \int_0^k a_2(x)\,dx = \beta(a_1) + \beta(a_2)$ and $\delta : A \to B$ by $\delta(a) = \int_{-k}^k a(x)\,dx, \forall a \in A$. Since $a_1, a_2 \in A$, then $\delta(a_1 + a_2) = \int_{-k}^k (a_1 + a_2)(x)\,dx = \int_{-k}^k (a_1(x) + a_2(x))\,dx = \int_{-k}^k a_1(x)\,dx + \int_{-k}^k a_2(x)\,dx = \delta(a_1) + \delta(a_2)$.
The system is non-linear.
Since $F((q_1, a_1) + (q_2, a_2)) = F(q_1 + q_2, a_1 + a_2) = log(1 - (q_1 + q_2)) + \int_0^k (a_1 + a_2)(x)\,dx = log(1 - q_1 - q_2) + \int_0^k a_1(x)\,dx + \int_0^k a_2(x)\,dx$. $F(q_1, a_1) + F(q_2, a_2) =$

$log\,(1-q_1)+\int_0^k a_1\,(x)\,dx+log\,(1-q_2)+\int_0^k a_2\,(x)\,dx = log\,((1-q_1)\,(1-q_2))+\int_0^k a_1\,(x)\,dx+\int_0^k a_2\,(x)\,dx = log\,(1-q_1-q_2+q_1q_2)+\int_0^k a_1\,(x)\,dx+\int_0^k a_2\,(x)\,dx$. So $F\,((q_1,a_1)+(q_2,a_2)) \neq F\,(q_1,a_1)+F\,(q_2,a_2)$. Again, $G\,((q_1,a_1)+(q_2,a_2)) = G\,(q_1+q_2,a_1+a_2) = Sin^{-1}\,(q_1+q_2)+\int_{-K}^k (a_1+a_2)\,(x)\,dx = Sin^{-1}\,(q_1+q_2)+\int_{-k}^k a_1\,(x)\,dx+\int_{-k}^k a_2\,(x)\,dx$. $G\,(q_1,a_1)+G\,(q_2,a_2) = Sin^{-1}\,(q_1)+\int_{-k}^k a_1\,(x)\,dx+Sin^{-1}\,(q_2)+\int_{-k}^k a_2\,(x)\,dx = Sin^{-1}\,(q_1)+Sin^{-1}\,(q_2)+\int_{-k}^k a_1\,(x)\,dx+\int_{-k}^k a_2\,(x)\,dx$. So, $G\,((q_1,a_1)+(q_2,a_2)) \neq G\,(q_1,a_1)+G\,(q_2,a_2)$.

Example 3.4. Consider $Q=A=(\mathbb{R}^2,+)\,,B=(\mathbb{R}^3,+)\,,F:Q\times A\to Q$ defined by $F\,((q_1,q_2)\,,(a_1,a_2)) = (q_1+q_2+3a_1+4a_2,q_1^2+a_1) = ((q_1+q_2,q_1^2)\,,(3a_1+4a_2,a_1))\,,$ $G:Q\times A\to B$ defined by $G\,((q_1,q_2)\,,(a_1,a_2)) = (q_1+a_1+a_2+1,q_2+1,0)$ $=(q_1+1,q_2+1,0)+(a_1+a_2,0,0)$, Then $(Q,A,B,\alpha,\beta,\gamma,\delta)$ is a separable systems.

Since $Q=A=(\mathbb{R}^2,+)\,,B=(\mathbb{R}^3,+)$ are groups and there are two maps $\alpha: Q\to Q$ defined by $\alpha\,(q_1,q_2)=(q_1+q_2,q_1^2)\,,\gamma:Q\to B$ defined by $\gamma\,(q_1,q_2)=(q_1+1,q_2+1,0)$ and two homomorphisms $\beta:A\to Q$ defined by $\beta\,(a_1,a_2)=(3a_1+4a_2,a_1)$ hold, $\forall\,(a_1,a_2)\,,(b_1,b_2)\in A$

Since

$\beta\,((a_1,a_2)+(b_1,b_2)) = \beta\,((a_1+b_1)\,,(a_2+b_2)) = (3\,(a_1+b_1)+4\,(a_2+b_2)\,,(a_1+b_1))$
$=(3a_1+4a_2+3b_1+4b_2,a_1+b_1) = (3a_1+4a_2,a_1)+(3b_1+4b_2,b_1)$
$=\beta\,(a_1,a_2)+\beta\,(b_1,b_2)$.

$\delta:A\to B$ by $\delta\,(a_1,a_2)=(a_1+a_2,0,0)$ hold, $\forall\,(a_1,a_2)\in A$.

Since $\delta\,((a_1,a_2)+(b_1,b_2)) = \delta\,((a_1+b_1)\,,(a_2+b_2)) = (a_1+b_1+a_2+b_2,0,0)$
$=(a_1+a_2+b_1+b_2,0,0) = (a_1+a_2,0,0)+(b_1+b_2,0,0) = \delta\,(a_1,a_2)+\delta\,(b_1,b_2)$.

The Separable system is non-linear. Since

$F\,((0,1,)\,,(0,2)) = (0+1+3.0+4.2,0^2+0)$
$=(9,0)\,,F\,((1,0)\,,(2,0)) = (1+0+3.2+4.0,1^2+2) = (7,3)$
$F\,(((0,1)\,,(0,2))+((1,0)\,,(2,0))) = F\,((0,1)+(1,0)\,,(0,2)+(2,0))$
$=F\,((0+1,1+0)\,,(0+2,2+0))$
$=F\,((1,1,)\,,(2,2)) = (1+1+3.2+4.2,1^2+2) = (16,3)$.

So, $F\,(((0,1)\,,(0,2))+((1,0)\,,(2,0))) \neq F\,((0,1)\,,(0,2))+F\,((1,0)\,,(2,0))$.

Similarly, $G\,(((0,1)\,,(0,2))+((1,0)\,,(2,0))) \neq G\,((0,1)\,,(0,2))+G\,((1,0)\,,(2,0))$.

Example 3.5. Consider $Q=A=B=(\mathbb{C},+)\,,F:Q\times A\to Q$ defined by $F\,(q,a)=(q+1)+\bar{a}$, where $\bar{a}=x-iy$ is a complex conjugate of $a=x+iy\in A,G:Q\times A\to B$ defined by $G\,(q,a)=q^2+3a$. Then $(Q,A,B,\alpha,\beta,\gamma,\delta)$ is a separable systems.

Since, $Q=A=B=(\mathbb{C},+)$ are groups and there are two maps $\alpha:Q\to Q$

defined by $\alpha(q) = q + 1$ and $\gamma : Q \to B$ defined by $\gamma(q) = q^2$ and two homomorphisms $\beta : A \to Q$ defined by $\beta(a) = \bar{a}, \forall a \in A$. Since $a_1, a_2 \in A$. Then
$\beta(a_1 + a_2) = \beta(x_1 + iy_1 + x_2 + iy_2)$
$= \beta((x_1 + x_2) + i(y_1 + y_2)) = (x_1 + x_2) - i(y_1 + y_2)$
$= x_1 - iy_1 + x_2 - iy_2 = \beta(a_1) + \beta(a_2)$,
and $\delta : A \to B$ by $\delta(a) = 3a, \forall a \in A$. Since, $a_1, a_2 \in A$,
Then $\delta(a_1 + a_2) = \delta(x_1 + iy_1 + x_2 + iy_2) = \delta((x_1 + x_2) + i(y_1 + y_2))$
$= 3\{(x_1 + x_2) + i(y_1 + y_2)\} = 3(x_1 + iy_1) + 3(x_2 + iy_2) = \delta(a_1) + \delta(a_2)$.
The separable system is non-linear. $F((q_1, a_1) + (q_2, a_2)) = F(q_1 + q_2, a_1 + a_2)$
$= q_1 + q_2 + 1 + \overline{a_1 + a_2} = q_1 + q_2 + 1 + \overline{a_1} + \overline{a_2}$,
$F(q_1, a_1) + F(q_2, a_2) = q_1 + 1 + \overline{a_1} + q_2 + 1 + \overline{a_2} = q_1 + q_2 + 2 + \overline{a_1} + +\overline{a_2}$
So, $F((q_1, a_1) + (q_2, a_2)) \neq F(q_1, a_1) + F(q_2, a_2)$.
Similarly, $G((q_1, a_1) + (q_2, a_2)) \neq G(q_1, a_1) + G(q_2, a_2)$.

Example 3.6. Consider $Q = (\mathbb{R}, +), A = B = (\mathbb{Z}, +), F : Q \times A \to Q$ defined by $F(q, a) = q^3 + 3a, G : Q \times A \to B$ defined by $G(q, a) = [q] + a$,where $[q]$ denote the greatest integer $\leq q$. Then $(Q, A, B, \alpha, \beta, \gamma, \delta)$ is a separable systems.

Since $Q = (\mathbb{R}, +), A = B = (\mathbb{Z}, +)$ are groups and there are two maps $\alpha : Q \to Q$ defined by $\alpha(q) = q^3$ and $\gamma : Q \to B$ defined by $\gamma(q) = [q]$ and two homomorphisms $\beta : A \to Q$ defined by $\beta(a) = 3a, \forall a \in A$. Since $a_1, a_2 \in A$, then
$\beta(a_1 + a_2) = 3(a_1 + a_2) = 3a_1 + 3a_2 = \beta(a_1) + \beta(a_2)$
and $\delta : A \to B$ by $\delta(a) = a, \forall a \in A$. Since, $a_1, a_2 \in A$, then $\delta(a_1 + a_2) = a_1 + a_2 = \delta(a_1) + \delta(a_2)$.
The separable system is non-linear. Let $q_1 = 3.6, q_2 = 3.7, a_1 = 4, a_2 = 5$,
$G((q_1, a_1) + (q_2, a_2)) = G(q_1 + q_2, a_1 + a_2) = G(3.6 + 3.7, 4 + 5) = G(7.3, 9) = [7.3] + 9 = 7 + 9 = 16$.
$G(q_1, a_1) + G(q_2, a_2) = G(3.6, 4) + G(3.7, 5) = [3.6] + 4 + [3.7] + 5 = 3 + 4 + 3 + 5 = 15$.
So, $G((q_1, a_1) + (q_2, a_2)) \neq G(q_1, a_1) + G(q_2, a_2)$.
Similarly, $F((q_1, a_1) + (q_2, a_2)) \neq F(q_1, a_1) + F(q_2, a_2)$.

Example 3.7. Consider $Q = (\mathbb{Z}_2, +_2), A = (\mathbb{Z}, +), B = (\mathbb{Z}_3, +), F : Q \times A \to Q$ defined by $F(q, a) = 1 + \begin{cases} [0] & \text{,if } a \text{ is even} \\ [1] & \text{,if } a \text{ is odd} \end{cases}, \forall a \in A$
$G : Q \times A \to B$ defined by $G(q, a) = 2 + a \pmod 3, \forall a \in A$.
Then $(Q, A, B, \alpha, \beta, \gamma, \delta)$ is a separable systems.
Since, $Q = (\mathbb{Z}_2, +_2), A = (\mathbb{Z}, +), B = (\mathbb{Z}_3, +_3)$ are groups and there are two maps $\alpha : Q \to Q$ defined by $\alpha(q) = 1, \forall q \in \mathbb{Z}$ and $\gamma : Q \to B$ defined by $\gamma(q) = 2$ and

two homomorphisms $\beta : A \to Q$ defined by $\beta(a) = \begin{cases} [0] & \text{,if } a \text{ is even} \\ [1] & \text{,if } a \text{ is odd} \end{cases}, \forall a \in A.$
Since, $a_1, a_2 \in A$ if $a_1 + a_2$ is even, then $\beta(a_1 + a_2) = [0] = \beta(a_1) + \beta(a_2)$,
if $a_1 + a_2$ is odd, then $\beta(a_1 + a_2) = [1] = \beta(a_1) + \beta(a_2)$
and $\delta : A \to B$ by $\delta(a) = a \pmod 3, \forall a \in A, \because a_1, a_2 \in A$.
Then $\delta(a_1 + a_2) = (a_1 + a_2) \pmod 3$
$= [a_1 \pmod 3 + a_2 \pmod 3] \pmod 3 = \delta(a_1) + \delta(a_2)$.
The separable system is non-linear. Since, α, γ are non-linear.
$\alpha(1+1) = \alpha(2) = 1$,
but $\alpha(1) + \alpha(1) = 1 + 1 = 0$, So $\alpha(1+1) \neq \alpha(1) + \alpha(1)$.
Similarly, $\gamma(1+1) = \gamma(2) = 2$, but $\gamma(1) + \gamma(1) = 1$, So $\gamma(1+1) \neq \gamma(1) + \gamma(1)$.

Definition 3.1. Extension
The map $\alpha : Q \to Q$ i.e. $Q \to \alpha(q), \forall q \in Q, Q \subseteq \alpha(Q)$ can be extended [10] *to a map $\alpha' : L(Q) \to L(Q)$ defined by $\alpha'\left(\sum_{i \geq k} q_i z^i\right) = \sum_{i \geq k} \alpha'(q_i) z^i$. Also, we can extend a map β to a map $\beta' : L(A) \to L(Q)$ defined by $\beta'\left(\sum_{i \geq k} a_i z^i\right) = \sum_{i \geq k} \beta'(a_i) z^i$ and so on.*

Proposition 3.1. *For a separable system $\sum = (Q, A, B, \alpha, \beta, \gamma, \delta)$ the map $-\alpha' + \bar{z} : L(Q) \to L(Q)$ is always bijective. Also $-\alpha' + \bar{z}$ is zero symmetric iff α' is zero symmetric. In these, z bar is shift to the left operator* [10].

Theorem 3.1. *In a separable system $\sum = (Q, A, B, \alpha, \beta, \gamma, \delta)$ the following relations hold, if the first non-zero input arrives at time $k \in \mathbb{Z}$ in which the system is in state* 0. [10]
(i) $\sum_{i \geq k} q_i z^i = \left(-\alpha' + \bar{z}\right)^{-1} \beta'\left(\sum_{i \geq k} a_i z^i\right)$,
(ii) $\sum_{i \geq k} b_i z^i = \{\gamma'\left(-\alpha' + \bar{z}\right)^{-1} \beta' + \delta'\}\left(\sum_{i \geq k} a_i z^i\right)$.

Definition 3.2. *In a separable system $\sum = (\alpha, \beta, \gamma, \delta)$, the function,*
$f_{\sum} = \gamma'\left(-\alpha' + \bar{z}\right)^{-1} \beta' + \delta' : L(A) \to L(B)$ is called the transfer function of $\sum$. If α', γ' are zero-symmetric then $f_{\sum}$ is zero-symmetric [10].
$f_{\sum}$ Completely characterizes the input-output behavior of $\sum$ if $\sum$ starts in state 0. *If $\sum$ starts in state $q \neq 0$ which is reachable from* 0 *by means of an input sequence $a_1, a_2, a_3, ...a_r$. We simply start at time $k - r$ and then in state* 0. *Since it doesn't make much sense to start from non-reachable states, $f_{\sum}$ "characterizes" $\sum$ itself. We call zero-symmetric if this applies to $f_{\sum}$.*

Definition 3.3. Series and Parallel connection
Two systems $\sum_1 = (Q_1, A_1, B_1, F_1, G_1)$ and $\sum_2 = (Q_2, A_2, B_2, F_2, G_2)$ are connected. The series connection $\sum_1 \ddagger \sum_2$ requires $B_1 = A_2$. Then $\sum_1 \ddagger \sum_2 =$

$(Q_1 \times Q_2, A_1, B_2, F, G)$ with $F((q_1, q_2), a) = (F_1(q_1, a), F_2(q_2, G_1(q_1, a)))$ and $G((q_1, q_2), a) = G_2(q_2, G_1(q_1, a))$.
The parallel connections $\sum_1 \| \sum_2$ works with $A_1 = A_2 = A, B_1 = B_2 = B$ and gives $\sum_1 \| \sum_2 = \left(Q_1 \times Q_2, A, B, F', G'\right)$ with $F'((q_1, q_2), a) = (F_1(q_1, a), F_2(q_2, a))$ and $G'((q_1, q_2), a) = G_1(q_1, a) + G_2(q_2, a)$. [10], [15].

Proposition 3.2. *If $\sum_1$ and $\sum_2$ are separable systems, then $\sum_1 \ddagger \sum_2$ and $\sum_1 \| \sum_2$ (if the output groups are abelian) are also separable systems. If $\sum_1$ and $\sum_2$ are zero-symmetric, then $\sum_1 \ddagger \sum_2$ and $\sum_1 \| \sum_2$ are also zero-symmetric.* [10], [15].
Proof. Given that $\sum_i = (Q_i, A_i, B_i, F_i, G_i), (i = 1, 2)$ are separable. So we have Q_i, A_i, B_i are groups, there are maps $\alpha_i : Q_i \to Q_i, \gamma_i : Q_i \to B_i$ and two homomorphisms $\beta_i : A_i \to Q_i, \delta_i : A_i \to B_i, \forall q_i \in Q_i, a \in A_i, i = 1, 2.$ such that $F_i(q_i, a_i) = \alpha_i(q_i) + \beta_i(a_i), G_i(q_i, a_i) = \gamma_i(q_i) + \delta_i(a_i)$, $\sum_1 \ddagger \sum_2 = (Q_1 \times Q_2, A_1, B_2, F, G)$ with $F((q_1, q_2), a) = (F_1(q_1, a), F_2(q_2, G_1(q_1, a)))$ and $G((q_1, q_2), a) = G_2(q_2, G_1(q_1, a))$ is a separable system. Since $Q_1 \times Q_2, A_1, B_2$ are groups (not necessarily abelian), there are maps
$\alpha : Q_1 \times Q_2 \to Q_1 \times Q_2, \gamma : Q_1 \times Q_2 \to B_2$.
Also, $F((q_1, q_2), a) = (F_1(q_1, a), F_2(q_2, G_1(q_1, a)))$
$= (\alpha_1(q_1) + \beta_1(a), \alpha_2(q_2) + \beta_2(G_1(q_1, a))), (\because \sum_1$ and $\sum_2$ are separable systems.)
$= (\alpha_1(q_1) + \beta_1(a), \alpha_2(q_2) + \beta_2(\gamma_1(q_1) + \delta_1(a))), (\because \sum_1$ is a separable systems.)
$= (\alpha_1(q_1) + \beta_1(a), \alpha_2(q_2) + \beta_2(\gamma_1(q_1)) + \beta_2(\delta_1(a))), (\because \beta_2$ is a homomorphism in $\sum_2)$
$= (\alpha_1(q_1), \alpha_2(q_2) + \beta_2\gamma_1(q_1)) + (\beta_1(a), \beta_2(\delta_1(a)))$,
and $G((q_1, q_2), a) = G_2(q_2, G_1(q_1, a))$
$= \gamma_2(q_2) + \delta_2(G_1(q_1, a)), (\because \sum_2$ is a separable systems.)
$= \gamma_2(q_2) + \delta_2(\gamma_1(q_1) + \delta_1(a)), (\because \sum_1$ is a separable systems.)
$=\gamma_2(q_2) + \delta_2(\gamma_1(q_1)) + \delta_2(\delta_1(a)), (\because \delta_2$ is a homomorphism.)
Also $\beta : A_1 \to Q_1 \times Q_2$ defined by $\beta(a) = (\beta_1(a), \beta_2\delta_1(a))$ is a homomorphism.
Since $\forall a, b \in A_1, \beta(a + b) = (\beta_1(a + b), \beta_2\delta_1(a + b))$
$= (\beta_1(a) + \beta_1(b), \beta_2\delta_1(a) + \beta_2\delta_1(b)), (\because \beta_1, \beta_2$ are homomorphism)
$=(\beta_1(a), \beta_2\delta_1(a)) + (\beta_1(b), \beta_2\delta_1(b)) = \beta(a) + \beta(b)$
$\delta : A_1 \to B_2$ defined by $\delta(a) = \delta_2\delta_1(a)$ is a homomorphism.
Since, $\forall a, b \in A_1, \delta(a + b) = \delta_2\delta_1(a + b) = \delta_2(\delta_1(a) + \delta_1(b)), (\because \delta_1$ is a homomorphism in $\sum_1$)
$=\delta_2\delta_1(a) + \delta_2\delta_1(b)$, (Since, δ_2 is a homomorphism in $\sum_2$)$=\delta(a) + \delta(b)$. Hence $\sum_1 \ddagger \sum_2$ forms a separable system. Again
$\sum_1 \| \sum_2 = \left(Q_1 \times Q_2, A, B, F', G'\right)$ with $F'((q_1, q_2), a) = (F_1(q_1, a), F_2(q_2, a))$

and $G'((q_1,q_2),a) = G_1(q_1,a) + G_2(q_2,a)$ is a separable system.
Since, $F'((q_1,q_2),a) = (F_1(q_1,a), F_2(q_2,a))$
$= (\alpha_1(q_1) + \beta_1(a), \alpha_2(q_2) + \beta_2(a)) = (\alpha_1(q_1), \alpha_2(q_2)) + (\beta_1(a), \beta_2(a))$ and
$G'((q_1,q_2),a) = G_1(q_1,a) + G_2(q_2,a) = \gamma_1(q_1) + \delta_1(a) + \gamma_2(q_2) + \delta_2(a)$
$= (\gamma_1(q_1) + \gamma_2(q_2)) + (\delta_1 + \delta_2)(a)$, [If B is abelian].
Therefore $\sum_1 \| \sum_2$ forms a separable system.
$\sum_i = (Q_i, A_i, B_i, F_i, G_i), (i = 1,2)$ are zero-symmetric then we have
$\alpha_i(0) = 0, \gamma_i(0) = 0$. So, $\sum_1 \ddagger \sum_2$ and $\sum_1 \| \sum_2$ are also zero-symmetric. As $\alpha : Q_1 \times Q_2 \to Q_1 \times Q_2$, So $\alpha(0,0) = (\alpha_1(0), \alpha_2(0)) = (0,0), \gamma : Q_1 \times Q_2 \to B_2$. So $\gamma(0,0) = \gamma_1(0) = 0$.

Example 3.8. We verify the above result with an example
We consider $\sum_1 = (Q_1, A_1, B_1, F_1, G_1) = (\mathbb{R}, \mathbb{Z}, \mathbb{Z}, F_1, G_1)$ where $Q_1 = (\mathbb{R}, +), A_1 = (\mathbb{Z}, +), B_1 = (\mathbb{Z}, +), F_1 : Q_1 \times A_1 \to Q_1$, i.e. $F_1 : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ defined by $F_1(q_1, a_1) = q_1^3 + 3a_1, G_1 : Q_1 \times A_1 \to B_1, i.e. G_1 : \mathbb{R} \times \mathbb{Z} \to \mathbb{Z}$ defined by $G_1(q_1, a_1) = [q_1] + a_1$,Where $[q_1]$ denote the greatest integer $\le q_1$,
Then $(Q_1, A_1, B_1, \alpha, \beta, \gamma, \delta)$ is a separable systems.
Again, we consider, $\sum_2 = (Q_2, A_2, B_2, F_2, G_2) = (\mathbb{Z}_2, \mathbb{Z}, \mathbb{Z}_3, F_2, G_2)$ where $Q_2 = (\mathbb{Z}_2, +), A_2 = (\mathbb{Z}, +), B_2 = (\mathbb{Z}_3, +_3), F_2 : Q_2 \times A_2 \to Q_2, i.e. F_2 : \mathbb{Z}_2 \times \mathbb{Z} \to \mathbb{Z}_2$
defined by $F_2(q_2, a_2) = 1 + \begin{cases} [0] & \text{,if } a_2 \text{ is even} \\ [1] & \text{,if } a_2 \text{ is odd} \end{cases}$ $\forall a_2 \in A_2$ and $G_2 : Q_2 \times A_2 \to B_2$, i.e. $G_2 : \mathbb{Z}_2 \times \mathbb{Z} \to \mathbb{Z}_3$ defined by $G_2(q_2, a_2) = 2 + a_2 (mod 3)$. Then $(Q_2, A_2, B_2, \alpha, \beta, \gamma, \delta)$ is a separable systems.
Now, consider $\sum_1 \ddagger \sum_2 = (Q_1 \times Q_2, A_1, B_2, F, G) = (\mathbb{R} \times \mathbb{Z}_2, \mathbb{Z}, \mathbb{Z}_3, F, G)$,
$F : (Q_1 \times Q_2) \times A_1 \to Q_1 \times Q_2$, i.e.$F : (\mathbb{R} \times \mathbb{Z}_2) \times \mathbb{Z} \to \mathbb{R} \times \mathbb{Z}_2$ with
$F((q_1,q_2),a) = (F_1(q_1,a), F_2(q_2, G_1(q_1,a))) = (q_1^3 + 3a_1, F_2(q_2, [q_1] + a_1))$
$= (q_1^3 + 3a_1,\ 1 + \begin{cases} [0] & \text{,if } [q_1] + a_1 \text{ is even} \\ [1] & \text{,if } [q_1] + a_1 \text{ is odd} \end{cases}$
and $G : (Q_1 \times Q_2) \times A_1 \to B_2$, i.e. $G : (\mathbb{R} \times \mathbb{Z}_2) \times \mathbb{Z} \to \mathbb{Z}_3$ with $G((q_1,q_2),a_1) = G_2(q_2, G_1(q_1,a_1))$
$= G_2(q_2, [q_1] + a_1) = 2 + ([q_1] + a_1)(mod 3)$. Then $\sum_1 \ddagger \sum_2 = (Q_1 \times Q_2, A_1, B_2, F, G) = (\mathbb{R} \times \mathbb{Z}_2, \mathbb{Z}, \mathbb{Z}_3, F, G)$ is a separable system. Since $Q_1 \times Q_2 = (\mathbb{R} \times \mathbb{Z}_2, +_2), A_1 = (\mathbb{Z}, +), B_2 = (\mathbb{Z}_3, +_3)$ are groups (not necessarily abelian), there are maps $\alpha : Q_1 \times Q_2 \to Q_1 \times Q_2$ i.e. $\alpha : \mathbb{R} \times \mathbb{Z}_2 \to \mathbb{R} \times \mathbb{Z}_2$ defined by $\alpha(q_1,q_2) = (q_1^3, 1), \gamma : Q_1 \times Q_2 \to B_2$ i.e. $\gamma : \mathbb{R} \times \mathbb{Z}_2 \to \mathbb{Z}_3$ defined by $\gamma(q_1,q_2) = 2$
and two homomorphisms $\beta : A_1 \to Q_1 \times Q_2$ i.e. $\beta : \mathbb{Z} \to \mathbb{R} \times \mathbb{Z}_2$

by $\beta(a_1) = \left(3a_1, \begin{cases} [0] & \text{,if } [q_1]+a_1 \text{ is even} \\ [1] & \text{,if } [q_1]+a_1 \text{ is odd} \end{cases}\right)$.
Since $a_1, a_2 \in \mathbb{Z}$ if $a_1 + a_2$ is even, then $\beta(a_1+a_2) = (3(a_1+a_2), 0;$ if $[q_1]$ is even $) = (3a_1, 0;$ if $[q_1]$ is even $) + (3a_2, 0;$ if $[q_1]$ is even $) = \beta(a_1) + \beta(a_2)$,
If $a_1 + a_2$ is odd Then $\beta(a_1+a_2) = (3(a_1+a_2), 1;$ if $[q_1]$ is odd$) = (3a_1, 1;$ if $[q_1]$ is odd$) + (3a_2, 1;$ if $[q_1]$ is odd $) = \beta(a_1) + \beta(a_2)$,
and $\delta : A_1 \to B_2$ i.e. $\delta : \mathbb{Z} \to \mathbb{Z}_3$ by $\delta(a_1) = ([q_1] + a_1)(\mod 3), \forall a_1 \in \mathbb{Z}$. Since $a_1, a_2 \in A = \mathbb{Z}$ Then $\delta(a_1+a_2) = ([q_1] + (a_1+a_2))(\mod 3)$
$= ([q_1]+a_1)(\mod 3) + ([q_1]+a_2)(\mod 3) = \delta(a_1) + \delta(a_2)$
Again, consider $\sum_1 = (Q_1, A_1, B_1, F_1, G_1) = (\mathbb{Z}_2, \mathbb{Z}, \mathbb{Z}_3, F_1, G_1)$ where
$Q_1 = (\mathbb{Z}_2, +_2), A_1 = (\mathbb{Z}, +), B_1 = (\mathbb{Z}_3, +_3), F_1 : Q_1 \times A_1 \to Q_1, i.e. F_1 : \mathbb{Z}_2 \times \mathbb{Z} \to \mathbb{Z}_2$
defined by $F_1((q_1, a_1)) = 1 + \begin{cases} [0] & \text{,if } a_1 \text{ is even} \\ [1] & \text{,if } a_1 \text{ is odd} \end{cases}$,
$G_1 : Q_1 \times A_1 \to B_1, i.e. G_1 : \mathbb{Z}_2 \times \mathbb{Z} \to \mathbb{Z}_3$ defined by $G_1(q_1, a_1) = 2 + a_1 (mod 3)$,
Then $(Q_1, A_1, B_1, \alpha, \beta, \gamma, \delta)$ is a separable systems.
And consider $\sum_2 = (Q_2, A_2, B_2, F_2, G_2) = (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_3, F_2, G_2)$ where $Q_2 = (\mathbb{Z}, +), A_2 = (\mathbb{Z}, +), B_2 = (\mathbb{Z}_3, +_3), F_2 : Q_2 \times A_2 \to Q_2, i.e. F_2 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $F_2((q_2, a_2)) = q_2^2 + a_2, \forall a_2 \in A_2$
and $G_2 : Q_2 \times A_2 \to B_2$, i.e. $G_2 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_3$ defined by $G_2(q_2, a_2) = q_2 + a_2 m, m = \delta(1)$, Then $(Q_2, A_2, B_2, \alpha, \beta, \gamma, \delta)$ is a separable systems.
Consider $\sum_1 \| \sum_2 = \left(Q_1 \times Q_2, A, B, F', G'\right), F' : (Q_1 \times Q_2) \times A \to Q_1 \times Q_2$,
i.e. $F' : (\mathbb{Z}_2 \times \mathbb{Z}) \times \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}$ with $F'((q_1, q_2), a) = (F_1(q_1, a), F_2(q_2, a))$
$= \left(1 + \begin{cases} [0] & \text{,if } a \text{ is even} \\ [1] & \text{,if } a \text{ is odd} \end{cases}, q_2^2 + a\right), \forall a \in A$ and $G' : (Q_1 \times Q_2) \times A \to B, i.e. G' : (\mathbb{Z}_2 \times \mathbb{Z}) \times \mathbb{Z} \to \mathbb{Z}_3$ with $G'((q_1, q_2), a) = G_1(q_1, a) + G_2(q_2, a) = 2 + a(mod 3) + q_2 + am, m = \delta(1)$. Then $\sum_1 \| \sum_2 = \left(Q_1 \times Q_2, A, B, F', G'\right) = \left(\mathbb{Z}_2 \times \mathbb{Z}, \mathbb{Z}, \mathbb{Z}_3, F', G'\right)$ is a separable system. Since $Q_1 \times Q_2 = (\mathbb{Z}_2 \times \mathbb{Z}, +_2), A = (\mathbb{Z}, +), B = (\mathbb{Z}_3, +_3)$ are groups and there are two maps $\alpha : Q_1 \times Q_2 \to Q_1 \times Q_2$ i.e. $\alpha : \mathbb{Z}_2 \times \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}$ defined by $\alpha(q_1, q) = (1, q^2), \gamma : Q_1 \times Q_2 \to B_2$ i.e. $\gamma : \mathbb{Z}_2 \times \mathbb{Z} \to \mathbb{Z}_3$ defined by $\gamma(q_1, q) = 2 + q$ and two homomorphisms $\beta : A \to Q_1 \times Q_2$ i.e. $\beta : \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}$
by $\beta(a) = \left(\begin{cases} [0] & \text{,if } a \text{ is even} \\ [1] & \text{,if } a \text{ is odd} \end{cases}, a\right)$
Since $a_1, a_2 \in \mathbb{Z}$ if $a_1 + a_2$ is even, Then $\beta(a_1+a_2) = ([0], a_1+a_2) = ([0], a_1) + ([0], a_2) = \beta(a_1) + \beta(a_2)$,
If $a_1 + a_2$ is odd Then $\beta(a_1+a_2) = ([1], a_1+a_2) = ([1], a_1) + ([1], a_2) = \beta(a_1) +$

$\beta(a_2)$,
and $\delta : A \to B$ i.e. $\delta : \mathbb{Z} \to \mathbb{Z}_3$ by $\delta(a) = a(mod3) + am$, where $m = \delta(1)$. Since $a_1, a_2 \in A = \mathbb{Z}$, then $\delta(a_1 + a_2) = (a_1 + a_2)(mod3) + (a_1 + a_2)m = (a_1)(mod3) + (a_2)(mod3) + (a_1 + a_2)\delta(1) = a_1(mod3) + a_1\delta(1) + a_2(mod3) + a_2\delta(1) = a_1(mod3) + a_1 m + a_2(mod3) + a_2 m = \delta(a_1) + \delta(a_2), m = \delta(1)$.

Theorem 3.2. *If* $\sum_1$ *and* $\sum_2$ *are separable* [10] *then*
(i) $f_{\sum_1 \| \sum_2} = f_{\sum_1} + f_{\sum_2}$,*(if the output groups are abelian)*
(ii) $f_{\sum_1 \ddagger \sum_2} = f_{\sum_1} \circ f_{\sum_2}$.

Proof. From above proposition, we have, $\left(-\alpha' + \bar{z}\right)^{-1}(q_k, q_{k+1}, q_{k+2}, ...)$
$= \left(0, \alpha'(0) + q_k, \alpha'\left(\alpha' 0 + q_k\right) + q_{k+1}, ...\right)$ where the first 0 appears at time k.
(i) Let $\sum_1 \| \sum_2 = (\alpha, \beta, \gamma, \delta)$ where $\alpha, \beta, \gamma, \delta$ is given as in the lines preceding above proposition.
Now, $f_{\sum_1 \| \sum_2}(a_k, a_{k+1}, ...) = \left(\gamma(-\alpha + \bar{z})^{-1}\beta + \delta\right)(a_k, a_{k+1}, ...)$
$= \left(\gamma(-\alpha + \bar{z})^{-1}\beta(a_k, a_{k+1}, ...)\right) + (\delta(a_k, a_{k+1}, ...))$
$= \left(\gamma(-\alpha + \bar{z})^{-1}((\beta_1(a_k), \beta_2(a_k)), (\beta_1(a_{k+1}), \beta_2(a_{k+1})), ...)\right) +$
$((\delta_1 + \delta_2)(a_k, a_{k+1}, ...))$
(If B is abelian then $\delta_1 + \delta_2$ is a homomorphism.)
$=\left(\gamma(-\alpha + \bar{z})^{-1}((\beta_1(a_k), \beta_2(a_k)), (\beta_1(a_{k+1}), \beta_2(a_{k+1})), ...)\right)$
$+ (\delta_1(a_k), \delta_1(a_{k+1}), ...) + (\delta_2(a_k), \delta_2(a_{k+1}), ...)$
$= \gamma(0, \alpha(0) + (\beta_1(a_k), \beta_2(a_k)), \alpha(\alpha(0) + (\beta_1(a_{k+1}), \beta_2(a_{k+1}))))$
$+ ((\beta_1(a_{k+1}), \beta_2(a_{k+1})), ...) + ((\delta_1(a_k), \delta_1(a_{k+1}), ...) + (\delta_2(a_k), \delta_2(a_{k+1}), ...))$
$= \gamma((0,0), (\alpha_1(0), \alpha_2(0)) + (\beta_1(a_k), \beta_2(a_k)), \alpha((\alpha_1(0), \alpha_2(0)) + (\beta_1(a_k), \beta_2(a_k)))$
$+ ((\beta_1(a_{k+1}), \beta_2(a_{k+1})), ... + ((\delta_1(a_k), \delta_1(a_{k+1}), ...) + (\delta_2(a_k), \delta_2(a_{k+1}), ...))$
$= \gamma((0,0), (\alpha_1(0), \alpha_2(0)) + (\beta_1(a_k), \beta_2(a_k)), \alpha((\alpha_1(0) + \beta_1(a_k)), (\alpha_2(0), \beta_2(a_k)))$
$+ (\beta_1(a_{k+1}), \beta_2(a_{k+1})), ...) + ((\delta_1(a_k), \delta_1(a_{k+1}), ...) + (\delta_2(a_k), \delta_2(a_{k+1}), ...))$
$= \gamma((0,0), (\alpha_1(0) + \beta_1(a_k), (\alpha_2(0) + \beta_2(a_k))), ...) +$
$((\delta_1(a_k), \delta_1(a_{k+1}), ...) + (\delta_2(a_k), \delta_2(a_{k+1}), ...))$
$=(\gamma_1(0) + \gamma_2(0), \gamma_1(\alpha_1(0) + \beta_1(a_k)) + \gamma_2(\alpha_2(0) + \beta_2(a_k)) ......)$
$+ ((\delta_1(a_k), \delta_1(a_{k+1}), ...) + (\delta_2(a_k), \delta_2(a_{k+1}), ...))$
$= (\gamma_1(0) + \delta_1(a_k), \gamma_1(\alpha_1(0) + \beta_1(a_k)) + \delta_1(a_{k+1}), ...)$
$+ (\gamma_2(0) + \delta_2(a_k), \gamma_2(\alpha_2(0) + \beta_2(a_k)) + \delta_2(a_{k+1}), ...)$
$= (\gamma_1(0), \gamma_1(\alpha_1(0) + \beta_1(a_k)), ...) + (\delta_1(a_k), \delta_1(a_{k+1}), ...)$
$+ (\gamma_2(0), \gamma_2(\alpha_2(0) + \beta_2(a_k)), ...)$
$+ (\delta_2(a_k), \delta_2(a_{k+1}), ...)$
$= \gamma_1(0, \alpha_1(0) + \beta_1(a_k), ...) + \delta_1(a_k, a_{k+1}, ...) + \gamma_2(0, \alpha_2(0) + \beta_2(a_k), ...) + \delta_2(a_k, a_{k+1}, ...)$
$= \gamma_1(-\alpha_1 + \bar{z})^{-1}(\beta_1(a_k), \beta_1(a_{k+1}), ...) + (\delta_1(a_k), \delta_1(a_{k+1}), ...)$
$+\gamma_2(-\alpha_2 + \bar{z})^{-1}(\beta_2(a_k), \beta_2(a_{k+1}), ...) + (\delta_2(a_k), \delta_2(a_{k+1}), ...)$

$= \gamma_1 (-\alpha_1 + \bar{z})^{-1} \beta_1 (a_k, a_{k+1}, ...) + \delta_1 (a_k, a_{k+1}, ...)$
$+\gamma_2 (-\alpha_2 + \bar{z})^{-1} \beta_2 (a_k, a_{k+1}, ...) + \delta_2 (a_k, a_{k+1}, ...)$
$= \left(\gamma_1 (-\alpha_1 + \bar{z})^{-1} \beta_1 + \delta_1\right) (a_k, a_{k+1}, ...)$
$+ \left(\gamma_2 (-\alpha_2 + \bar{z})^{-1} \beta_2 + \delta_2\right) (a_k, a_{k+1}, ...)$
$= f_{\sum_1} (a_k, a_{k+1}, ...) + f_{\sum_2} (a_k, a_{k+1}, ...)$
$= \left(f_{\sum_1} + f_{\sum_2}\right) (a_k, a_{k+1}, ...)$.

(ii) Let $\sum_1 \ddagger \sum_2$ by the maps $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ as indicated before proposition .
$f_{\sum_1 \ddagger \sum_2} (a_k, a_{k+1}, ...) = \left(\bar{\gamma} (-\bar{\alpha} + \bar{z})^{-1} \bar{\beta} + \bar{\delta}\right) (a_k, a_{k+1}, ...)$
$= \left(\bar{\gamma} (-\bar{\alpha} + \bar{z})^{-1} \bar{\beta} (a_k, a_{k+1}, ...)\right) + \left(\bar{\delta} (a_k, a_{k+1}, ...)\right)$
$= \bar{\gamma} (-\bar{\alpha} + \bar{z})^{-1} \left(\bar{\beta} (a_k), \bar{\beta} (a_{k+1}), ...\right)$
$+ \left(\bar{\delta} (a_k), \bar{\delta} (a_{k+1}), ...\right)$
$= \bar{\gamma} (-\bar{\alpha} + \bar{z})^{-1} ((\beta_1 (a_k), \beta_2\delta_1 (a_k)), (\beta_1 (a_{k+1}), \beta_2\delta_1 (a_{k+1})), ...)$
$+ (\delta_2\delta_1 (a_k), \delta_2\delta_1 (a_{k+1}), ...)$
$= \bar{\gamma}(0, \bar{\alpha}(0) + (\beta_1(a_k), \beta_2\delta_1(a_k)), \bar{\alpha}(\bar{\alpha}0 + (\beta_1(a_k), \beta_2\delta_1(a_k))$
$+(\beta_1(a_{k+1}), \beta_2\delta_1(a_{k+1})), ...$ $+(\delta_2\delta_1(a_k), \delta_2\delta_1(a_{k+1}), ...)$
$= \bar{\gamma} ((0, 0), (\alpha_1 (0), \alpha_2 (0)) + (\beta_1 (a_k), \beta_2\delta_1 (a_k)), ...)+(\delta_2\delta_1 (a_k), \delta_2\delta_1 (a_{k+1}), ...)$
$= (\gamma_2 (0) + \delta_2\gamma_1 (0), \gamma_2 (\alpha_2 (0) + \beta_2\gamma_1 (0) + \beta_2\delta_1 (a_k)) + \delta_2\gamma_1 (\alpha_1 (0) + \beta_1 (a_k)), ...)$
$+ (\delta_2\delta_1 (a_k), \delta_2\delta_1 (a_{k+1}), ...) = (\gamma_2(0) + \delta_2\gamma_1(0) + \delta_2\delta_1(a_k), \gamma_2(\alpha_2(0) + \beta_2\gamma_1(0) + \beta_2\delta_1(a_k)) + \delta_2\gamma_1(\alpha_1(0) + \beta_1(a_k))$ $+\delta_2\delta_1 (a_{k+1}), ...)$
$= (\gamma_2(0) + \delta_2(\gamma_1(0) + \delta_1(a_k)), \gamma_2(\alpha_2(0) + \beta_2(\gamma_1(0) + \delta_1(a_k))) + \delta_2(\gamma_1(\alpha_1(0) + \beta_1(a_k))$
$+\delta_1 (a_{k+1})), ...)$
$= (\gamma_2 (0), \gamma_2 (\alpha_2 (0) + \beta_2 (\gamma_1 (0) + \delta_1 (a_k)), ...)$
$+ (\delta_2 (\gamma_1 (0) + \delta_1 (a_k)), \delta_2 (\gamma_1 (\alpha_1 (0) + \beta_1 (a_k)) + \delta_1 (a_{k+1})), ...)$
$= \gamma_2(0, \alpha_2(0) + \beta_2(\gamma_1(0) + \delta_1(a_k)), \alpha_2(\alpha_2 0 + (\beta_2(\gamma_1(0)$
$+\delta_1(a_k)))) + \gamma_1(\alpha_1(0) + \beta_1(a_k)) + \delta_1(a_{k+1}), ...)$
$+ (\delta_2 (\gamma_1 (0) + \delta_1 (a_k)), \delta_2 (\gamma_1 (\alpha_1 (0) + \beta_1 (a_k)) + \delta_1 (a_{k+1})), ...)$
$= \gamma_2 (-\alpha_2 + \bar{z})^{-1} (\beta_2 (\gamma_1 (0) + \delta_1 (a_k)), \beta_2 (\gamma_1 (\alpha_1 (0) + \beta_1 (a_k)) + \delta_1 (a_{k+1})), ...)$
$+ (\delta_2 (\gamma_1 (0) + \delta_1 (a_k)), \delta_2 (\gamma_1 (\alpha_1 (0) + \beta_1 (a_k)) + \delta_1 (a_{k+1})), ...)$
$= \gamma_2 (-\alpha_2 + \bar{z})^{-1} \beta_2 (\gamma_1 (0) + \delta_1 (a_k)), \gamma_1 (\alpha_1 (0) + \beta_1 (a_k)) + \delta_1 (a_{k+1}), ...)$
$+ (\delta_2 (\gamma_1 (0) + \delta_1 (a_k), \gamma_1 (\alpha_1 (0) + \beta_1 (a_k)) + \delta_1 (a_{k+1}), ...)$
$= \left(\gamma_2 (-\alpha_2 + \bar{z})^{-1} \beta_2 + \delta_2\right) (\gamma_1 (0) + \delta_1 (a_k), \gamma_1 (\alpha_1 (0) + \beta_1 (a_k)) + \delta_1 (a_{k+1}), ...)$
$= f_{\sum_2} (\gamma_1 (0) + \delta_1 (a_k), \gamma_1 (\alpha_1 (0) + \beta_1 (a_k)) + \delta_1 (a_{k+1}), ...)$
$= f_{\sum_2} (\gamma_1 (0), \gamma_1 (\alpha_1 (0) + \beta_1 (a_k)), ...) + (\delta_1 (a_k) \delta_1 (a_{k+1}), ...))$
$= f_{\sum_2} (\gamma_1 (0, \alpha_1 (0) + \beta_1 (a_k), ...) + (\delta_1 (a_k) \delta_1 (a_{k+1}), ...))$
$= f_{\sum_2}(\gamma_1(0, \alpha_1(0) + \beta_1(a_k), \alpha_1(\alpha_1 0 + \beta_1(a_k)) + \beta_1(a_{k+1}), ...) + (\delta_1(a_k)\delta_1(a_{k+1}), ...))$
$= f_{\sum_2}(\gamma_1(-\alpha_1 + \bar{z})^{-1}(\beta_1(a_k), \beta_1(a_{k+1}), ...) + (\delta_1(a_k)\delta_1(a_{k+1}), ...))$
$= f_{\sum_2} \left(\gamma_1 (-\alpha_1 + \bar{z})^{-1} \beta_1 (a_k, a_{k+1}, ...) + \delta_1 (a_k, a_{k+1}, ...)\right)$

$= f_{\sum_2}\left(\left(\gamma_1\left(-\alpha_1+\bar{z}\right)^{-1}\beta_1+\delta_1\right)\left(a_k, a_{k+1}, ...\right)\right)$
$= f_{\sum_2}\left(f_{\sum_1}\left(a_k, a_{k+1}, ...\right)\right) = \left(f_{\sum_2}\circ f_{\sum_1}\right)\left(a_k, a_{k+1}, ...\right).$

Theorem 3.3. *Separable systems itself form a near-ring by means of series/parallel connections of these systems* [10].

4. Discussion and Future works

In this article, we review some fundamental definitions of automata. Additionally, we discuss examples that meet all the conditions outlined in the provided definitions. Unlike the case with significant classes of linear dynamical systems, which form rings with respect to parallel and series connections, nonlinear systems exhibit a near-ring structure. Consequently, one can anticipate results regarding the stabilization of nonlinear dynamical systems. This study will guide us in addressing questions in automata/dynamical system theory, such as feedbacks, reachability, and invertibility [10]. In this paper, we examine the compound interest problem as a practical example of a dynamical system. However, much work remains to be done in exploring the compound interest problem within the framework of dynamical systems. Additionally, our efforts could be expanded beyond the study of near-rings to encompass various areas of computer science, such as system theory, coding theory, rubber sheet geometry, and more [3], [8], [11], [16].

5. Conclusion

In this paper, we have examined a broad class of nonlinear systems that allow for a transfer function entirely described by their input-output behavior, featuring unique characteristics related to automata/separable systems. Moving forward, we aim to delve into various aspects of near-rings, such as structure theory and radical theory, in our future work.

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