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CERTAIN IDENTITIES ASSOCIATED WITH EISENSTEIN SERIES, GÖLLNITZ–GORDON IDENTITIES AND COMBINATORIAL PARTITION IDENTITIES FOR THE CONTINUED FRACTIONS OF ORDER SIXTEEN

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Abstract: The objective of this paper is to establish six new identities which depict interrelationships between Eisenstein series identities, Göllnitz–Gordon identities and combinatorial partition identities.

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1. Introduction

Throughout this paper, we assume that $|q| < 1$ and use the standard product notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

For convenience, we sometimes use the multiple q-shifted factorial notation, which is defined as

$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

The celebrated Ramanujan–Göllnitz–Gordon continued fraction is defined as

$$G(q) = \frac{q^{1/2}}{1+q} \, \underset{+}{} \, \frac{q^2}{1+q^3} \, \underset{+}{} \, \frac{q^4}{1+q^5} \, \underset{+}{} \, \frac{q^6}{1+q^7} \, \underset{+}{} \, \cdots. \tag{1}$$

An interesting product representation of $G(q)$ is recorded in [14]

$$G(q) = q^{\frac{1}{2}} \frac{(q; q^8)_\infty (q^7; q^8)_\infty}{(q^3; q^8)_\infty (q^5; q^8)_\infty}. \tag{2}$$

The famous Göllnitz–Gordon functions $S(q)$ and $T(q)$ are defined by

$$S(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n^2} = \frac{1}{(q; q^8)_\infty (q^4; q^8)_\infty (q^7; q^8)_\infty} \tag{3}$$

and

$$T(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n^2+2n} = \frac{1}{(q^3; q^8)_\infty (q^4; q^8)_\infty (q^5; q^8)_\infty}, \tag{4}$$

where the two equalities on the right-hand of (3) and (4) are the celebrated Göllnitz–Gordon identities. It is noted that

$$G(q) = q^{1/2} \frac{S(q)}{T(q)}. \tag{5}$$

Without any knowledge of Ramanujan's work, Göllnitz [11] and Gordon [12] rediscovered and proved (2) independently. Later Andrews [2] proved (2) as a corollary of a more general result.

On page 299 of his second notebook [14], Ramanujan recorded the two identities,

$$\frac{1}{G(q)} - G(q) = \frac{(-q^2; q^4)_\infty^2 (q^4; q^4)_\infty (q^4; q^8)_\infty}{q^{1/2} (q^8; q^8)_\infty} \tag{6}$$

and

$$\frac{1}{G(q)} + G(q) = \frac{(-q; q^2)_\infty^2 (q^2; q^2)_\infty (q^4; q^8)_\infty}{q^{1/2} (q^8; q^8)_\infty}. \tag{7}$$

The identities (6) and (7) were first proved by B. C. Berndt [4] and then rediscovered by Chan and Huang [10].

We consider partitions studied by Göllnitz and Gordon. Let $\mathcal{GG}$ denote the set of Göllnitz–Gordon partitions, which are those partitions satisfying $\lambda_i - \lambda_{i+2} \geq 2$, with strict inequality if either part is even. Furthermore, let $\mathcal{GG}_t$ denote those partitions in $\mathcal{GG}$ with all parts at least t. A direct combinatorial argument shows that

$$f_{\mathcal{GG}}(x;q) := \sum_{\lambda \in \mathcal{GG}} x^{l(\lambda)} q^{|\lambda|} = \sum_{n \geq 0} \frac{x^n q^{n^2} (-q;q^2)_n}{(q^2;q^2)_n}. \tag{8}$$

Furthermore, if the sum is instead taken over partitions $\lambda \in \mathcal{GG}_t$ and t is odd, it is clear that the resulting generating function is $f_{\mathcal{GG}}(xq^{t-1};q)$. The Göllnitz–Gordon identities, which were independently proven in [11, 12], then state that for $t = 1$ or 3, we have the following product formulas:

$$f_{\mathcal{GG}_t}(1;q) = \frac{1}{(q^t, q^4, q^{8-t}; q^8)_\infty}. \tag{9}$$

Andrews *et al.* [3] investigated new double summation hypergeometric $q-$series representations for several families of partitions and further explored the role of double series in combinatorial partition identities by introducing the following general family:

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn} r(l,u,v,w;n), \tag{10}$$

where

$$r(l,u,v,w;n) := \sum_{j=0}^{\left[\frac{n}{u}\right]} (-1)^j \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}(q^{uv};q^{uv})_j}. \tag{11}$$

To illustrate this notation, we note that the double series associated to the partition identities described above may be written as

$$R(2,t,0,2,2,2) = f_{\mathcal{GG}}(q^{t-1};q). \tag{12}$$

The following interesting special cases of (10) are recalled [3, p. 106]

$$R(2,1,1,1,2,2) = (-q;q^2)_\infty; \tag{13}$$

$$R(2,2,1,1,2,2) = (-q^2;q^2)_\infty; \tag{14}$$

$$R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{15}$$

In [6], Chaudhary *et al.* have established the combinatorial partition identities for the identities (6) and (7) using (13) - (15). In [5, 7, 8, 9] Chaudhary *et al.* have established the several identities between q- product identities and combinatorial partition identities. Recently, Srivastava *et al.* [15, 16], continued fraction identities and combinatorial partition identities.
Surekha [17] and Vanitha [18] studied two continued fractions $I_1(q)$ and $I_2(q)$ of order sixteen, which are defined as follows:

$$I_1(q) := \frac{q^{1/2}(q^3, q^{13}; q^{16})_\infty}{(q^5, q^{11}; q^{16})_\infty} = \frac{q^{1/2}(1-q^3)}{(1-q^4)} + \frac{q^4(1-q)(1-q^7)}{(1-q^4)(1+q^8)} + \cdots \tag{16}$$

and

$$I_2(q) := \frac{q^{3/2}(q, q^{15}; q^{16})_\infty}{(q^7, q^9; q^{16})_\infty} = \frac{q^{3/2}(1-q)}{(1-q^4)} + \frac{q^4(1-q^3)(1-q^5)}{(1-q^4)(1+q^8)} + \cdots. \tag{17}$$

The continued fractions (16) and (17), are a special case of fascinating continued fraction identity recorded by Ramanujan in his second notebook [1, 14].

Park [20], studied the continued fractions $I_1(q)$ and $I_2(q)$ by using the theory of modular functions. He proved the modularities of $I_1(q)$ and $I_2(q)$. Further, he proved that $2(I_1(q)^2 + 1/I_1(q)^2)$ and $2(I_2(q)^2 + 1/I_2(q)^2)$ are algebraic integers for certain imaginary quadratic quantity q. In [13], S. Rajkhowa and N. Saikia have established the theta function identities, explicit values, partition-theoretic results and some matching coefficients of the continued fractions $I_1(q)$ and $I_2(q)$. Recently, Vanitha, Chaudhary and Bulkhali [19] have proved many new identities associated with Ramanujan's continued fraction of order sixteen and Ramanujan–Göllnitz–Gordon continued fraction. We further established several new Eisenstein series identities associated with Ramanujan's continued fraction of order sixteen.

The main purpose of this paper is to establish six new identities which depict interrelationships between Eisenstein series identities, Göllnitz–Gordon identities and combinatorial partition identities.

2. Preliminaries

For our purpose, here we recall some known results. Vanitha, Chaudhary and Bulkhali [19] presented the following Eisenstein series identities for the continued fractions of order sixteen:

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{3n}+q^{5n}}{1-q^{16n}} - \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{11n}+q^{13n}}{1-q^{16n}} = \frac{\eta^4(32\tau)}{\eta^2(16\tau)}\left(I_1(q^2) + \frac{1}{I_1(q^2)}\right), \tag{18}$$

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^n+q^{7n}}{1-q^{16n}} - \sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{9n}+q^{15n}}{1-q^{16n}} = \frac{\eta^4(32\tau)}{\eta^2(16\tau)}\left(I_2(q^2)+\frac{1}{I_2(q^2)}\right), \quad (19)$$

$$\sum_{n=1}^{\infty} \frac{n(q^n-q^{7n}-q^{9n}+q^{15n})}{1-q^{16n}} = \frac{q(q^{16};q^{16})_\infty^2(q^8;q^8)_\infty^2(q^6,q^{10};q^{16})_\infty}{(q,q^7,q^9,q^{15};q^{16})_\infty^2}, \quad (20)$$

$$\sum_{n=1}^{\infty} \frac{n(q^{3n}-q^{5n}-q^{11n}+q^{13n})}{1-q^{16n}} = \frac{q^3(q^{16};q^{16})_\infty^2(q^8;q^8)_\infty^2(q^2,q^{14};q^{16})_\infty}{(q^3,q^5,q^{11},q^{13};q^{16})_\infty^2}, \quad (21)$$

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n-q^{7n}-q^{9n}+q^{15n}}{1-q^{16n}} = \frac{q(q^8;q^8)_\infty^2(q^{48};q^{48})_\infty(q^6,q^{10};q^{16})_\infty(q,q^7,q^9,q^{15};q^{16})_\infty}{(q^{16};q^{16})_\infty(q^3,q^{21},q^{27},q^{45};q^{48})_\infty}, \quad (22)$$

and

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^{3n}-q^{5n}-q^{11n}+q^{13n}}{1-q^{16n}} = \frac{q^3(q^8;q^8)_\infty^2(q^{48};q^{48})_\infty(q^2,q^{14};q^{16})_\infty(q^3,q^5,q^{11},q^{13};q^{16})_\infty}{(q^{16};q^{16})_\infty(q^9,q^{15},q^{33},q^{39};q^{48})_\infty}. \quad (23)$$

where, $\eta(\tau) = q^{1/24}f(-q)$ where $q = e^{2\pi i\tau}$, Im$\tau > 0$, and $\eta(\tau)$ is the Dedekind-eta function and $\left(\frac{\cdot}{p}\right)$, $p-prime$ denote the Legendre symbol modulo p.

3. Main Results

Here, we present six interrelationships between Eisenstein series identities, Göllnitz–Gordon identities and combinatorial partition identities.

Theorem 3.1. *Each of the following identities holds true:*

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{3n}+q^{5n}}{1-q^{16n}} - \sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{11n}+q^{13n}}{1-q^{16n}} = q^4\ R(16,16,1,1,1,2)\left(I_1(q^2)+\frac{1}{I_1(q^2)}\right) \quad (24)$$

and

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^n+q^{7n}}{1-q^{16n}} - \sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{9n}+q^{15n}}{1-q^{16n}} = q^4\ R(16,16,1,1,1,2)\left(I_2(q^2)+\frac{1}{I_2(q^2)}\right). \quad (25)$$

Proof. First of all, in order to prove the assertion (24), we apply the identity (15) (with $m = 16$) in (18), after using little algebra, we are led to the desire identity (24). Proof of (25) is similar to the proof of (24), so we omit.
We thus have completed our proof of the Theorem (3.1).

Theorem 3.2. *Each of the following identities holds true:*

$$\sum_{n=1}^{\infty} \frac{n(q^n - q^{7n} - q^{9n} + q^{15n})}{1 - q^{16n}} = q \; f_{GG_1}(1;q^2) \; R(8,8,1,1,1,2) \frac{(q^2;q^2)_\infty (q^8;q^8)_\infty^3 (q^8;q^{16})_\infty}{(q^4;q^4)_\infty (q,q^7,q^9,q^{15};q^{16})_\infty^2}, \tag{26}$$

$$\sum_{n=1}^{\infty} \frac{n(q^{3n} - q^{5n} - q^{11n} + q^{13n})}{1 - q^{16n}} = q^3 \; f_{GG_2}(1;q^2) \; R(8,8,1,1,1,2) \frac{(q^2;q^2)_\infty (q^8;q^8)_\infty^3 (q^8;q^{16})_\infty}{(q^4;q^4)_\infty (q^3,q^5,q^{11},q^{13};q^{16})_\infty^2}, \tag{27}$$

$$\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n - q^{7n} - q^{9n} + q^{15n}}{1 - q^{16n}} = \frac{q \; f_{GG_1}(1;q^2)}{R(8,8,1,1,1,2)} \frac{(q^2;q^2)_\infty (q^8;q^8)_\infty^2 (q^{48};q^{48})_\infty (q,q^7,q^9,q^{15};q^{16})_\infty}{(q^4;q^4)_\infty (q^3,q^{21},q^{27},q^{45};q^{48})_\infty} \tag{28}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^{3n} - q^{5n} - q^{11n} + q^{13n}}{1 - q^{16n}} = \frac{q^3 \; f_{GG_2}(1;q^2)}{R(8,8,1,1,1,2)} \frac{(q^2;q^2)_\infty (q^8;q^8)_\infty^2 (q^{48};q^{48})_\infty (q^3,q^5,q^{11},q^{13};q^{16})_\infty}{(q^4;q^4)_\infty (q^9,q^{15},q^{33},q^{39};q^{48})_\infty}. \tag{29}$$

Proof. First of all, in order to prove the assertion (26), we use the identities of (15) (with $m = 8$) and (9) (with $q = q^2$; $t = 1$) in (20), by arranging the power of suitable terms and further using little algebra, we obtain the desire identity (26).
Proof of (27) is similar to the proof of (26), so we omit.
Further, we prove our next assertion (28), using the identities of (15) ($m = 8$) and (9) (with $q = q^2$; $t = 1$) in (22), by arranging the power of suitable terms and applying little algebra, we obtain the desire identity (28).
Proof of (29) is similar to the proof of (28), so we omit.

We thus have completed our proof of the Theorem 3.2.

4. Concluding Remarks

Recently, Vanitha, Chaudhary and Bulkhali [19] established relations between Eisenstein series identities and continued fractions of order sixteen. In present article, authors developed six interrelationships between Eisenstein series identities, Göllnitz–Gordon identities and combinatorial partition identities.

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References

[1] Adiga C., Berndt B. C., Bhargava S. and Watson G. N., Chapter 16 of Ramanujan's second notebook: Theta functions and q-series, Mem. Amer. Math. Soc., 315 (1985), 1-91.

[2] Andrews G. E., On $q-$ difference equations for certain well-poised basic hypergeometric series, Quart. J. Math., 19 (1968), 433-447.

[3] Andrews G. E., Bringman K. and Mahlburg K., Double series representation for Schur's partition function and related identities, J. Combin. Theory Ser. A, 132 (2015), 102-119.

[4] Berndt B. C., Ramanujan's Notebooks, Part III, Springer-Verlag, Heidelberg and New York, 1991.

[5] Chaudhary M. P., Some relationships between q-product identities, combinatorial partition identities and continued fractions identities III, Pacic J. Appl. Math., 7 (2) (2015), 87-95.

[6] Chaudhary M. P. and Choi J., Certain identities associated with Eisenstein series, Ramanujan–Göllnitz–Gordon continued fraction and combinatorial partition identities, Int. J. Math. Anal., 10 (2016), 237-244.

[7] Chaudhary M. P. and Salilew G. A. S., Note on q-product identities and combinatorial partition identities, Honam Math. J., 39 (2017), 267-273.

[8] Chaudhary M. P., A family of theta-function identities based upon R_α, R_β and R_m- functions related to Jacobi's triple product identity, Publ. Inst. Math. (Beograd), 108 No. 122, (2020), 23-32.

[9] Chaudhary M. P., Guiben Salem and Mazhouda Kamel, Relations between mock theta functions and combinatorial partition identities, J. Ramanujan Soc. Math. Math. Sci., 10 No. 1, (2022), 79-86.

[10] Chan H. H. and Huang S. -S., On the Ramanujan–Göllnitz–Gordon continued fraction, Ramanujan J., 1 (1997), 75-90.

[11] Göllnitz H., Partitionen mit Differenzenbedingungen, J. Reine Angew. Math., 225 (1967), 154-190.

[12] Gordon B., Some continued fractions of the Rogers-Ramanujan type, Duke Math. J., 32 (1965), 741-748.

[13] Rajkhowa S. and Saikia N., Theta-function identities, explicit values for Ramanujan's continued fractions for order sixteen and applications to partition theory, Integer, 23 (2023), 1-15.

[14] Ramanujan S., Notebooks (2 volumes), Tata Inst. Fund. Res., Bombay, 1957.

[15] Srivastava H. M. and Chaudhary M. P., Some relationships between q-product identities, combinatorial prtition identities and continued fractions identities, Adv. Stud. Contemporary Math., 2 No. 3, (2015), 265 - 272.

[16] Srivastava H. M., Chaudhary M. P., Chaudhary S. and Salilew G. A., Jacobi's triple product identity and an associated family of theta function identities, Math. Notes, 112 No. 5, (2022), 755-762.

[17] Surekha M. S., On the modular relations and dissections for a continued fraction of order sixteen, Palestine J. Math., 6 No. 1 (2017), 119-132.

[18] Vanitha A., On the expansion of Ramanujan's continued fraction of order sixteen, Tamsui Oxf. J. Inf. Math. Sci., 31 No. 1, (2017), 81-99.

[19] Vanitha A., Chaudhary M. P. and Bulkhali N. A. S., On a Ramanujan's continued fraction of order sixteen and new Eisenstein series identities, communicated for publication.

[20] Yoon Kyung Park, Ramanujan continued fractions of order sixteen, Int. J. Number Thy., 18 No. 5, (2022), 1097-1109.