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SOME QUENCHING PROBLEMS FOR ω-DIFFUSION EQUATIONS ON GRAPHS WITH A POTENTIAL AND A SINGULAR SOURCE

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Abstract: In this paper, we study the quenching phenomenon related to the ω-diffusion equation on graphs with a potential and a singular source

$$u_t(x,t) = \Delta_\omega u(x,t) + b(x)(1-u(x,t))^{-p},$$

where Δ_ω is called the discrete weighted Laplacian operator. Under some appropriate hypotheses, we prove the existence and uniqueness of the local solution via Banach fixed point theorem. We also show that the solution of the problem quenches in a finite time and that the time-derivative blows up at the quenching time. Moreover, we estimate the quenching time and the quenching rate. Finally, we verify our results through some numerical examples.

Keywords and Phrases: Quenching, ω - diffusion equation, quenching rate, graph.

2020 Mathematics Subject Classification: 35K57, 35A02, 05C62.

1. Introduction and Definitions

In this paper, we study the quenching properties on the graph $G(V, E, \omega)$ of the following problem :

$$\begin{cases} u_t(x,t) = \Delta_\omega u(x,t) + b(x)(1-u(x,t))^{-p}, & (x,t) \in S \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial S \times (0,T), \\ u(x,0) = u_0(x), & x \in V, \end{cases} \tag{1}$$

where p is a positive real. Following [18] V is the set of nodes (or vertices) of the graph $G(V, E, \omega)$. In this work, we suppose that the graph G is simple (i.e. without loops), finite, connected, undirected and weighted graph. Moreover, the set of vertices can be split into two disjoint subsets S and ∂S such that $V = S \cup \partial S$. The subsets S and ∂S of V are called the interior and the boundary of V , respectively. Moreover, $\omega : V \times V \to \mathbb{R}_+$ denotes the weighted function, which such as :

(i) $\omega(x, x) = 0$ for any $x \in V$,

(ii) $\omega(x, y) = \omega(y, x)$ for any $x,\ y \in V$,

(iii) $\omega(x, y) > 0$ if and only if $\{x, y\} \in E$,

where E is the set of edge of the graph G. The discrete weighted Laplacian operator Δ_ω is defined as follows :

$$\Delta_\omega u(x,t) = \sum_{y \in V} \omega(x,y)(u(y,t) - u(x,t)), \quad (x,t) \in V \times [0,T),$$

and $\sum_{y \in V} \omega(x, y) = h(x)$.

Now, let us state the basic assumptions in this work :

- the initial data $u_0 : V \to [0, 1)$ is a nontrivial function such as

$$\Delta_\omega u_0(x) + b(x)(1 - u_0(x,t))^{-p} \geq 0,$$

- the potential $b \in C(V)$ is a nonnegative function, such as

$$\min_{x \in S} b(x) > \max_{x \in S} h(x).$$

Here $[0,T)$ is the maximum time interval on which the solution u of (1) exists and satisfies

$$\|u(\cdot,t)\|_\infty < 1, \text{ where } \|u(\cdot,t)\|_\infty = \max_{x\in S} u(x,t).$$

When T is finite, we say that the solution u quenches in finite time, namely

$$\lim_{t\to T^-} \|u(\cdot,t)\|_\infty = 1$$

and T is called the quenching time of the solution u.

In our problem we have singular heat source, $b(x)(1-u(x,t))^{-p}$. The main goal of this paper is to study the quenching of the above problem and then to derive the quenching rate estimates. This mathematical problem can be considered as a model of heat diffusion (or energy) through networks which flows is influenced by proportional reactive forces to the power of its potential.

It should be added that the diffusion equations on graphs have been introduced as mathematical models of heat flowing (or energy) through an electric network or as models of informations on networks or vibration of molecules. For this reason, the study of ω-diffusion equation of the form $u_t(x,t) = \Delta_\omega u(x,t)$ has attracted many researchers attention in recent years (see [1, 4, 6, 9, 11, 16, 17, 21, 23] and the references cited therein). On the other hand, the long time behavior (extinction and positivity) of solutions to evolution Laplace equation with absorption on networks is studied in papers [5, 10, 19]. Yun-Sung Chung et al [3] considered the following problem

$$\begin{cases} u_t = \Delta_\omega u - u^q \text{ in } S\times(0,\infty), \\ u = 0 \quad \text{on} \quad \partial S\times[0,\infty), \\ u(x,0) = u_0(x) \geq 0, \qquad x\in S. \end{cases} \tag{2}$$

They proved that a nontrivial solution becomes extinct in finite time if $0<q<1$, while it remains positive for $q\geq 1$. In [12] Liu et al replaced the reaction term $-u^{-q}$ by $\lambda u^q - u^p$. They discussed of the quenching of the solution to the problem according to the parameters p, q and λ. Furthermore, Qiao Xin et al in [18] considered the following ω-diffusion equation with a reaction term

$$\begin{cases} u_(x,t) = \Delta_\omega u(x,t) + u^p(x,t), & (x,t)\in S\times(0,\infty), \\ u(x,t) = 0, & (x,t)\in \partial S\times(0,\infty), \\ u(x,0) = u_0(x), & x\in V. \end{cases} \tag{3}$$

They proved that if $p\leq 1$, every solution is global. They also obtained that if $p>1$, the nonnegative and nontrivial solution blows up in finite time and that the blow-up rate on L^∞-norm only depends on p, under some suitable conditions. For more details about the study of ω-diffusion equation on graphs, we refer to [11, 16, 17, 20]. Moreover, in the continuous case, solutions of diffusion equations which quench in a finite time have been the subject of many authors investigations [7, 8, 13-15]. In particular, Boni et al [2] studied the reaction-diffusion equation (1) on a continuous domain $\Omega \subset \mathbb{R}^n$, defined as follows :

$$\begin{cases} u_t(x,t) = Lu(x,t) + r(x)(b-u(x,t))^{-p}, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{4}$$

where $r(x)$ is a nonnegative potential and L, an elliptic operator defined by

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right).$$

The authors have given some conditions under which the solution of the above problem quenches in a finite time and estimated its quenching time. Moreover, D. Xiaoqiang et al [15] are interested in the asymptotic behavior of the quenching time of a similar problem.

Our work has also been motivated by the paper of Yun-Sung Chung et al in [5]. They have considered the problem (3) in the case where the reaction term u^p is replaced by $-u^q$ with $q > 1$. They proved that a nontrivial solution becomes extinct in finite time if $0 < q < 1$, while it remains positive for $q \geq 1$. For papers concerning other ω-diffusion equations, we also refer readers to [16], [22].

Motivated by the above researches, we study problem (1) on graphs. By using the Banach fixed point theorem, we prove the local existence of the solution. Then, we construct the comparison principle and use it to study the quenching and quenching rate. Finally, we give several numerical experiments to illustrate our results.

The rest of the paper is organized as follows. In Section 2, we consider the local existence of the solution of the problem (1). In section 3, we prove that the solution u of problem (1) quenches in a finite time and estimate its upper bound. On the other hand, we estimate the quenching rate and prove that u_t blows up at quenching time. In Section 4, we give some numerical experiments to illustrate our analysis.

2. Local existence

In this section, we prove the existence of local solution for problem (1) via Banach fixed point theorem. Since $0 \leq u_0(x) < 1$, we can choose $0 < \alpha < 1$ such that $0 \leq u_0(x) < \dfrac{\alpha}{2}$. We also consider the Banach space defined as follows :

$$X_{t_0} = \{u \in C(V \times [0,t_0];\mathbb{R}) \mid 0 \leq u(x,t) \leq \alpha,\ u(x,t) \equiv 0,\ \text{for any } x \in \partial S\},$$

with the norm

$$\|u\|_{X_{t_0}} = \max_{t\in[0,t_0]} \max_{x\in S} |u(x,t)|,$$

where $t_0 > 0$ is a fixed constant.

Now, let's define the nonlinear operator T_{u_0} by

$$T_{u_0}[u](x,t) = \begin{cases} u_0(x) + \int_0^t \Delta_\omega u(x,\tau)d\tau + \int_0^t b(x)(1-u(x,\tau)^{-p})d\tau, & x \in S, t \in [0,t_0], \\ 0, & x \in \partial S,\ t \in [0,t_0]. \end{cases}$$

In the following, under some appropriate conditions, we present the properties of the nonlinear operator T_{u_0} and the local existence result of solution of the problem (1).

Lemma 1. *The nonlinear operator T_{u_0} is well defined, mapping X_{t_0} into X_{t_0} and verifies the following properties :*

- *For any $u,\ v \in X_{t_0}$, $0 \leq u_0(x),\ v_0(x) < \dfrac{\alpha}{2}$, there exists a positive constant C which only depends on $h(x)$, p, $\|u\|_{X_{t_0}}$, $\|v\|_{X_{t_0}}$ such that*

$$\|T_{u_0}[u] - T_{v_0}[v]\|_{X_{t_0}} \leq \|u_0 - v_0\|_\infty + Ct_0\|u-v\|_{X_{t_0}}. \tag{5}$$

- *The mapping T_{u_0} is strictly contractive.*

Proof.

Step 1. We prove that T_{u_0} maps X_{t_0} into X_{t_0}. By definition of T_{u_0}, it is easy to check that $T_{u_0}[u](x,\cdot)$ is continuous mapping on $[0,t_0]$ for any $x \in V$. Now, we show that $0 \leq T_{u_0}[u](x,t) \leq \alpha$ for any $u \in X_{t_0}$. For this, we consider $u \in X_{t_0}$. We consider $t_0 > 0$ such that :

$$t_0 \leq \frac{\alpha/2}{\alpha \max_{x\in S} h(x) + \max_{x\in S} b(x)(1-\alpha)^{-p}}.$$

By the definition of the discrete Laplacian operator and the nonnegative of the weighted function $\omega(x,y)$, we have :

$$\begin{aligned} T_{u_0}[u](x,t) &= u_0(x) + \int_0^t \sum_{y\in V} \omega(x,y)(u(y,\tau)-u(x,\tau))d\tau + \int_0^t b(x)(1-u(x,\tau))^{-p}d\tau \\ &\leq \frac{\alpha}{2} + \int_0^t \sum_{y\in V} \omega(x,y)u(y,\tau)d\tau + \int_0^t b(x)(1-u(x,\tau))^{-p}d\tau \\ &\leq \frac{\alpha}{2} + \int_0^t \alpha \max_{x\in S} h(x)d\tau + \int_0^t \max_{x\in S} b(x)(1-\alpha)^{-p}d\tau \\ &\leq \frac{\alpha}{2} + \left(\alpha \max_{x\in S} h(x) + \max_{x\in S} b(x)(1-\alpha)^{-p}\right) t_0 \\ &\leq \alpha. \end{aligned}$$

Since $\min\limits_{x\in S} b(x) > \max\limits_{x\in S} h(x)$, we obtain

$$\begin{aligned} T_{u_0}[u](x,t) &= u_0(x) + \int_0^t \sum_{y\in V} \omega(x,y)(u(y,\tau)-u(x,\tau))d\tau + \int_0^t b(x)(1-u(x,\tau))^{-p}d\tau \\ &\geq -\int_0^t \sum_{y\in V} \omega(x,y)u(x,\tau)d\tau + \int_0^t b(x)d\tau \\ &\geq \left(-\alpha \max_{x\in S} h(x) + \min_{x\in S} b(x)\right) t \\ &\geq 0. \end{aligned}$$

Therefore, the nonlinear operator T_{u_0} maps X_{t_0} to X_{t_0}.

Step 2. Since, the function $s \mapsto (1-s)^{-p}$ is locally Lipschitz continuous on $[0,\alpha]$, for any $(x,t) \in S \times [0,t_0]$, $u_0(x),\ v_0(x) \in C(V)$ and $u(x,t),\ v(x,t) \in X_{t_0}$, we get :

$$\begin{aligned} |T_{u_0}[u](x,t) - T_{v_0}[v](x,t)| &\leq \|u_0 - v_0\|_\infty + \int_0^t |\Delta_\omega(u(x,\tau) - v(x,\tau))|d\tau \\ &+ \int_0^t |b(x)||(1-u(x,\tau))^{-p} - (1-v(x,\tau))^{-p}|d\tau \\ &\leq \|u_0 - v_0\|_\infty + Ct\|u-v\|_{X_{t_0}}, \end{aligned}$$

where $C = 2\max_{x\in S}|h(x)| + p(1-\eta)^{-p-1}$, with $\eta = \max\left\{\|u\|_{X_{t_0}}, \|v\|_{X_{t_0}}\right\}$.

Step 3. Now, we prove that T_{u_0} is strictly contractive in the ball $B\left(u_0, 2\|u_0\|_{L^\infty(V)}\right)$. For any $u,\ v \in B\left(u_0, 2\|u_0\|_{L^\infty(V)}\right)$, we have

$$\|u\|_{X_{t_0}} \leq 3\|u_0\|_{L^\infty(V)}, \quad \|v\|_{X_{t_0}} \leq 3\|u_0\|_{L^\infty(V)}.$$

Thus, we get

$$\|T_{u_0}[u] - T_{u_0}[v]\|_{X_{t_0}} \leq C_2 t_0 \|u - v\|_{X_{t_0}},$$

where C_2 only depends on ω, u_0 and p. Taking t_0 small enough such that $C_2 t_0 < 1$, we obtain the desired results.

By the Banach fixed point theorem and the Lemma 1, we can prove the existence and the uniqueness of solution u to (1) in the time interval $[0, t_0]$. Thus, if $\|u\|_{X_{t_0}} < 1$, taking initial data $u(x, t_0)$, we can extend the solution to some interval $[0, t_1]$, where $t_1 > t_0$. We repeat the above processing until $\lim_{t \to T^-} u(x, t) = 1$.

3. Quenching, quenching rate and blow-up of time-derivative

In this section, we investigate the quenching phenomenon of the solution of (1). We also study the blow-up of the time-derivative of its solution under some appropriate assumptions and estimate the time of quenching.
Now, we give some auxiliary results for the problem (1).

Definition 2. *We say that v is an upper solution of (1) if*

$$\begin{aligned}
&v_t(x,t) - \Delta_\omega v(x,t) - b(x)(1 - v(x,t))^{-p} \geq 0, && (x,t) \in S \times (0,T),\\
&v(x,t) \geq 0, && (x,t) \in \partial S \times (0,T),\\
&v_0(x) \geq u_0(x), && x \in V.
\end{aligned}$$

On the other hand, we say that v is a lower solution of (1) if these inequalities are reversed.

The following lemma is a discrete form of the maximum principle.

Lemma 3. *Let $a(x, \cdot) \in C^0([0, T_0]; V)$ and v such that :*

$$\begin{aligned}
&v_t(x,t) - \Delta_\omega v(x,t) - b(x)a(x,t)v(x,t) \geq 0, && (x,t) \in S \times (0,T),\\
&v(x,t) \geq 0, && (x,t) \in \partial S \times (0,T),\\
&v_0(x) \geq 0, && x \in V.
\end{aligned}$$

Then we have

$$v(x,t) \geq 0, \quad (x,t) \in V \times [0,T).$$

Proof. Let $0 < T_0 < T$ and define the function $z(x,t) = e^{\lambda t} v(x,t)$ where λ is small enough such that $b(x)a(x,t) + \lambda < 0$, for $t \in [0, T_0]$, $x \in V$. We have :

$$\begin{aligned}
&z_t(x,t) - \Delta_\omega z(x,t) - (b(x)a(x,t) + \lambda) z(x,t) \geq 0, && (x,t) \in S \times (0,T), && \text{(6a)}\\
&z(x,t) \geq 0, && x \in \partial S \times (0,T), && \text{(6b)}\\
&z(x,0) = z_0(x) \geq 0, && x \in V. && \text{(6c)}
\end{aligned}$$

Let $m = \min_{x\in V} \min_{t\in[0,T_0]} z(x,t)$. Since for $x \in V$, $z(x,t)$ is continuous function, we can assume that $m = z(x_0, t_{x_0})$ for a certain $x_0 \in V$.
Assume $m < 0$.

If $t_{x_0} = 0$, then $z(x_0, t_{x_0}) = z_0(x_0) < 0$, which contradicts (6c), hence $t_{x_0} \neq 0$. If $x_0 \in \partial S$, we have $z(x_0, t_{x_0}) < 0$, which contradicts (6b), thereby $x_0 \in S$. Moreover, we get

$$z_t(x_0, t_{x_0}) = \lim_{k\to 0} \frac{z(x_0, t_{x_0}) - z(x_0, t_{x_0} - k)}{k} \leq 0,$$
$$-\sum_{y\in V} \omega(x_0, y)(z(y, t_0) - z(x_0, t_0)) \leq 0.$$

On the other hand, for λ small enough, we get

$$z_t(x_0, t_{x_0}) - \Delta_\omega z(x_0, t_{x_0}) - (b(x_0)a(x_0, t_{x_0}) + \lambda)z(x_0, t_{x_0}) < 0,$$

but this inequality contradicts (6a) and the proof is complete.

The following comparison lemma will be used throughout the paper.

Lemma 4. *Let v and w be upper and lower solutions of (1) respectively, then for $(x,t) \in S \times (0,T)$*
$$v(x,t) \geq w(x,t)$$

Proof. Let us introduce $z(x,t) = v(x,t) - w(x,t)$. We get :

$$z_t(x,t) - \Delta_\omega z(x,t) - p(b(x)(1-\eta(x,t))^{-p-1}z(x,t) \geq 0, \quad (x,t) \in S \times (0,T), \quad (7a)$$
$$z(x,t) \geq 0, \quad x \in \partial S \times (0,T), \quad (7b)$$
$$z_0(x) \geq 0, \quad x \in V, \quad (7c)$$

where $\eta(x,t)$ lies between $v(x,t)$ and $w(x,t)$, for $(x,t) \in S \times (0,T)$.
We can rewrite (7a)-(7c) as follows :

$$z_t(x,t) - \Delta_\omega z(x,t) - (b(x)a(x,t)z(x,t) \geq 0, \quad (x,t) \in S \times (0,T),$$

where $a(x,t) = p(1-\eta(x,t))^{-p-1}$ for $(x,t) \in S \times (0,T)$. According to lemma 3, $z(x,t) \geq 0$, for $(x,t) \in S \times (0,T)$ and the proof is complete.

We need the following two important lemmas.

Lemma 5. *Let u be solution of (1) with a positif initial data u_0 lower solution such as for some $x_0 \in S$, $u_0(x_0) \geq u_0(x) > 0$ and $b(x_0) \geq b(x)$ for $x \in S$. Then we have* *(i)* $u(x,t) \geq u_0(x)$, *for* $(x,t) \in S \times (0,T)$ *;*

(ii) $u(x_0,t) \geq u(x,t)$, *for* $(x,t) \in S \times (0,T)$ *;*

(iii) $u_t(x,t) > 0$, *for* $(x,t) \in S \times (0,T)$.

Proof. Here we follow the ideas of [6].

(i) Since u_0 is a lower solution of (1), by the lemma 4 we get:

$$u(x,t) \geq u_0(x) \geq 0, \quad \text{for} \quad (x,t) \in S \times (0,T).$$

(ii) Denote $z(x,t) = u(x_0,t) - u(x,t)$, for $(x,t) \in V \times (0,T)$. It follows from the definition of z that $z(x,0) \geq 0$ for any $x \in V$ and $z(x,t) \geq 0$ for any $(x,t) \in \partial S \times (0,T)$. Moreover, a straightforward calculation yields

$$z_t(x,t) - \Delta_\omega z(x,t) - pb(x)(1-a(x,t))^{-p-1} z(x,t) \geq 0, \quad \forall (x,t) \in S \times (0,T),$$

where $a(x,t)$ lies between $u(x_0,t)$ and $u(x,t)$. By virtue of Lemma 3, we have

$$u(x_0,t) \geq u(x,t) \quad \forall (x,t) \in S \times (0,T).$$

(iii) The proof of (iii) is similar to that of the lemma 4. So we omit the details here.

Now, we prove the quenching of u.

Theorem 6. *Let $u_0 \in C(V)$ a nonnegative and nontrivial data, compatible with the boundary condition such as $u_0(x_0) \geq u_0(x) > 0$ and $b(x_0) \geq b(x)$ for $x \in S$. Then the corresponding solution to (1) quenches in finite time T, and $T \leq \dfrac{(1-u_0(x_0))^{p+1}}{(p+1)(b(x_0)-h(x_0))}$. Moreover, there exists two positive constants C_1, C_2, such that*

$$C_1(T-t)^{\frac{1}{p+1}} \leq 1 - \|u(\cdot,t)\|_\infty \leq C_2(T-t)^{\frac{1}{p+1}}.$$

Proof.
Step 1 (Quenching). Set $h(x) = \sum_{y \in V} \omega(x,y)$, we get :

$$u_t(x,t) = \sum_{y \in V} \omega(x,y)(u(y,t) - u(x,t)) + b(x))(1-u(x,t))^{-p},$$

since $0 \leq u(y,t) < 1$ for $y \in V$ and $t \in (0,T)$, we obtain :

$$\begin{aligned} u_t(x,t) &\geq -\sum_{y\in V}\omega(x,y)u(x,t) + b(x)(1-u(x,t))^{-p}, \\ u_t(x,t) &\geq -h(x)(1-u(x,t))^{-p} + b(x)(1-u(x,t))^{-p}, \end{aligned}$$

thus,

$$(1-u(x,t))^p u_t(x,t) \geq b(x) - h(x). \tag{8}$$

Since $\min_{x\in S} b(x) \geq \max_{x\in S} h(x)$, we have $b(x) - h(x) > 0$.
Integrating inequality (8) from 0 to t, we get

$$(1-u(x,t))^{p+1} \leq (1-u_0(x))^{p+1} - (p+1)(b(x)-h(x))t.$$

According to Lemma 4 for $t \in (0,t)$, $u(\cdot,t)$ reaches its maximum at x_0, which implies that there exists $T \leq T_0 = \dfrac{(1-u_0(x_0))^{p+1}}{(p+1)(b(x_0)-h(x_0))}$ such that

$$\lim_{t\to T^-} u(x_0,t) = 1.$$

Step 2 (Quenching rates). Considering inequality (8) at point x_0, integrating from t to T, we get

$$(1-u(x_0,t))^{p+1} \geq (p+1)(b(x_0)-h(x_0))(T-t),$$

therefore

$$1-u(x_0,t) \geq C_1(T-t)^{\frac{1}{p+1}}, \text{ with } C_1 = ((p+1)(b(x_0)-h(x_0)))^{\frac{1}{p+1}}.$$

We still consider

$$u_t(x_0,t) = \sum_{y\in V}\omega(x_0,y)(u(y,t)-u(x_0,t)) + b(x_0)(1-u(x_0,t))^{-p}.$$

Due to $u(x_0,t) \geq u(x,t)$ for all $x \in V$, we get

$$u_t(x_0,t) \leq b(x_0)(1-u(x_0,t))^{-p}.$$

Multiplying both sides by $(1-u(x_0,t))^p$ and integrating from t to T, we have

$$(1-u(x_0,t))^{p+1} \leq (p+1)b(x_0) - (T-t),$$

therefore we get :

$$1 - u(x_0, t) \le C_2(T - t)^{\frac{1}{p+1}}, \text{ with } C_2 = ((p+1)b(x_0))^{\frac{1}{p+1}}.$$

Theorem 7. *Let $u_0 \in C(V)$ a nonnegative and nontrivial data, compatible with the boundary condition such as $u_0(x_0) \ge u_0(x) > 0$ and $b(x_0) \ge b(x)$ for $x \in S$. Then u_t blows up at the quenching time.*

Proof. We prove that u_t blows up at quenching time, as in [2] and [6].
Suppose that u_t is bounded. Then, a positive constant M exists, such that $u_t(x,t) < M$ for $(x,t) \in S \times (0, T]$. For $(x,t) \in S \times (0,T)$, we get :

$$u_t(x,t) = \Delta_\omega u(x,t) + b(x)(1 - u(x,t))^{-p} < M,$$

$$\begin{aligned} \Delta_\omega u(x_0,t) + b(x_0)(1 - u(x_0,t))^{-p} &< M, \\ -h(x_0)u(x_0,t) + b(x_0)(1 - u(x_0,t))^{-p} &< M, \end{aligned}$$

which implies

$$b(x_0)(1 - u(x_0,t))^{-p} < h(x_0)u(x_0,t) + M.$$

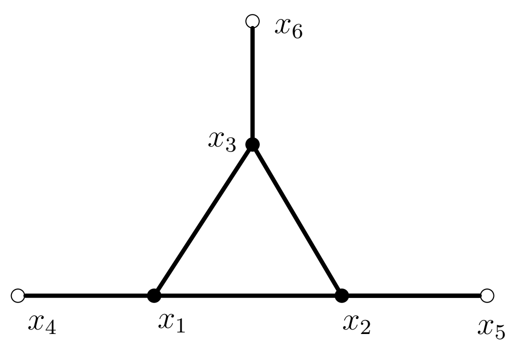
Therefore, we obtain

$$(1 - u(x_0,t))^{-p} < \frac{h(x_0) + M}{b(x_0)}.$$

When $t \to T^-$, the left-hand side tends to infinity while the right-side is finite. We then obtain a contradiction, which shows that u_t blows up when u quenches.

4. Examples and numerical experiments

In this section, we present examples to illustrate the quenching of the solution of (1) as well as the blow-up of the time-derivative of its solution. We consider a simple graph $G(V,E,\omega)$ such that $\omega(x,y) = 1$ if $\{x,y\} \in E$ and vertices $S = \{x_1,\ x_2,\ x_3\}$, $\partial S = \{x_4,\ x_5,\ x_6\}$ are linked as the following figure shows.



Thus, the problem (1) can be rewritten as:

$$\begin{cases} u_t(x_1,t) = -3u(x_1,t) + u(x_2,t) + u(x_3,t) + b(x_1)(1-u(x_1,t))^{-p}, & t \in (0,T), \\ u_t(x_2,t) = u(x_1,t) - 3u(x_2,t) + u(x_3,t) + b(x_2)(1-u(x_2,t))^{-p}, & t \in (0,T), \\ u_t(x_3,t) = u(x_1,t) + u(x_2,t) - 3u(x_3,t) + b(x_3)(1-u(x_3,t))^{-p}, & t \in (0,T), \\ u_0(x_1) = a, \\ u_0(x_2) = b, \\ u_0(x_2) = c. \end{cases} \tag{9}$$

This system is nonlinear. In this case, it is difficult to obtain its analytic solutions. This motives the following numerical study.

We present numerical experiments of (9) conducted with the linearly explicit method given by :

$$u^{(n+1)} = A_n u^{(n)} + \Delta t_n f^{(n)},$$

where $u^{(n)} = (u(x_1, n\Delta t_n), u(x_2, n\Delta t_n), u(x_3, n\Delta t_n))^T$,

$$A_n = \begin{bmatrix} 1-3\Delta t_n & \Delta t_n & \Delta t_n \\ \Delta t_n & 1-3\Delta t_n & \Delta t_n \\ \Delta t_n & \Delta t_n & 1-3\Delta t_n \end{bmatrix}, \ \Delta t_n = 0.0001(1-\|u^n\|_\infty)^{p+1},$$

$f^{(n)} = (b(x_1)(1-u(x_1,n\Delta t_n))^{-p}, b(x_2)(1-u(x_2,n\Delta t_n))^{-p}, b(x_3)(1-u(x_3,n\Delta t_n))^{-p})^T$, $b(x_1) = 4,\ b(x_2) = 4.5,\ b(x_3) = 4.5$ and $a = 0.05,\ b = 0.07,\ c = 0.08$.

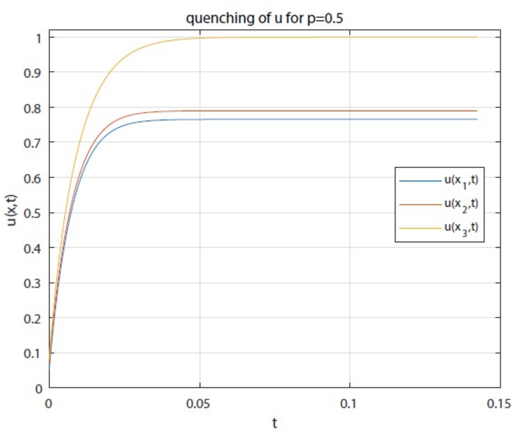


Figure 1: Evolution of u for $p = 0.5$

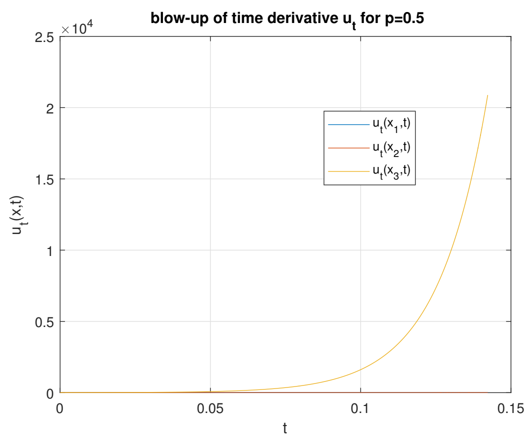


Figure 2: Evolution of u_t for $p = 0.5$

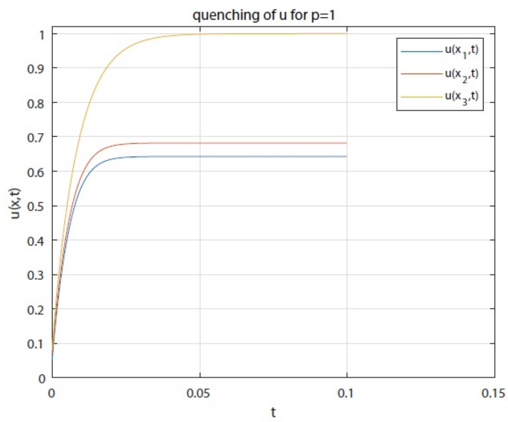


Figure 3: Evolution of u for $p = 1$

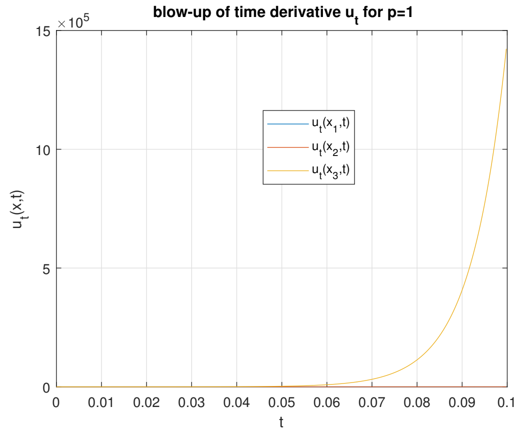


Figure 4: Evolution of u_t for $p = 1$

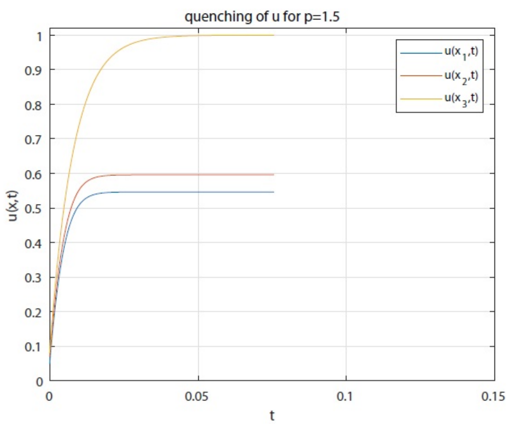


Figure 5: Evolution of u for $p = 1.5$

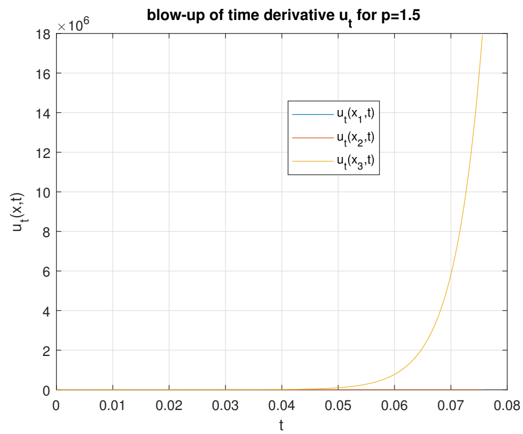


Figure 6: Evolution of u_t for $p = 1.5$

Remark 8. *We consider the problem (9) in the case where the initial data* $u_0(x_1) = 0.05,\ u_0(x_2) = 0.07,\ u_0(x_3) = 0.08$ *and* $b(x_1) = 4,\ b(x_2) = 4.5,\ b(x_3) = 4.5$. *From figures 1-6, we observe that* u_t *blows up while* u *quenches in a finite time. By the numerical simulation, when* $p = 0.1$ *(resp.* $p = 1$ *and* $p = 1.5$ *) we obtain approximately* $T = 0.1423$ *(resp.* $T = 0.0998$ *and* $T = 0.0756$*). Thus, we can say that when* p *increases, we have an acceleration of quenching of the solution. This result is not a surprise due to the result established in the previous section.*

5. Conclusion

In this paper, we consider the quenching phenomenon for the ω-diffusion equations on graphs with potential and singular source and we obtain the existence of local solution for problem (1) via Banach fixed point theorem. We also show that the solution of the problem quenches, whereas its time derivate blows up in a finite time. Moreover, we give the upper quenching time and the quenching rate on L^∞-norm. Examples were proposed to illustrate our results. In the future, we will further consider its lower blow-up time and also the blow-up set.

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