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AN OVERVIEW OF RECENT DEVELOPMENTS IN FUNCTIONS OF MATRIX ARGUMENT

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Abstract: The purpose of this article is to introduce the readers, especially the researchers, to some topics connected with functions of matrix argument, scaling models, distributions of products and ratios, Bayesian structures, symmetric products and symmetric ratios of matrices, scalar and matrix-variate fractional integrals, functions of matrix argument through entropy optimization, singular matrix-variate gamma and beta functions etc which are currently active so that interested readers can get into these classes of problems for their current research or teaching. Let X be a $p \times q, p \leq q$ matrix of rank p in the real domain. If the function $f(X)$, associated with X, is a function of XX', where a prime denotes the transpose, then such a function appears in a number of different disciplines. This paper examines the recent developments in such matrix-variate functions when X is in the real or complex domain. Connections to Bayes procedures, quantum physics, scalar and matrix texture models in communication and engineering problems, fractional integrals, distributions of symmetric products and symmetric ratios of matrices, singular matrix-variate gamma and beta functions and other related areas are pointed out. Only an overview of the current research in these topics with some illustrative examples are given in this paper. Since the material is summarized from the author's own works, most of the references are author's own papers, and hence similarity index, similarity with author's own works, may be high. The materials also cover some current results which are being published.

Keywords and Phrases: Functions of matrix argument, real and complex domains, scale mixtures, Bayesian structure, distribution of products, symmetric product and symmetric ratio of matrices, texture models. communication theory, fractional integrals.

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1. Introduction

In this paper, we consider only real-valued scalar function $f(X)$, where the argument X may be scalar (1×1 matrix), vector ($1 \times m$ or $m \times 1$ matrix), matrix or a sequence of matrices in the real or complex domain where f is scalar and real-valued. One can also make a statistical density out of $f(X)$ by imposing additional conditions. When $f(X)$ is a real-valued scalar function of X such that $f(X) \geq 0$ for all X in the domain of X and $\int_X f(X)\mathrm{d}X = 1$, then such an $f(X)$ is called a statistical density of X where X may be a scalar, vector, matrix, collection of matrices, in the real or complex domain. Here, $\mathrm{d}X$ represents the wedge product of differentials of all distinct real scalar variables in X. For two real scalar variables x and y the wedge product of differentials is defined as $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$ so that $\mathrm{d}x \wedge \mathrm{d}x = 0, \mathrm{d}y \wedge \mathrm{d}y = 0$. If $X = (x_{jk})$ is a $p \times q$ matrix in the real domain with x_{jk}'s being distinct real scalar variables, then $\mathrm{d}X = \wedge_{j=1}^{p} \wedge_{k=1}^{q} \mathrm{d}x_{jk}$. If $\tilde{X} = (\tilde{x}_{jk})$ is a $p \times q$ matrix in the complex domain with distinct scalar complex variables $\tilde{x}_{jk}$'s as elements, then one can write $\tilde{X} = X_1 + iX_2, i = \sqrt{(-1)}, X_1, X_2$ are real $p \times q$ matrices, then $\mathrm{d}\tilde{X}$ will be defined as $\mathrm{d}X_1 \wedge \mathrm{d}X_2$. The integral of a real-valued scalar function $f(X)$, over X, will be denoted as $\int_X f(X)\mathrm{d}X$.

When we deal with matrix-variate functions in the real and complex domains, a multiplicity of symbols and notations are required to represent items uniquely. Hence, the following simplified notations will be used. Real scalar variables, whether mathematical variables or random variables, will be denoted by lower-case letters such as x, y, z and vector/matrix variables, mathematical or random, will be denoted by capital letters such as X, Y, Z. Scalar constants will be denoted by a, b, c etc and vector/matrix constants by A, B etc. Variables in the complex domain will be written with a tilde such as $\tilde{x}, \tilde{y}, \tilde{X}, \tilde{Y}$. No tilde will be used on constants. Greek letters and other symbols will be explained when they occur for the first time. For a $p \times p$ matrix A, the determinant will be written as $|A|$ or $\det(A)$. When A is in the complex domain, then $\det(A) = a + ib, i = \sqrt{(-1)}, a, b$ are real scalar quantities, then the absolute value of the determinant will be written as $|\det(A)| = \sqrt{a^2 + b^2}$. If B is in the complex domain then B^* will represent the conjugate transpose of B. If $B = B^*$, then B is Hermitian. A positive definite $p \times p$ matrix A will be written as $A > O$ and $A = A'$ when A is real, and $B = B^* > O$ (Hermitian positive definite) when B is in the complex domain, where the prime denotes the transpose.

Let $X = (x_{jk})$ be $p \times q, p \leq q$ matrix of rank p in the real domain, that is, X is a full rank matrix. Then, $XX' > O$ (positive definite). If $\tilde{X}$ is $p \times q, p \leq q$ matrix of rank p in the complex domain, then $\tilde{X}\tilde{X}^* > O$ (Hermitian positive definite) observing that $\tilde{X}$ is a full rank matrix. If X is real, then $S = XX' > O$ is $p \times p$ when X is a full rank matrix of rank $p, p \leq q$ and when X is $p \times q$. If $p > q$ and X is of rank $r \leq q$, then XX' will be positive semi-definite in the real case and $\tilde{X}\tilde{X}^*$ will be Hermitian positive semi-definite in the complex domain. Then, $f(XX')$ or $f(\tilde{X}\tilde{X}^*)$ will be a singular matrix-variate case. A glimpse into the singular case will be given in Section 7. If a function of X is a function of $XX' = S$, then one can convert the differential element $\mathrm{d}X$ into the differential element $\mathrm{d}S$, if necessary. This can be achieved by a result from Mathai [8] which will be given here as a lemma.

Lemma 1.1. *Let X be a $p \times q, p \leq q$ matrix of rank p in the real domain. Let $S = XX'$. Then, going through a transformation involving a lower triangular matrix with positive diagonal elements and a unique semi-orthonormal matrix and then integrating out the differential element corresponding to the semi-orthonormal matrix, we have the following connection:*

$$\mathrm{d}X = \frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}|S|^{\frac{q}{2}-\frac{p+1}{2}}\mathrm{d}S$$

where, for example, $\Gamma_p(\alpha)$ is the real matrix-variate gamma function defined as the following where Y is a $p \times p$ real positive definite matrix:

$$\begin{aligned}\Gamma_p(\alpha) &= \int_{Y>O} |Y|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(Y)}\mathrm{d}Y, \Re(\alpha) > \frac{p-1}{2} \\ &= \pi^{\frac{p(p-1)}{4}}\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})...\Gamma(\alpha-\frac{p-1}{2}), \Re(\alpha) > \frac{p-1}{2}\end{aligned}$$

where $\mathrm{tr}(\cdot)$ means the trace of the square matrix $(\cdot)$ and $\Re(\cdot)$ means the real part of $(\cdot)$. When $\tilde{X}$ is $p \times q, p \leq q$ matrix of rank p in the complex domain and when $\tilde{S} = \tilde{X}\tilde{X}^* > O$ we have the following connection between the differential elements of $\tilde{X}$ and $\tilde{S}$, which is obtained by going through a transformation involving a lower triangular matrix with real and positive diagonal elements and a unique semi-unitary matrix and then integrating out the differential element corresponding to the semi-unitary matrix:

$$\mathrm{d}\tilde{X} = \frac{\pi^{pq}}{\tilde{\Gamma}_p(q)}|\det(\tilde{S})|^{q-p}\mathrm{d}\tilde{S}$$

where, for example, $\tilde{\Gamma}_p(\alpha)$ *is a complex matrix-variate gamma function defined as the following where* $\tilde{Z} > O$ *is a* $p \times p$ *Hermitian positive definite matrix:*

$$\tilde{\Gamma}_p(\alpha) = \int_{\tilde{Z}>O} |\det(\tilde{Z})|^{\alpha-p} e^{-\mathrm{tr}(\tilde{Z})} d\tilde{Z}, \Re(\alpha) > p-1$$
$$= \pi^{\frac{p(p-1)}{2}} \Gamma(\alpha)\Gamma(\alpha-1)...\Gamma(\alpha-p+1), \Re(\alpha) > p-1.$$

For the proof and for other related Jacobians of matrix transformations, see Mathai [8]. For example, if we have a function of the following form, where X is real $p \times q, p \leq q$ matrix of rank p,

$$f(X)dX = c|XX'|^{\gamma}[\mathrm{tr}(XX')]^{\eta} e^{-[\mathrm{tr}(XX')]^{\delta}} dX \tag{1.1}$$

then, we can write it as

$$f(X)dX = c\frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}|S|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}[\mathrm{tr}(S)]^{\eta} e^{-[\mathrm{tr}(S)]^{\delta}} dS$$

where c can be taken as the normalizing constant if $f(X)$ is taken as a statistical density. Now, there are techniques of integrating out S, see Mathai [14], to obtain the normalizing constant c here. Corresponding will be the procedure in the complex domain. Note that (1.1) is the most general form of a rectangular matrix-variate function of the gamma or Gaussian type in the standard form. When $\gamma = 0$ the model in (1.1) is often called Kotz model. Some people call (1.1) with $\gamma \neq 0$ as Kotz model also. This model is widely used in communication problems. When return signals in radar transmissions are captured, a cross section of the multi-look data will have pepper dust like contaminants called freckles and the cross section itself is a random quantity called texture. Then, a cross section with the contaminants is of the form AX where X represents the freckle component and A represents the texture component. One can look into the distribution of X at preassigned A, then the problem goes into Bayes analysis. One can look at A as a scaling parameter then AX is a scaling model or scale mixture. If both A and X are considered as independent random variables then it is a problem of the distribution of a product. Usually, in communication problems involving multi-look data, X is a $p \times q$ matrix. Usually, each elements will have a phase part and a time part and hence the proper representation of each element is through a scalar complex variable. Thus, usually, $\tilde{X}$ will be in the complex domain and the texture parameter A can be a scalar or vector or matrix. Very often the density of $\tilde{X}$ is taken as the complex analogue of (1.1). Starting from the 1960's, the model in (1.1) with $\gamma \neq 0$ was being used in physics, statistics, communication and engineering problems etc. The normalizing constants traditionally used may be seen from Diaz-Garcia and Gutiérrez-Jáimez [7]. Detailed derivations of the normalizing constants in all the models involving determinant with the power $\gamma \neq 0$ and at the same time the exponential trace having a power of the form $[\mathrm{tr}(XX')]^{\delta}, \delta > 0$ are given recently in Mathai (2023).

In (1.1) if $\gamma = 0, \eta = 0, \delta = 1$ then it is the Gaussian model in the standard form. A general matrix-variate Gaussian model is of the following form, where X is $p \times q, p \leq q$ and of rank p:

$$f_1(X)\mathrm{d}X = c_1 \mathrm{e}^{-\mathrm{tr}(AXBX')}\mathrm{d}X \tag{1.2}$$

where $A > O$ is $p \times p$ and $B > O$ is $q \times q$ constant positive definite matrices. If a location parameter is to be included, then we replace $AXBX'$ by $A(X-M)B(X-M)', M$ is a location parameter matrix and if $f_1(X)$ is a statistical density, then $M = E[X]$ where $E[\cdot]$ means the expected value of $[\cdot]$. There will not be any change in the normalizing constant c_1 by incorporating M into the model. In order to evaluate the normalizing constant in (1.2), we have to convert $\mathrm{tr}(AXBX')$ into a form $\mathrm{tr}(UU')$ and then UU' into S, that is $\mathrm{d}U$ into $\mathrm{d}S$ by using Lemma 1.1 and then integration is possible with the help of a matrix-variate type-1 beta integral. In order to convert to UU' form we need another matrix transformation, which will be given here as a lemma.

Lemma 1.2. *Let $X = (x_{jk})$ be $p \times q$ matrix with distinct real scalar elements x_{jk}'s. Let A be a $p \times p$ and B be a $q \times q$ nonsingular constant matrices. Then,*

$$Y = AXB, |A| \neq 0, |B| \neq 0, \Rightarrow \mathrm{d}Y = |A|^q |B|^p \mathrm{d}X$$

If $\tilde{X} = (\tilde{x}_{jk})$ is a $p \times q$ matrix in the complex domain with distinct scalar complex variables as elements, then

$$\tilde{Y} = A\tilde{X}B, |A| \neq 0, |B| \neq 0, \Rightarrow \mathrm{d}\tilde{Y} = |\mathrm{det}(A)|^{2q} |\mathrm{det}(B)|^{2p} \mathrm{d}\tilde{X}$$

where the $p \times p$ matrix A and the $q \times q$ matrix B are constant nonsingular matrices in the real or complex domain, observing that one can write, for example, $|\mathrm{det}(A)|^{2q} = |\mathrm{det}(AA^*)|^q$ where A^* is the conjugate transpose of A or $(A^c)' = (A')^c = A^*$, where A^c is the conjugate of A.

For the proof and other details, see Mathai [8]. As an application of Lemma 1.2, consider the exponent in the Gaussian model (1.2). One can write $\mathrm{tr}(AXBX') = \mathrm{tr}(A^{\frac{1}{2}}XBX'A^{\frac{1}{2}})$ where $A^{\frac{1}{2}}$ is the positive definite square root of the positive definite matrix $A > O$. Now, write $U = A^{\frac{1}{2}}XB^{\frac{1}{2}} \Rightarrow \mathrm{d}U = |A|^{\frac{q}{2}}|B|^{\frac{p}{2}}\mathrm{d}X$ by Lemma 1.2 and thus the exponent has become $\mathrm{tr}(UU')$. Note that once we have a form UU' which is symmetric. If we need any further transformation involving UU', then we need Jacobians involving symmetric matrices. One such result will be listed here as a lemma.

Lemma 1.3. *Let $X = X'$ be a $p \times p$ nonsingular symmetric matrix with distinct real scalar variables as elements. Let A be a $p \times p$ nonsingular constant matrix. Then,*

$$Y = AXA', |A| \neq 0, \Rightarrow \mathrm{d}Y = |A|^{p+1}\mathrm{d}X.$$

When $\tilde{X} = \tilde{X}^*$ is $p \times p$ Hermitian and when A is a $p \times p$ constant nonsingular matrix in the real or complex domain, then

$$\tilde{Y} = A\tilde{X}A^*, |A| \neq 0, \Rightarrow \mathrm{d}\tilde{Y} = |\det(A)|^{2p}\mathrm{d}\tilde{X} = |\det(AA^*)|^{p}\mathrm{d}\tilde{X}.$$

For the proof, see Mathai [8]. As an illustration, let us consider the evaluation of the following integral:

$$\int_{X>O} |X|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(BX)}\mathrm{d}X$$

where $X = X' > O$ is $p \times p$ positive definite, $B = B' > O$ is a constant $p \times p$ positive definite matrix. We may write $\mathrm{tr}(BX) = \mathrm{tr}(B^{\frac{1}{2}}XB^{\frac{1}{2}})$. Now, let $Y = B^{\frac{1}{2}}XB^{\frac{1}{2}} \Rightarrow \mathrm{d}Y = |B|^{\frac{p+1}{2}}\mathrm{d}X$ by Lemma 1.3. Now, we have an integral of the form

$$|B|^{-\alpha}\int_{Y>O} |Y|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(Y)}\mathrm{d}Y.$$

This is the matrix-variate gamma integral defined in Lemma 1.1. The value of this integral is also given in Lemma 1.1. How do we evaluate the above real matrix-variate gamma integral? This can be evaluated by using another result which is given here as a lemma.

Lemma 1.4. *Let the $p \times p$ matrix $X = (x_{jk})$ be real and positive definite. Let $T = (t_{jk})$ be a lower triangular matrix with positive diagonal elements, that is, $t_{jk} = 0, j < k, t_{jj} > 0, j = 1, ..., p$. Then,*

$$X = TT' \Rightarrow \mathrm{d}X = 2^p\{\prod_{j=1}^{p} t_{jj}^{p+1-j}\}\mathrm{d}T.$$

Let $\tilde{X} = \tilde{X}^* > O$ be $p \times p$ Hermitian positive definite with distinct scalar complex variables as elements. Let $\tilde{T} = (\tilde{t}_{jk})$ be a lower triangular matrix in the complex *domain with real and positive diagonal elements, that is* $\tilde{t}_{jk} = 0, j < k, \tilde{t}_{jj} = t_{jj} > 0, j = 1, ..., p$. *Then,*

$$\mathrm{d}\tilde{X} = 2^p\{\prod_{j=1}^{p} t_{jj}^{2(p-j)+1}\}\mathrm{d}\tilde{T}.$$

For the proof, see Mathai [8]. Now, let us evaluate the real matrix-variate gamma integral, namely

$$\Gamma_p(\alpha) = \int_{X>O} |X|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X$$

where X is $p \times p$ real and positive definite with distinct real scalar variables as elements. Let $X = TT'$ where T is lower triangular with positive diagonal elements. Note that $|X| = |TT'| = \prod_{j=1}^{p} t_{jj}^2$ and $\mathrm{tr}(X) = \mathrm{tr}(TT') = \sum_{j\geq k} t_{jk}^2$ because for any matrix $B = (b_{jk})$, square or rectangular, $\mathrm{tr}(BB') = \mathrm{tr}(B'B) = \sum_{jk} b_{jk}^2 =$ the sum of squares of all elements in B. If B is in the complex domain, then $\mathrm{tr}(BB^*) = \mathrm{tr}(B^*B) = \sum_{jk} |b_{jk}|^2 =$ sum of squares of the absolute values of all elements in B. Now, $\Gamma_p(\alpha)$ splits into integrals over t_{jj}'s and integrals over t_{jk}'s for $j > k$. That is,

$$\Gamma_p(\alpha) = \{\prod_{j=1}^{p} 2\int_0^\infty (t_{jj}^2)^{\alpha-\frac{j}{2}}\mathrm{e}^{-t_{jj}^2}\mathrm{d}t_{jj}\}\{\prod_{j>k}\int_{-\infty}^{\infty} \mathrm{e}^{-t_{jk}^2}\mathrm{d}t_{jk}\}.$$

Observe that

$$2\int_0^\infty (t_{jj}^2)^{\alpha-\frac{j}{2}}\mathrm{e}^{-t_{jj}^2}\mathrm{d}t_{jj} = \Gamma(\alpha - \frac{j-1}{2}), \Re(\alpha) > \frac{j-1}{2}$$

for $j = 1, ..., p$ which means

$$\Gamma(\alpha)\Gamma(\alpha - \frac{1}{2})...\Gamma(\alpha - \frac{p-1}{2}), \Re(\alpha) > \frac{p-1}{2}$$

and

$$\prod_{j>k}\int_{-\infty}^{\infty} \mathrm{e}^{-t_{jk}^2}\mathrm{d}t_{jk} = \prod_{j>k}\sqrt{\pi} = \pi^{\frac{p(p-1)}{4}}.$$

These establish that

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}}\Gamma(\alpha)\Gamma(\alpha - \frac{1}{2})...\Gamma(\alpha - \frac{p-1}{2}), \Re(\alpha) > \frac{p-1}{2}.$$

In a similar manner, one can establish the result that the integral representation in the complex matrix-variate gamma in Lemma 1.1 has the explicit expression as given there. For the $p \times p$ real positive definite matrix one can define the real matrix-variate beta function as

$$B_p(\alpha,\beta) = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} = B_p(\beta,\alpha), \Re(\alpha) > \frac{p-1}{2}, \Re(\beta) > \frac{p-1}{2}$$

and the corresponding complex matrix-variate beta function will be defined as

$$\tilde{B}_p(\alpha,\beta) = \frac{\tilde{\Gamma}_p(\alpha)\tilde{\Gamma}_p(\beta)}{\tilde{\Gamma}_p(\alpha+\beta)} = \tilde{B}_p(\beta,\alpha), \Re(\alpha) > p-1, \Re(\beta) > p-1.$$

From here, one can derive type-1 and type-2 beta integrals as the following, see the details from Mathai [8]: For $p \times p$ real positive definite matrices $X > O$, $Y > O$, $\Re(\alpha) > \frac{p-1}{2}$, and $\Re(\beta) > \frac{p-1}{2}$

$$\int_{O<X<I} |X|^{\alpha-\frac{p+1}{2}} |I-X|^{\beta-\frac{p+1}{2}} \mathrm{d}X = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.3}$$

which is a type-1 real matrix-variate beta integral, where $O < X < I$ means $X > O$ and $I - X > O$ (both positive definite), and

$$\int_{Y>O} |Y|^{\alpha-\frac{p+1}{2}} |I+Y|^{-(\alpha+\beta)} \mathrm{d}Y = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.4}$$

which is called a type-2 real matrix-variate beta integral. The corresponding integrals in the complex domain are the following for $\Re(\alpha) > p-1, \Re(\beta) > p-1$:

$$\int_{O<\tilde{X}<I} |\det(\tilde{X})|^{\alpha-p} |\det(I-\tilde{X})|^{\beta-p} \mathrm{d}\tilde{X} = \frac{\tilde{\Gamma}_p(\alpha)\tilde{\Gamma}_p(\beta)}{\tilde{\Gamma}_p(\alpha+\beta)} \tag{1.5}$$

which is called a type-1 complex matrix-variate beta integral, and

$$\int_{\tilde{Y}>O} |\det(\tilde{Y})|^{\alpha-p} |\det(I+\tilde{Y})|^{-(\alpha+\beta)} \mathrm{d}\tilde{Y} = \frac{\tilde{\Gamma}_p(\alpha)\tilde{\Gamma}_p(\beta)}{\tilde{\Gamma}_p(\alpha+\beta)} \tag{1.6}$$

which is called a complex matrix-variate type-2 beta integral. Properties of type-1 and type-2 beta integrals, connections to gamma integrals and other related properties may be seen from Mathai [8].

This paper is organized as follows: Section 1 deals with some preliminaries and notations to be used in the manuscript. Section 2 evaluates some general integrals and introduces some general densities, mainly of the matrix-variate type. Section 3 details some recent procedures involving logistic type models for vector/matrix variables in the real and complex domains. Section 4 concentrates on the evaluation of the normalizing constants in some general matrix-variate densities and introduces some general scaling models and general Bayesian structures. Section 5 shows the connection of Mellin convolutions of products and ratios and statistical distribution theory to fractional calculus, especially to fractional integrals of the first and second kinds. Section 6 deals with the derivations of matrix-variate functions through optimization of Mathai entropy. Section 7 gives a glimpse into singular matrix-variate gamma and beta functions and Section 8 provides some concluding remarks.

2. Evaluation of Some General Integrals and Some General Densities

We will consider several models here. Once a model is introduced, for studying the properties of such a model, the procedures will be parallel to the procedure in computing the normalizing constant there. Hence, in what follows, we will introduce various models and compute the normalizing constants in each case and stop the discussion of that model.

Problem 2.1. A real multivariate model Let X be a $p\times 1$ real vector, $X'=[x_1,...,x_p]$ with x_j's real scalar variables. Then, $X'X=x_1^2+...+x_p^2$. Consider the function

$$f_1(X)\mathrm{d}X=c_1(X'X)^{\gamma}\mathrm{e}^{-b(X'X)^{\delta}}\mathrm{d}X.$$

Let us take $f_1(X)$ as a density and let us evaluate the normalizing constant c_1. Since $X'X$ is invariant under rotation of the axes of coordinates or under orthonormal transformations, $f_1(X)$ here is a spherically symmetric distribution. It is also the density for isotropic gamma distributed random points in a p-dimensional Euclidean space. For isotropic and non-isotropic random points in geometrical probability problems, see Mathai [9]. Here we have isotropic gamma distributed random points. Let us evaluate the normalizing constant in $f_1(X)$. Let $u=X'X$. Note that u is a scalar variable. Then, from Lemma 1.1 for a $1\times p$ matrix, we have

$$\mathrm{d}X=\frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}u^{\frac{p}{2}-1}\mathrm{d}u.$$

Then, the total integral

$$1=c_1\int_X[X'X]^{\gamma}\mathrm{e}^{-b(X'X)^{\delta}}\mathrm{d}X=c_1\frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}\int_0^{\infty}u^{\gamma+\frac{p}{2}-1}\mathrm{e}^{-bu^{\delta}}\mathrm{d}u$$
$$=c_1\frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}\frac{1}{\delta}\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))b^{-\frac{1}{\delta}(\gamma+\frac{p}{2})}\Rightarrow$$

$$c_1 = \frac{\delta\Gamma(\frac{p}{2})b^{\frac{1}{\delta}(\gamma+\frac{p}{2})}}{\pi^{\frac{p}{2}}\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))}$$

for $\Re(\gamma) > -\frac{p}{2}, \delta > 0, b > 0$. The last steps are done by making the substitution $v = u^\delta$ and then integrating out v by using a real scalar variable gamma integral. We will denote the corresponding density in the complex domain by incorporating the letter c into the subscript and write

$$f_{1c}(\tilde{X})\mathrm{d}\tilde{X} = \tilde{c}_1[\tilde{X}^*\tilde{X}]^\gamma \mathrm{e}^{-b[\tilde{X}^*\tilde{X}]^\delta}\mathrm{d}\tilde{X}.$$

Let

$$u = \tilde{X}^*\tilde{X} = |\tilde{x}_1|^2 + ... + |\tilde{x}_p|^2 = (x_{11}^2 + x_{12}^2) + ... + (x_{p1}^2 + x_{p2}^2)$$

where $\tilde{x}_j = x_{j1} + ix_{j2}, i = \sqrt{(-1)}, x_{j1}, x_{j2}$ are real scalar quantities. Here, we have the sum of squares of $2p$ real scalar variables, as opposed to p scalar variables in the real case. Then,

$$\mathrm{d}\tilde{X} = \frac{\pi^{\frac{2p}{2}-1}}{\Gamma(\frac{2p}{2})}u^{\frac{2p}{2}-1}\mathrm{d}u = \frac{\pi^{p-1}}{\Gamma(p)}u^{p-1}\mathrm{d}u.$$

Thus, $\frac{p}{2}$ in the real case is replaced by p in the complex case. This is the only difference. Hence, in the complex case the normalizing constant

$$\tilde{c}_1 = \frac{\delta\Gamma(p)b^{\frac{1}{\delta}(\gamma+p)}}{\pi^p\Gamma(\frac{1}{\delta}(\gamma+p))}$$

for $\Re(\gamma) > -p, \delta > 0, b > 0$. Note that the density $f_1(X)$ is also the multivariate Maxwell-Boltzmann and Raleigh densities in the standard forms in Physics considered in Mathai and Princty [17] and the corresponding density in the complex domain is considered in Mathai, Provost and Haubold [19].

Problem 2.2. A rectangular matrix-variate model. Let X be $p \times q, p \leq q$ matrix of rank p in the real domain. Consider the density

$$f_2(X)\mathrm{d}X = c_2[\mathrm{tr}(XX')]^\eta \mathrm{e}^{-b[\mathrm{tr}(XX')]^\delta}\mathrm{d}X.$$

Let us evaluate the normalizing constant c_2. Since X has pq distinct real scalar variables, $\mathrm{tr}(XX')$ = sum of squares of pq real scalar variables. Hence if $u = \mathrm{tr}(XX')$, then consider a string of pq real scalar variables, and now the wedge product of the differentials of the elements in this string is $\mathrm{d}X$ itself. Hence

$$\mathrm{d}X = \frac{\pi^{\frac{pq}{2}}}{\Gamma(\frac{pq}{2})}u^{\frac{pq}{2}-1}\mathrm{d}u.$$

Now, proceeding as in Problem 1, we have the normalizing constant

$$c_2 = \frac{\delta\Gamma(\frac{pq}{2})b^{\frac{1}{\delta}(\gamma+\frac{pq}{2})}}{\pi^{\frac{pq}{2}}\Gamma(\frac{1}{\delta}(\gamma+\frac{pq}{2}))}, \Re(\gamma) > -\frac{pq}{2}, b > 0, \delta > 0.$$

In the corresponding complex case, replace $\frac{pq}{2}$ from the real case by pq for the complex case, to obtain the final result.

Problem 2.3. Another rectangular matrix-variate model. Let X be a $p \times q, p \leq q$ matrix of rank p in the real domain. Consider the model

$$f_3(X)\mathrm{d}X = c_3|XX'|^{\gamma}\mathrm{e}^{-b[\mathrm{tr}(XX')]^{\delta}}\mathrm{d}X.$$

Let us evaluate the normalizing constant c_3. Let the $p\times p$ matrix $S = XX'$. Then, $S = S' > O$. From Lemma 1.1,

$$\mathrm{d}X = \frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}|S|^{\frac{q}{2}-\frac{p+1}{2}}\mathrm{d}S.$$

Then

$$\int_X f_3(X)\mathrm{d}X = c_3\frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}\int_{S>O}|S|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}\mathrm{e}^{-b[\mathrm{tr}(S)]^{\delta}}\mathrm{d}S.$$

For $\delta = 1$ one can integrate out by using a real matrix-variate gamma integral of Lemma 1.1. Hence, for $\delta = 1$

$$\int_{S>O}|S|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}\mathrm{e}^{-b[\mathrm{tr}(S)]}\mathrm{d}S = \Gamma_p(\gamma+\frac{q}{2})b^{-p(\gamma+\frac{q}{2})}$$

and from here c_3 is available. Parallel steps will give the normalizing constant in the corresponding complex case for $\delta = 1$. When $\delta \neq 1$ then the normalizing constant is available from the general problem next.

Problem 2.4. Another rectangular matrix-variate model. Let X be $p \times q, p \leq q$ matrix of rank p in the real domain. Consider the model

$$f_4(X) = c_4|XX'|^{\gamma}[\mathrm{tr}(XX')]^{\eta}\mathrm{e}^{-b[\mathrm{tr}(XX')]^{\delta}}, \delta > 0, b > 0$$

for $\Re(\eta) > 0, \Re(\gamma) > -\frac{pq}{2}$. Let us evaluate the normalizing constant c_4. One method of doing it is to first use Lemma 1.1 and convert to $S = XX'$ and $\mathrm{d}X$ to $\mathrm{d}S$. Then, use Lemma 1.4 and convert S into $S = TT'$ where T is lower triangular with positive diagonal elements and $\mathrm{d}S$ into $\mathrm{d}T$. Then, there will be a total of

$p(p+1)/2$ squares of the form t_{jk}^2. Make a general polar coordinate transformation on these $p(p+1)/2$, t_{jk}'s. Then, integrate out the polar radius r and the polar angles $\theta_1, ..., \theta_{k-1}, k = p(p+1)/2$. One can use a parallel procedure in the complex case. But in the complex case, there will be only p^2 sum of squares of t_{jk}^2's. Hence, here $k = p^2$ when using general polar coordinate transformation. Instead of going through two lemmas, one can go directly to t_{jk}'s by using the following lemma.

Lemma 2.1. *Let X be $p \times q, p \leq q$ matrix of rank p in the real domain. Let $X = TU'$ where T is a lower triangular matrix with positive diagonal elements and U is a unique semi-orthonormal matrix $UU' = I_p$. Then, after integrating out the differential element corresponding to U, one has the following connection between $\mathrm{d}X$ and $\mathrm{d}T$:*

$$\mathrm{d}X = \frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}\{\prod_{j=1}^{p} t_{jj}^{q-j}\}\mathrm{d}T.$$

Let $\tilde{X}$ be $p \times q, p \leq q$ matrix of rank p in the complex domain. Let $\tilde{X} = \tilde{T}\tilde{U}^*$ where T is a lower triangular matrix with real and positive diagonal elements and $\tilde{U}$ is a unique semi-unitary matrix $\tilde{U}\tilde{U}^* = I_p$ with diagonal elements real. Then, after integrating out the differential element corresponding to $\tilde{U}$, we have the following connection, see the details from Mathai[8]:

$$\mathrm{d}\tilde{X} = \frac{\pi^{\frac{p(p-1)}{2}}}{\tilde{\Gamma}_p(q)}\{\prod_{j=1}^{p} t_{jj}^{2(p-j)+1}\}\mathrm{d}\tilde{T}.$$

Now, if we apply Lemma 2.1 in the real domain, then $\mathrm{tr}(XX') = \mathrm{tr}(TT') =$ sum of squares of $p(p+1)/2$, t_{jk}'s and $|XX'| = |TT'| = \prod_{j=1}^{p} t_{jj}^2$. Now, apply a general polar coordinate transformation on the t_{jk}'s, namely the following, which will be stated as a lemma:

Lemma 2.2. *Let $x_1, ..., x_k$ be distinct real scalar variables. Consider the polar coordinates $r, \theta_1, ..., \theta_{k-1}, r > 0, -\frac{\pi}{2} < \theta_j \leq \frac{\pi}{2}, j = 1, ..., k-2, -\pi < \theta_{k-1} \leq \pi$. Consider the transformation*

$$x_1 = r \ \sin\theta_1$$
$$x_j = r \ \cos\theta_1 ... \cos\theta_{j-1} \sin\theta_j, j = 2, ..., k-2$$
$$x_k = r \ \cos\theta_1 ... \cos\theta_{k-1}$$

Then

$$\mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_k = r^{k-1}\{\prod_{j=1}^{k-1} |\cos\theta_j|^{p-j-1}\}\mathrm{d}r \wedge \mathrm{d}\theta_1 \wedge ... \wedge \mathrm{d}\theta_{k-1}.$$

In our case, $k = p(p+1)/2$ in the real domain and $k = p^2$ in the complex domain. Note that $\text{tr}(TT') = r^2$ under Lemma 2.2. $|XX'| = |TT'| = \prod_{j=1}^{p} t_{jj}^2$. When applying Lemma 2.2 to our t_{jk}'s, take the first p, x_j's of Lemma 2.2 as $t_{11}, t_{22}, ..., t_{pp}$. Take the remaining x_j's as $t_{21}, t_{31}, t_{32}, ..., t_{p1}, ..., t_{pp-1}$ so that $k = p + p(p-1)/2 = p(p+1)/2$. Now, $t_{11}^2 = r^2(\sin\theta_1)^2, t_{22}^2 = r^2(\cos\theta_1 \sin\theta_2)^2$ etc. Then, integrate out r and then integrate out the sine and cosine product coming from $\prod_{j=1}^{p}(t_{jj}^2)^{\frac{q}{2}-\frac{j}{2}}(t_{jj}^2)^{p-j+\frac{1}{2}}$. Details of the integration is given explicitly in Mathai [14] and the final result is the following:

$$c_4 = \frac{\delta\Gamma_p(\frac{q}{2})b^{\frac{1}{\delta}(p(\gamma+\frac{q}{2})+\eta))}\Gamma(p(\gamma+\frac{q}{2}))}{\pi^{\frac{pq}{2}}\Gamma[\frac{1}{\delta}(p(\gamma+\frac{q}{2})+\eta)]\Gamma_p(\gamma+\frac{q}{2})} \tag{2.1}$$

for $\Re(\gamma) > -\frac{q}{2} + \frac{p-1}{2}$ and the corresponding normalizing constant in the complex domain is the following:

$$\tilde{c}_4 = \frac{\delta\tilde{\Gamma}_p(q)b^{\frac{1}{\delta}(p(\gamma+q)+\eta))}\Gamma(p(\gamma+q))}{\pi^{pq}\Gamma[\frac{1}{\delta}(p(\gamma+q)+\eta)]\tilde{\Gamma}_p(\gamma+q)} \tag{2.2}$$

for $\Re(\gamma) > -q + p - 1$.

Note 2.1. One can extend or generalize all the above models by replacing XX' by $A^{\frac{1}{2}}(X-M)B(X-M)'A^{\frac{1}{2}}$ in the real case, where $M = E[X], A > O$ and $B > O$ are $p \times p$ and $q \times q$ constant positive definite matrices. In the complex domain, replace $\tilde{X}\tilde{X}^*$ by $A^{\frac{1}{2}}(\tilde{X}-\tilde{M})B(\tilde{X}-\tilde{M})^*A^{\frac{1}{2}}$ where $\tilde{M} = E[\tilde{X}], A = A^* > O, B = B^* > O$ are $p\times p$ and $q\times q$ constant hermitian positive definite matrices. The only difference will be that the normalizing constant has to be multiplied by $|A|^{\frac{q}{2}}|B|^{\frac{p}{2}}$ in the real case, and in the complex case multiply the normalizing constant by $|\det(A)|^q|\det(B)|^p$.

3. Logistic Type Models

In each section we will list the functions in serial order as f_1, f_2 etc with the corresponding normalizing constants by c_1, c_2 etc in order the avoid multiplicity of notations and symbols. For a real scalar variable x the logistic density is the following:

$$f(x) = \frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2} = \frac{\mathrm{e}^{x}}{(1+\mathrm{e}^{x})^2}, -\infty < x < \infty.$$

The graph of this model looks exactly like that of a real standard normal density but the logistic density has a thicker tail compared to that of the standard normal density. Hence, large deviations have higher probabilities compared to that of the standard normal. Therefore, for industrial applications, a logistic model is preferred to a standard Gaussian model. The logistic model can also be generated from a type-2 beta density by exponentiation. Consider a type-2 beta density for a real scalar variable $y \geq 0$, namely,

$$f_1(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1+y)^{-(\alpha+\beta)}, y \geq 0, \Re(\alpha) > 0, \Re(\beta) > 0.$$

Let $y = \mathrm{e}^{-x}$, then the density, denoted by $f_2(x)$, is the following:

$$f_2(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\mathrm{e}^{-\alpha x}}{(1+\mathrm{e}^{-x})^{\alpha+\beta}} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\mathrm{e}^{\beta x}}{(1+\mathrm{e}^{\alpha x})^{\alpha+\beta}}, -\infty < x < \infty$$

under the same conditions $\Re(\alpha) > 0, \Re(\beta) > 0$. When $\alpha = 1, \beta = 1$ in $f_2(x)$, one has the logistic model $f(x)$. In a statistical density, the parameters are usually real and then the conditions will be $\alpha > 0, \beta > 0$. Advantage of $f_2(x)$ over $f(x)$ is that $f_2(x)$ is more viable due to the presence of α and β. Note that $f_2(x)$ can be symmetric or non-symmetric whereas $f(x)$ is symmetric only. Here, we will consider logistic type models in the multivariate and matrix-variate cases so that these may be better models in applications compared to the corresponding Gaussian or gamma models. Let X be a $p \times 1$ real vector, $X' = [x_1, ..., x_p]$ where the x_j's are distinct real scalar variables. Consider the model

$$f_3(X) = c_3[X'X]^{\gamma} \frac{\mathrm{e}^{-b[X'X]^{\delta}}}{(1+a\mathrm{e}^{-[X'X]^{\delta}})^{\alpha+\beta}}, 0 < a < 1, \Re(\alpha) > 0, \Re(\beta) > 0, \delta > 0.$$

What is the normalizing constant c_3? We can expand the denominator by using a binomial expansion.

$$(1+a\mathrm{e}^{-[X'X]^{\delta}})^{-(\alpha+\beta)} = \sum_{k=0}^{\infty} (\alpha+\beta)_k \frac{(-a)^k}{k!} \mathrm{e}^{-k[X'X]^{\delta}}$$

where, for example, $(g)_k$ is the Pochhammer symbol given by $(g)_k = g(g+1)...(g+k-1), g \neq 0, (g)_0 = 1$. Then,

$$f_3(X) = c_3 \sum_{k=0}^{\infty} (\alpha+\beta)_k \frac{(-k)^k}{k!} [X'X]^{\gamma} \mathrm{e}^{-(b+k)[X'X]^{\delta}}.$$

Make the transformation $u = X'X \Rightarrow \mathrm{d}X = \frac{\pi^{\frac{p}{2}-1}}{\Gamma(\frac{p}{2})} u^{\frac{p}{2}-1}\mathrm{d}u$ from Lemma 1.1. Now,

$$\int_0^{\infty} u^{\gamma+\frac{p}{2}-1} \mathrm{e}^{-(b+k)u^{\delta}} \mathrm{d}u = \frac{1}{\delta}\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))(b+k)^{-\frac{1}{\delta}(\gamma+\frac{p}{2})}, \Re(\gamma) > -\frac{p}{2}.$$

Therefore,

$$c_3^{-1} = \frac{\pi^{\frac{p}{2}}}{\delta\Gamma(\frac{p}{2})}\sum_{k=0}^{\infty}(\alpha+\beta)_k\frac{(-a)^k}{k!}(b+k)^{-\frac{1}{\delta}(\gamma+\frac{p}{2})}.$$

This can be represented in terms of an extended zeta function defined recently by Mathai [15]. The ordinary zeta function $\zeta(\rho)$ and the generalized zeta function $\zeta(\rho,\alpha)$, available in the literature, are the following:

$$\zeta(\rho)=\sum_{k=1}^{\infty}\frac{1}{k^{\rho}}, \zeta(\rho,\alpha)=\sum_{k=0}^{\infty}\frac{1}{(\alpha+k)^{\rho}}, \Re(\rho)>1, \alpha\neq 0,-1,...$$

The extended zeta function is the following:

$$\begin{aligned}\zeta_{r,n}^{m}(z) &= \zeta[\{(\alpha_1+k)^{m_1}...(\alpha_r+k)^{m_r}\} : a_1,...,a_m; b_1,...,b_n; z]\\ &= \sum_{k=0}^{\infty}\frac{1}{(\alpha_1+k)^{m_1}...(\alpha_r+k)^{m_r}}\frac{(a_1)_k...(a_m)_k}{(b_1)_k...(b_n)_k}\frac{z^k}{k!}\end{aligned}$$

for $n\geq m, \Re(m_1+...+m_r)>1$ or $m=n+1, |z|<1, \Re(m_1+....+m_r)>1, \alpha_j\neq 0,-1,...,j=1,...,r$. Then, in terms of extended zeta function, the normalizing constant c_3 is the following:

$$c_3^{-1}=\frac{\pi^{\frac{p}{2}}}{\delta\Gamma(\frac{p}{2})}\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))\zeta[\{((\frac{1}{\delta}(\gamma+\frac{p}{2})),b)\}:\alpha+\beta;\quad;-a].$$

Now, we will consider a very general model in this category. Let X be $p\times q, p\leq q$ real matrix of rank p. Consider the model

$$f_4(X)=c_4|XX'|^{\gamma}[\mathrm{tr}(XX')]^{\eta}\frac{\mathrm{e}^{-b[\mathrm{tr}(XX')]^{\delta}}}{(1+a\mathrm{e}^{-[\mathrm{tr}(XX')]^{\delta}})^{\alpha+\beta}}.$$

We would like to evaluate the normalizing constant c_4. Note that we can write

$$f(X)=\sum_{k=0}^{\infty}(\alpha+\beta)_k\frac{(-a)^k}{k!}|XX'|^{\gamma}[\mathrm{tr}(XX')]^{\eta}\mathrm{e}^{-(b+k)[\mathrm{tr}(XX')]^{\delta}}.$$

Hence, the integral part is available from equations (2.1) in the real case and (2.2) for the complex case, by replacing b by $b+k$. That is, denoting the integral in the real case by ϵ and in the complex case by $\tilde{\epsilon}$ we have the following:

$$\begin{aligned}\epsilon &= \int_X |XX'|^{\gamma}[\mathrm{tr}(XX')]^{\eta}\mathrm{e}^{-(b+k)[\mathrm{tr}(XX')]^{\delta}}\mathrm{d}X\\ &= \frac{\pi^{\frac{pq}{2}}\Gamma[\frac{1}{\delta}(p(\gamma+\frac{q}{2})+\eta)]\Gamma_p(\gamma+\frac{q}{2})}{\delta\Gamma_p(\frac{q}{2})\Gamma(p(\gamma+\frac{p}{2}))(b+k)^{\frac{1}{\delta}(p(\gamma+\frac{q}{2})+\eta)}}\end{aligned}$$

for $\Re(\gamma) > -\frac{q}{2} + \frac{p-1}{2}$ from (2.1). Then,

$$c_4^{-1} = \frac{\pi^{\frac{pq}{2}}\Gamma[\frac{1}{\delta}(p(\gamma+\frac{q}{2})+\eta)]\Gamma_p(\gamma+\frac{q}{2})}{\delta\Gamma_p(\frac{q}{2})\Gamma(p(\gamma+\frac{q}{2}))} \times \zeta[\{(((\frac{1}{\delta}(p(\gamma+\frac{q}{2})+\eta)),b)\} : \alpha+\beta; \ \ ; -a] \tag{3.2}$$

for $b > 0, \delta > 0, \Re(\eta) > 0, \Re(\gamma) > -\frac{q}{2} + \frac{p-1}{2}, p \leq q$. The corresponding normalizing constant in the corresponding model in the complex domain, denoted by $\tilde{c}_4$, is the following from (2.2):

$$\tilde{c}_4^{-1} = \frac{\pi^{pq}\Gamma[\frac{1}{\delta}(p(\gamma+q)+\eta)]\tilde{\Gamma}_p(\gamma+q)}{\delta\tilde{\Gamma}_p(q)\Gamma(p(\gamma+q))} \times \zeta[\{(((\frac{1}{\delta}(p(\gamma+q)+\eta)),b)\} : \alpha+\beta; \ \ ; -a] \tag{3.3}$$

for $b > 0, \delta > 0, \Re(\eta) > 0, \Re(\gamma) > -q + p - 1, p \leq q$.

Note 3.1. The first model in Section 3 can be generalized by replacing X by $X - \mu, \mu = E[X]$ and multiplying the whole thing by a scalar constant also, in the real case, and replacing $\tilde{X}$ by $\tilde{X} - \tilde{\mu}, \tilde{\mu} = E[\tilde{X}]$ and the whole thing is multiplied by a real scalar constant also. The other models, where X in the real case is a $p \times q$ matrix can be generalized by replacing XX' by $A^{\frac{1}{2}}(X-M)B(X-M)'A^{\frac{1}{2}}$ where $M = E[X], A > O$ and $B > O$ are $p \times p$ and $q \times q$ constant positive definite matrices. In the complex case, replace $\tilde{X}\tilde{X}^*$ by $A^{\frac{1}{2}}(\tilde{X}-\tilde{M})B(\tilde{X}-\tilde{M})^*A^{\frac{1}{2}}$ where $\tilde{M} = E[\tilde{X}]$, $A = A^* > O, B = B^* > O$ are $p \times p$ and $q \times q$ Hermitian positive definite matrices.

4. Scaling Models and Bayesian Procedures

At the very outset, one should point out that as per Benavoli, Facchini and Zaffalon [1], quantum physics is nothing but Bayesian analysis of Hermitian positive definite matrices in a Hilbert space. Hence, all the models that we discuss here have direct connection to quantum physics. Consider a variable X, where X may be scalar or vector or matrix. If the density or function associated with X is a function of XX' or $X'X$ in the real case or $\tilde{X}\tilde{X}^*$ or $\tilde{X}^*\tilde{X}$ in the complex case, then a scaling constant can be $\sqrt{a}$ if $a > 0$ is a real scalar or $A^{\frac{1}{2}}$ if $A > O$ is a real positive definite matrix because, for example, $\sqrt{a}X \Rightarrow aXX'$. Consider the real case first. Let $U = \sqrt{a}X$ and let the function associated with X, or the density of X if X is a random variable, be a function of XX'. Here X is scaled by $\sqrt{a}$ or U is a scaled variable. Sometimes U is also called a scale mixture. If $f(X)$ is the function or density associated with X, then $f(a^{-\frac{1}{2}}U)$ is the function or density associated with U. When a is a pre-assigned constant, then we may take $f(a^{-\frac{1}{2}}U)$ as the conditional density of U, given $a > 0$, and we may denote it as $f(U|a)$ which is available from $f(X)$ by replacing X by $a^{-\frac{1}{2}}U$. Here, we will consider three situations. (1): Computation of the distribution of $\sqrt{a}X$ when the density of X is available, where $a > 0$ is a constant. (2): Computation of the distribution of U or the unconditional distribution of U when $f(U|a)$ is taken as a conditional density of U, given a, with some prior distribution for a. (3): Computation of the density of $U = \sqrt{a}X$ when a and X are independently distributed. A variable such as $U = \sqrt{a}X$ appears in many practical situations. Deng [3] gives the details of the analysis of PolSAR (Polarimetric Synthetic Aparture Radar) data where when a cross section of the multi-look return signal is taken, the cross section has pepper dust like contaminants called freckles and the cross section variable itself called texture and the structure is of the form $U = \sqrt{a}X$ when the density of X is a function of XX' or $X'X$. In radar problems, usually $\tilde{X}$ is in the complex domain and $\tilde{X}$ may be scalar/vector/matrix variable. In a scalar texture model, $A > 0$ is a real scalar quantity having its own distribution. Usually complex Gaussian model is taken for $\tilde{X}$. But it is shown that when the surface is not smooth such as forests, urban areas, sea surfaces etc, non-Gaussian models are preferred, see for example, Bombrun and Beaulieu [2], Frery, Muller, Yannasse and Saint'Anne [4], Yueh, Kong, Jao, Shin and Novak [6]. Here, we will consider various distributions belonging to Mathai's pathway family [Mathai [10]], namely generalized gamma, type-1 beta and type-2 beta forms, where X may be scalar, vector or matrix in the real or complex domain. At the same time we will consider $a > 0$ having densities belonging to the pathway family. This is the situation of scalar texture models in PolSAR data. If we cover all different combinations, then it will take up a lot of pages and hence one or two typical situations will be given in detail here and the same procedure can be applied to derive the unconditional densities for other combinations of a and X.

Problem 4.1. Let X be $p \times q, p \leq q$ matrix of rank p in the real domain. Let X have an extended rectangular matrix-variate gamma function or a gamma density $f(X)$ of the following form:

$$f(X) = c|XX'|^{\gamma}[\mathrm{tr}(XX')]^{\eta}\mathrm{e}^{-b[\mathrm{tr}(XX')]^{\delta}}, b > 0, \delta > 0$$

where c is the corresponding normalizing constant, when $f(X)$ is taken as a statistical density, which we have evaluated in Section 2, equations (2.1) and (2.2). Let $U = \sqrt{a}X$ where $a > 0$ have a prior generalized gamma density, denoted by $g(a)$

as follows:

$$g(a)\mathrm{d}a=\frac{\rho_1\beta^{\frac{\rho_2}{\rho_1}}}{\Gamma(\frac{\rho_2}{\rho_1})}a^{\rho_2-1}\mathrm{e}^{-\beta a^{\rho_1}}\mathrm{d}a,\rho_1>0,\beta>0,a\geq 0,\Re(\rho_2)>0.$$

Then, as per our notation, the conditional density of U, given a, is

$$f(U|a)\mathrm{d}U=c\ a^{-p\gamma-pq/2}|UU'|^{\gamma}[\frac{1}{a}\mathrm{tr}(UU')]^{\eta}\mathrm{e}^{-b[\frac{\mathrm{tr}(UU')}{a}]^{\delta}}\mathrm{d}U$$

where $a^{-p\gamma}$ is coming from the determinant and $a^{-\frac{pq}{2}}$ is coming from the conversion of $\mathrm{d}X$ into $\mathrm{d}U$. Then the unconditional density of U, denoted by $f_u(U)$, is the following:

$$\begin{aligned}f_u(U)&=\int_{a=0}^{\infty}f(U|a)g(a)\mathrm{d}a\\&=\frac{\rho_1\beta^{\frac{\rho_2}{\rho_1}}c}{\Gamma(\frac{\rho_2}{\rho_1})}|UU'|^{\gamma}\int_0^{\infty}a^{\rho_2-p\gamma-pq/2-1}[\frac{w}{a}]^{\eta}\mathrm{e}^{-\beta a^{\rho_1}-b(\frac{w}{a})^{\delta}}\mathrm{d}a.\end{aligned}$$

In the integral, we have a factor $\mathrm{e}^{-\beta a^{\rho_1}-b(\frac{w}{a})^{\delta}}$. This will produce a Bessel type or Krätzel type integral. For $\rho_1=1,\delta=1$ the integrand normalized is inverse Gaussian density for appropriate parameter for the exponent of a. For $\rho_1=1,\delta=\frac{1}{2}$ is the reaction-rate probability integral in nuclear reaction-rate theory, see Mathai and Haubold [16]. For $\rho_1=1$ and general δ, it is the Krätzel integral and associated with it is the Krätzel transform. This integral can be evaluated by using the following statistical procedure. Let $x_1>0$ and $x_2>0$ be two independently distributed real scalar variables with the densities $f_1(x_1)$ and $f_2(x_2)$ respectively. Let $w=x_1x_2$ the product. Then, taking $w=x_1x_2$ and $v=x_1$ or $v=x_2$, the Jacobian is $\frac{1}{v}$, and the density of w, denoted by $h(w)$, is the following:

$$h(w)=\int_v\frac{1}{v}f_1(\frac{w}{v})f_2(v)\mathrm{d}v=\int_v\frac{1}{v}f_1(v)f_2(\frac{w}{v})\mathrm{d}v \tag{4.1}$$

and then taking the expected values $E[w^{s-1}]=E[x_1^{s-1}]E[x_2^{s-1}]=M_{f_1}(s)M_{f_2}(s)$ where $M_f(s)$ is the Mellin transform of f with Mellin parameter s. That is,

$$\begin{aligned}M_h(s)&=\int_0^{\infty}w^{s-1}h(w)\mathrm{d}w;\quad M_{f_1}(s)=\int_0^{\infty}x_1^{s-1}f_1(x_1)\mathrm{d}x_1;\\M_{f_2}(s)&=\int_0^{\infty}x_2^{s-1}f_2(x_2)\mathrm{d}x_2\end{aligned}$$

whenever the Mellin transforms exist or whenever the respective integrals are convergent, and from (4.1) by taking the Mellin transform we have

$$M_h(s) = M_{f_1}(s)M_{f_2}(s). \tag{4.2}$$

These equations (4.1) and (4.2) are called the Mellin convolution of a product property. The above statistical procedure is nothing but the application of the Mellin convolution of a product property. We will make use of the Mellin convolution of a product property to evaluate our integral over a in $f_u(U)$. Once, (4.2) is available, we can take the inverse Mellin transform to obtain the explicit form of the function $h(w)$ or the density $h(w)$ when random variables are involved. In Mellin transform, the functions involved need not be densities, the only restriction is that the Mellin transforms exist. We will evaluate our integral over a by using the following lemma, the derivation and other details may be seen from Mathai [14].

Lemma 4.1. *For $\delta > 0, \Re(\nu) > 0, \Re(\gamma) > 0$ the first part of the lemma follows. For the second part the following additional conditions are needed. $\beta = \delta, \frac{\nu}{\delta} \neq \pm 0, 1, 2, ...,$ and $\nu = \rho_2 - p\gamma - pq/2$ in our example.*

$$\begin{aligned}\int_0^\infty x^{\nu-1} \mathrm{e}^{-\alpha x^\beta - \frac{\gamma}{x^\delta}} \mathrm{d}x &= \frac{1}{\delta\beta\alpha^{\frac{\nu}{\beta}}} H_{0,2}^{2,0}\left[\gamma^{\frac{1}{\delta}}\alpha^{\frac{1}{\beta}}\big|_{(0,\frac{1}{\delta}),(\frac{\nu}{\beta},\frac{1}{\beta})}\right], o < \gamma^{\frac{1}{\delta}}\alpha^{\frac{1}{\beta}} < \infty \\ &= \frac{\Gamma(\frac{\gamma}{\delta})}{\delta\alpha^{\frac{\nu}{\delta}}} {}_0F_1(\ ;1 - \frac{\nu}{\delta}; \alpha\gamma) + \frac{\gamma^{\frac{nu}{\delta}}\Gamma(-\frac{\nu}{\delta})}{\delta} {}_0F_1(\ ;1 + \frac{\nu}{\delta}; \alpha\gamma).\end{aligned}$$

For the theory and applications of H-function, see Mathai, Saxena and Haubold [21]. The above representation is in terms of ${}_0F_1$ series. For a representation in terms of Bessel function of the second kind, see Jeffrey and Zwillinger [5]. By using Lemma 4.1, our integral over a is the following for $\nu = \rho_2 - p\gamma - pq/2$ and $w = \mathrm{tr}(UU')$:

$$\begin{aligned}\int_0^\infty a^{\nu-1}(\frac{w}{a})^\eta \mathrm{e}^{-\beta a^{\rho_1} - (\frac{w}{a})^\delta} \mathrm{d}a &= \frac{1}{\delta\rho_1\beta^{\frac{\nu}{\rho_1}}} H_{0,2}^{2,0}\left[\beta^{\frac{1}{\rho_1}}w\big|_{(\frac{\eta}{\delta},\frac{1}{\delta}),(\frac{\nu}{\rho_1},\frac{1}{\rho_1})}\right], 0 < \beta^{\frac{1}{\rho_1}}w < \infty \\ &= \frac{1}{\delta\beta^{\frac{\nu}{\delta}}}[(\beta^{\frac{1}{\delta}}w)^\eta\Gamma(\frac{\nu-\eta}{\delta}){}_0F_1(\ ;1 + \frac{\eta-\nu}{\delta}; \beta w^\delta) \\ &+ (\beta^{\frac{1}{\delta}}w)^\nu\Gamma(\frac{\eta-\nu}{\delta}){}_0F_1(\ ;1 + \frac{\nu-\eta}{\delta}; \beta w^\delta)].\end{aligned}$$

For the series part to hold, one needs the conditions $\nu - \eta \neq \pm 0, 1, ..., \rho_1 = \delta$ and then in the contour integral $\frac{s}{\delta}$ is replaced by s, s by δs and $\mathrm{d}s$ by $\delta\mathrm{d}s$.

Problem 4.2. Let X and $f(X)$ be as defined in Problem 4.1. Then

$$f(U|a) = ca^{-p\gamma-\frac{pq}{2}}|UU'|^{\gamma}(\frac{w}{a})^{\eta}\mathrm{e}^{-b(\frac{w}{a})^{\delta}}, w = \mathrm{tr}(UU')$$

where the normalizing constant c is evaluated in Section 2 and $a^{-pq/2}$ is coming from converting dX into dU. Let the marginal density for a be of a type-2 beta type, again denoted by $g(a)$, that is,

$$g(a) = \frac{\rho_1 k^{\frac{\alpha}{\rho_1}}\Gamma(\beta+\frac{\alpha}{\rho_1})}{\Gamma(\beta)\Gamma(\frac{\alpha}{\rho_1})}\frac{a^{\alpha-1}}{(1+ka^{\rho_1})^{\beta+\frac{\alpha}{\rho_1}}}, \rho_1 > 0, k > 0, \Re(\alpha) > 0, \Re(\beta) > 0.$$

Then, the unconditional density of U, again denoted by $f_u(U)$, is the following:

$$\begin{aligned} f_u(U) &= \int_0^{\infty} f(U|a)g(a)\mathrm{d}a \\ &= \frac{c\rho_1\Gamma(\beta+\frac{\alpha}{\rho_1})k^{\frac{\alpha}{\rho_1}}|UU|^{\gamma}}{\Gamma(\beta)\Gamma(\frac{\alpha}{\rho_1})}\int_0^{\infty}\frac{a^{\alpha-p\gamma-\frac{pq}{2}-1}}{(1+ka^{\rho_1})^{\beta+\frac{\alpha}{\rho_1}}}(\frac{w}{a})^{\eta}\mathrm{e}^{-b(\frac{w}{a})^{\delta}}\mathrm{d}a. \end{aligned}$$

Comparing the integral part with (4.1) and (4.2) we can take

$$f_1(x) = x^{\eta}\mathrm{e}^{-bx^{\delta}} \Rightarrow M_{f_1}(s) = \frac{1}{\delta}\Gamma(\frac{\eta+s}{\delta})b^{-\frac{\eta+s}{\delta}}, \Re(s) > -\Re(\eta), \delta > 0, b > 0 \qquad (i)$$

and

$$\begin{aligned} f_2(x) &= \frac{x^{\alpha-p\gamma-pq/2}}{(1+kx^{\rho_1})^{\beta+\frac{\alpha}{\rho_1}}} \Rightarrow \\ M_{f_2}(s) &= \frac{1}{\rho_1}\frac{\Gamma(\frac{\alpha-p\gamma-pq/2+s}{\rho_1})\Gamma(\beta+\frac{p\gamma+pq/2}{\rho_1}-\frac{s}{\rho_1})}{\Gamma(\beta+\frac{\alpha}{\rho_1})}k^{-\frac{\alpha-p\gamma-pq/2+s}{\rho_1}}, \end{aligned}$$

for $\Re(s) > p\Re(\gamma)+\frac{pq}{2}-\Re(\alpha), \Re(s) < p\Re(\gamma)+pq/2+\Re(\beta), k > 0, \rho_1 > 0$. Now, by taking the inverse Mellin transform, we have the unconditional density as the following:

$$\begin{aligned} f_u(U) &= c_1\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\frac{\eta+s}{\delta})\Gamma(\frac{\alpha-p\gamma-\frac{pq}{2}+s}{\rho_1})\Gamma(\beta+\frac{p\gamma+\frac{pq}{2}-s}{\rho_1})(b^{\frac{1}{\delta}}k^{\frac{1}{\rho_1}}w)^{-s}\mathrm{d}s \\ &= H_{1,2}^{2,1}\left[b^{\frac{1}{\delta}}k^{\frac{1}{\rho_1}}w\Big|_{(\frac{\eta}{\delta},\frac{1}{\delta}),(\frac{\alpha-p\gamma-pq/2}{\rho_1},\frac{1}{\rho_1})}^{(1-\beta-\frac{p\gamma+pq/2}{\rho_1},\frac{1}{\rho_1})}\right] \end{aligned}$$

where

$$c_1 = \frac{ck^{\frac{p\gamma+pq/2}{\rho_1}}|UU'|^{\gamma}}{\delta\Gamma(\beta)\Gamma(\frac{\alpha}{\rho_1})b^{\frac{\eta}{\delta}}}$$

and c in the contour is such that $-\min\Re(\frac{\eta}{\delta},\frac{\alpha-p\gamma-\frac{pq}{2}}{\rho_1}) < c < \Re(\beta)+\frac{p\gamma+pq/2}{\rho_1}$. One can reduce the H-function in terms of confluent hypergeometric series for the following situation: $\delta=\rho_1, \eta-\alpha+p\gamma+pq/2 \neq \pm 0, 1, ...$ and then replace $\frac{s}{\delta}$ by s. Then, we have the following series by evaluating the sum of residues at the poles of $\Gamma(\frac{\eta}{\delta}+s)$ and $\Gamma(\frac{\alpha-p\gamma-pq/2}{\delta}+s)$. The final result is the following:

$$\begin{aligned} f_u(U) &= c_2[(bkw^{\delta})^{\frac{\eta}{\delta}}\Gamma(\frac{p\gamma+\frac{pq}{2}+\eta}{\delta})\Gamma(\frac{\alpha-p\gamma-pq/2-\eta}{\delta}) \\ &\quad\times {}_1F_1(\frac{p\gamma+\eta}{\delta};1+\frac{\eta+p\gamma+\frac{pq}{2}-\alpha}{\delta};bkw^{\delta}) \\ &\quad+(kbw^{\delta})^{\frac{\alpha-p\gamma-pq/2}{\delta}}\Gamma(\frac{\eta+p\gamma+\frac{pq}{2}-\alpha}{\delta})\Gamma(\frac{\alpha}{\delta}) \\ &\quad\times {}_1F_1(\frac{\alpha}{\delta};1+\frac{\alpha-p\gamma-\frac{pq}{2}-\eta}{\delta};bkw^{\delta})] \end{aligned}$$

where

$$c_2 = \frac{ck^{\frac{p\gamma+pq/2}{\delta}}|UU'|^{\gamma}}{\Gamma(\beta)\Gamma(\frac{\alpha}{\delta})b^{\frac{\eta}{\delta}}},$$

for $k>0, b>0, \delta>0, \Re(\gamma)>0, \Re(\eta)>0, \Re(\alpha)>p\gamma+\frac{pq}{2}+\Re(\eta), \eta+p\gamma+\frac{pq}{2}-\alpha \neq \pm 0, 1, ...$

Note 4.1. One can consider other functions as prior function for a. Also, one can consider other functions as associated functions for X. The procedure will be parallel. For all these cases, one can consider the corresponding functions in the complex domain. For example, in Problems 4.1 and 4.2 if the $p\times q, p\leq q$ matrix of rank p is in the complex domain, then the final results are available by going parallel to the derivations in the real case. If $\tilde{X}$ is in the complex domain, then replace $w=\mathrm{tr}(UU')$ in the real case by $\tilde{w}=\mathrm{tr}(\tilde{X}\tilde{X}^*)$ in the complex case, where $\tilde{w}$ will also be real. Replace $a^{\frac{pq}{2}}$ in the differential element when converting $\mathrm{d}X$ into $\mathrm{d}U$ in the real case by a^{pq} in the complex case. All other parameters will remain the same. Thus, the results in the complex domain, corresponding to the results in the real domain of Problems 4.1 and 4.2, are available from the real case results by changing $pq/2$ to pq and $w=\mathrm{tr}(UU')$ to $\tilde{w}=\mathrm{tr}(\tilde{U}\tilde{U}^*)$. In the normalizing constant c, the matrix-variate gamma functions $\Gamma_p(\cdot)$ in the real domain have to be replaced by the matrix-variate gamma functions $\tilde{\Gamma}_p(\cdot)$ in the complex domain when considering the distributions in the complex domain.

5. Connection to Fractional Calculus

Let $x_1 > 0$ and $x_2 > 0$ be two real scalar variables. Let the functions associated with x_1 and x_2 be $f_1(x_1)$ and $f_2(x_2)$ respectively. Let the joint function $f(x_1, x_2)$ be of the form $f(x_1, x_2) = f_1(x_1)f_2(x_2)$. If we wish to draw a parallel to statistical distribution theory, then one can take $x_1 > 0$ and $x_2 > 0$ be statistically independently distributed real scalar random variables with the density functions $f_1(x_1)$ and $f_2(x_2)$ respectively. Then, what is the function associated with the product $u = x_1x_2$? If x_1 and x_2 are random variables, then we are asking the question: what is the density of $u = x_1x_2$? If $g(u)$ is the density of u, then we have already shown that

$$g(u) = \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v)\mathrm{d}v$$

whenever the integral is convergent. Let $f_1(x_1)$ be of the form

$$f_1(x_1) = \frac{\Gamma(\gamma+1+\alpha)}{\Gamma(\gamma+1)\Gamma(\alpha)} x_1^{\gamma}(1-x_1)^{\alpha-1}$$

for $0 \le x_1 \le 1$, $\Re(\gamma) > -1, \Re(\alpha) > 0$ and $f_1(x_1) = 0$ elsewhere. If $x_1 > 0$ is a random variable, then we say that x_1 is type-1 beta distributed with the parameters $\gamma + 1$ and α. Let $f_2(x_2)$ be an arbitrary function $f(x_2)$. If $f(x_2)$ is a density, then we are considering $f(x_2)$ to be some arbitrary density function. Then,

$$\begin{aligned} g(u) &= \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v)\mathrm{d}v \qquad \text{(i)} \\ &= \frac{\Gamma(\gamma+1+\alpha)}{\Gamma(\alpha)\Gamma(\gamma+1)} \int_v \frac{1}{v}(\frac{u}{v})^{\gamma}(1-\frac{u}{v})^{\alpha-1} f(v)\mathrm{d}v = \frac{\Gamma(\gamma+1+\alpha)}{\Gamma(\gamma+1)} K_{2,\gamma,u}^{-\alpha}(f) \end{aligned}$$

where

$$K_{2,\gamma,u}^{-\alpha}(f) = \frac{u^{\gamma}}{\Gamma(\alpha)} \int_{v\ge u} v^{-\alpha-\gamma}(v-u)^{\alpha-1} f(v)\mathrm{d}v.$$

This $K_{2,\gamma,u}^{-\alpha}(f)$ is Erdélyi-Kober fractional integral of the second kind of order α and parameter γ. Note that the above procedure holds even if f_1 and f_2 are not statistical densities. The only condition is that the corresponding integrals are convergent. One can take $f_1(x_1)$ to be of the form $\frac{1}{\Gamma(\alpha)}\phi(x_1)(1-x_1)^{\alpha-1}$ where $\phi(x_1)$ is some function of x_1. In the above example, we have taken $\phi(x_1)$ to be a constant multiple of x_1^{γ}. By taking different forms of $\phi(x_1)$ we can show that all the various fractional integrals in the real scalar case available in the literature such as Riemann-Liouville fractional integral of the second kind, Weyl fractional integral of the second kind and all other fractional integrals of the second kind are available from the general definition introduced by Mathai [11]. In the general definition, $f_1(x_1)$ is taken as $\frac{1}{\Gamma(\alpha)}\phi(x_1)(1-x_1)^{\alpha-1}, \Re(\alpha) > 0$, f_2 is taken as an arbitrary function and then Mellin convolution of a product, namely (i) above, is taken to obtain all fractional integrals of the second kind. Also, in the series of papers which appeared in the journal: Linear Algebra and Applications, starting from 2013, Mathai has extended fractional integrals and fractional derivatives to functions of matrix argument and to the complex domain for the first time and at the same time giving a general definition along with geometrical interpretations, encompassing all the various definitions available in the literature. He had given most of the materials in CMS (Centre for Mathematical Sciences) Module 10. Later, Hans Haubold took the initiative to get all CMS Modules published by international publishers. This Module 10 was taken as such with some recent materials of Mathai and his co-workers added, and published by Nova Science Publishers under the title: Introduction to Fractional Calculus (A.M. Mathai and Hans J. Haubold), Nova Science Publishers, New York, 2017. Since the ideas were innovative, Springer wanted to bring out a Springer Brief on the topic. Springer Briefs are 100-paged publications by Springer on a current innovative idea, to expose the idea to the scientific world. A Springer Brief from Japan appeared in 2018 under the title: Erdélyi-Kober Fractional Integrals from a Statistical Perspective, Inspired by Solar Neutrino Problem (A.M. Mathai and Hans J. Haubold, 2018). Modules 1,2,3 of CMS were combined and published as a book by De Gruyter, Germany in 2017, under the title: Linear Algebra (A.M. Mathai and Hans J. Haubold) and Modules 6,7,9 of CMS were combined and published by De Gruyter, Germany, under the title: Probability and Statistics (A.M. Mathai and Hans J. Haubold, 2017). These two books from De Gruyter, Germany, are made free download. All Statistics students and faculty should have a copy of each of these basic materials, with a lot of subtle points indicated therein, on his/her desk.

Mathai has combined all available definitions on fractional integrals of the first kind and has shown that fractional integrals of the first kind can be constructed by considering the density of a ratio of positive random variables or as Mellin convolution of a ratio. Let $x_1 > 0$ and $x_2 > 0$ be as defined above with the associated functions $f_1(x_1)$ and $f_2(x_2)$ and with the joint function as $f_1(x_1)f_2(x_2)$. Now, consider the ratio $u_1 = \frac{x_2}{x_1}$ and let $v = x_2$, that is, $x_2 = v, x_1 = \frac{v}{u_1}$. Consider the transformation $(x_1, x_2) \to (u_1, v)$. Then, we see that $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = -\frac{v}{u_1^2}\mathrm{d}u_1 \wedge \mathrm{d}v$. Let

$$f_1(x_1) = \frac{\Gamma(\alpha+\gamma)}{\Gamma(\gamma)\Gamma(\alpha)} x_1^{\gamma-1}(1-x_1)^{\alpha-1}$$

for $0 \le x_1 \le 1, \Re(\gamma) > 0, \Re(\alpha) > 0$ and let $f_1(x_1) = 0$ elsewhere. Let $f_2(x_2) =$

$f(x_2)$ an arbitrary function. Then, let $g_1(u_1)$ be the function associated with u_1. When, x_1 and x_2 are real scalar positive random variables, then we are considering the density of the ratio $u_1 = \frac{x_2}{x_1}$. That is,

$$\begin{aligned} g_1(u_1) &= \frac{\Gamma(\gamma+\alpha)}{\Gamma(\gamma)\Gamma(\alpha)} \int_v (\frac{v}{u_1^2}) f_1(\frac{v}{u_1}) f_2(v) \mathrm{d}v \\ &= \frac{\Gamma(\gamma+\alpha)}{\Gamma(\gamma)} K_{1,\gamma,u_1}^{-\alpha}(f) \text{ where} \\ K_{1,\gamma,u_1}^{-\alpha}(f) &= \frac{u_1^{-\gamma-\alpha}}{\Gamma(\alpha)} \int_{v\le u_1} v^{\gamma}(u_1-v)^{\alpha-1} f(v) \mathrm{d}v, \end{aligned}$$

whenever the integral is convergent, is Erdélyi-Kober fractional integral of the first kind of order α and parameter γ. Thus, one can see that this first kind fractional integral is nothing but a constant multiple of a statistical density of a ratio when f_1 and f are statistical densities. Mathai has given a general definition for fractional integral of the first kind, encompassing all available definitions in the literature, by taking $f_1(x_1)$ as $\frac{1}{\Gamma(\alpha)}\psi(x_1)(1-x_1)^{\alpha-1}, \Re(\alpha) > 0$ and then taking the Mellin convolution of a ratio. Then, by specializing $\psi(x_1)$ one can obtain various fractional integrals of the first kind such as Riemann-Liouville integral, Weyl integral etc. Also, Mathai [11-14] has extended the ideas and defined fractional integrals of the first kind for functions of matrix arguments in the real and complex domains.

5.1. Symmetric product and symmetric ratio of matrices

Here we will examine the matrix-variate versions of the real scalar variable product $u = x_1x_2, x_1 > 0, x_2 > 0$ and the scalar variable ratio $u_1 = \frac{x_2}{x_1}$. Consider two $p \times p$ real positive definite matrices $X_1 > O, X_2 > O$. Let $X_2^{\frac{1}{2}}$ be the real positive definite square root of the real positive definite matrix $X_2 > O$. Let $U = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}$ and $U_1 = X_2^{\frac{1}{2}} X_1^{-1} X_2^{\frac{1}{2}}$. Then, U is a symmetric product of $X_1 > O$ and $X_2 > O$, and U_1 is a symmetric ratio of X_2 to X_1. In the real scalar case $u = x_1x_2 = x_2x_1$ but in the matrix case $X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}} \neq X_1^{\frac{1}{2}} X_2 X_1^{\frac{1}{2}}$ unless X_1 and X_2 commute. Similarly, $X_2^{\frac{1}{2}} X_1^{-1} X_2^{\frac{1}{2}} \neq X_1^{-\frac{1}{2}} X_2 X_1^{-\frac{1}{2}}$. From Mathai (1997) we have the following results on Jacobians:

$$U = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}, V = X_2 \Rightarrow \mathrm{d}X_1 \wedge \mathrm{d}X_2 = |V|^{-\frac{p+1}{2}} \mathrm{d}U \wedge \mathrm{d}V \quad \text{(i)}$$

$$U_1 = X_2^{\frac{1}{2}} X_1^{-1} X_2^{\frac{1}{2}}, V = X_2 \Rightarrow \mathrm{d}X_1 \wedge \mathrm{d}X_2 = |V|^{\frac{p+1}{2}} |U_1|^{-(p+1)} \mathrm{d}U_1 \wedge \mathrm{d}V \quad \text{(ii)}$$

In the complex case, consider the $p \times p$ Hermitian positive definite matrices $\tilde{X}_1 = \tilde{X}_1^* > O$ and $\tilde{X}_2 = \tilde{X}_2^* > O$. Let $\tilde{U} = \tilde{X}_2^{\frac{1}{2}} \tilde{X}_1 \tilde{X}_2^{\frac{1}{2}}$ be the symmetric product and let

$\tilde{U}_1 = \tilde{X}_2^{\frac{1}{2}}\tilde{X}_1^{-1}\tilde{X}_2^{\frac{1}{2}}$ be the symmetric ratio of $\tilde{X}_2$ to $\tilde{X}_1$. Then, from Mathai [8], we have the following:

$$\tilde{U} = \tilde{X}_2^{\frac{1}{2}}\tilde{X}_1\tilde{X}_2^{\frac{1}{2}}, \tilde{V} = \tilde{X}_2 \Rightarrow \mathrm{d}\tilde{X}_1 \wedge \mathrm{d}\tilde{X}_2 = |\det(\tilde{V})|^p \mathrm{d}\tilde{U} \wedge \mathrm{d}\tilde{V} \tag{iii}$$

$$\begin{aligned}\tilde{U}_1 = \tilde{X}_2^{\frac{1}{2}}\tilde{X}_1^{-1}\tilde{X}_2^{\frac{1}{2}}, \tilde{V} = \tilde{X}_2 &\Rightarrow \mathrm{d}\tilde{X}_1 \wedge \mathrm{d}X_2 \\ &= |\det(\tilde{V})|^p|\det(\tilde{U}_1)|^{-2p}\mathrm{d}\tilde{U}_1 \wedge \mathrm{d}\tilde{V}.\end{aligned} \tag{iv}$$

In the real case, let $f_1(X_1)$ and $f_2(X_2)$ be the real-valued scalar functions associated with X_1 and X_2 respectively and let the joint function be $f_1(X_1)f_2(X_2)$, the product, where $f_j(X_j)$ is a real-valued scalar function of $X_j, j = 1, 2$. If $f_j(X_j)$ is a statistical density, then we have the additional conditions $f_j(X_j) \ge 0$ for all X_j in the domain of X_j and $\int_{X_j} f_j(X_j)\mathrm{d}X_j = 1, j = 1, 2$ where the differential element $\mathrm{d}X_j$ is defined in Section 1. Then, when the joint function is the product, we have X_1 and X_2 statistically independently distributed. In the complex case, let $\tilde{X}_j$ be a $p \times p$ Hermitian positive definite matrix, $\tilde{X}_j = \tilde{X}_j^* > O$. Let the associated function be $f_j(\tilde{X}_j)$ which is a real-valued scalar function of the complex matrix argument $\tilde{X}_j$ for $j = 1, 2$. Let the joint function be the product $f_1(\tilde{X}_1)f_2(\tilde{X}_2)$. Again, if $f_j(\tilde{X}_j), j = 1, 2$ are statistical densities then the additional conditions are $f_j(\tilde{X}_j) \ge 0$ for all $\tilde{X}_j$ in the domain of $\tilde{X}_j$ and $\int_{\tilde{X}_j} f_j(\tilde{X}_j)\mathrm{d}\tilde{X}_j = 1, j = 1, 2$. When the joint density is a product then the matrix-variate random variables in the complex domain are statistically independently distributed.

5.2. Fractional integral of the second kind in the real and complex matrix-variate cases

Let the real matrix variables $X_1 > O$ and $X_2 > O$ and the associated functions $f_1(X_1)$ and $f_2(X_2)$ be as in Section 5.1. Let the symmetric product U, and V be as defined in Section 5.1. Then, the joint function of U and V, denoted by $h(U, V)$, is the following:

$$\begin{aligned}h(U, V)\mathrm{d}U \wedge \mathrm{d}V &= f_1(X_1)f_2(X_2)\mathrm{d}X_1 \wedge \mathrm{d}X_2 \\ &= |V|^{-\frac{p+1}{2}} f_1(V^{-\frac{1}{2}}UV^{-\frac{1}{2}})f_2(V)\mathrm{d}U \wedge \mathrm{d}V\end{aligned}$$

Then, the marginal function of U, denoted by $g(U)$, is the following:

$$g(U) = \int_V h(U, V)\mathrm{d}V = \int_V |V|^{-\frac{p+1}{2}} f_1(V^{-\frac{1}{2}}UV^{-\frac{1}{2}})f_2(V)\mathrm{d}V. \tag{v}$$

Let us examine this (v) for some special cases. Let

$$f_1(X_1) = \frac{|X_1|^\gamma}{\Gamma_p(\alpha)}|I - X_1|^{\alpha-\frac{p+1}{2}}, \Re(\alpha) > \frac{p-1}{2},$$

for $I - X_1 > O, X_1 > O$, and let $f_2(X_2) = f(X_2)$ an arbitrary function. Then,

$$\begin{aligned} g(U) &= \frac{1}{\Gamma_p(\alpha)} \int_V |V|^{-\frac{p+1}{2}} |V^{-\frac{1}{2}} U V^{-\frac{1}{2}}|^{\gamma} |I - V^{-\frac{1}{2}} U V^{-\frac{1}{2}}|^{\alpha - \frac{p+1}{2}} \mathrm{d}V \\ &= \frac{|U|^{\gamma}}{\Gamma_p(\alpha)} \int_{V-U>O} |V|^{-\gamma-\alpha} |V - U|^{\alpha - \frac{p+1}{2}} f(V) \mathrm{d}V \\ &= K^{-\alpha}_{2,\gamma,U}(f), \Re(\alpha) > \frac{p-1}{2} \end{aligned} \tag{5.1}$$

where $K^{-\alpha}_{2,\gamma,U}(f)$ for $p = 1$ is Erdélyi-Kober fractional integral of the second kind of order α and parameter γ. Hence Mathai [11], who introduced fractional integrals in the matrix-variate case, called (5.1) as Erdélyi-Kober fractional integral of the second kind of order α and parameter γ.

In the corresponding complex case, proceeding parallel to the derivation in the real case, we have the following, denoting the marginal function of $\tilde{U}$ as $g(\tilde{U})$:

$$\begin{aligned} g(\tilde{U}) &= \frac{|\det(\tilde{U})|^{\gamma}}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{V}-\tilde{U}>O} |\det(\tilde{V})|^{-\gamma-\alpha} |\det(\tilde{V} - \tilde{U})|^{\alpha - p} \mathrm{d}\tilde{V} \\ &= \tilde{K}^{-\alpha}_{2,\gamma,\tilde{U}}(f), \Re(\alpha) > p - 1 \end{aligned} \tag{5.2}$$

where, Mathai [11-14] called (5.2) as the Erdélyi-Kober fractional integral of the second kind of order α and parameter γ in the complex domain, and $|\det(\cdot)|$ denotes the absolute value of the determinant of $(\cdot)$ as explained in Section 1 and $\tilde{\Gamma}_p(\cdot)$ is the complex matrix-variate gamma as defined in Lemma 1.1.

One can establish a connection between fractional integral of the second kind and density of a symmetric product of independently distributed real or complex matrix-variate random variables. In the this case, take $f_1(X_1)$ in the real case as a real matrix-variate type-1 beta density with the parameters $(\gamma + \frac{p+1}{2}, \alpha)$ and $f_2(X_2) = f(X_2)$ an arbitrary density. In the complex case, take $f_1(\tilde{X}_1)$ as a complex matrix-variate type-1 beta density with the parameters $(\gamma + p, \alpha)$ and $f_2(\tilde{X}_2) = f(\tilde{X}_2)$ an arbitrary density. Then, the fractional integral in (5.1) is obtained as a constant multiple of the density of a symmetric product in the real domain and the fractional integral in (5.2) as a constant multiple of the density of a symmetric product in the complex domain.

5.3. Fractional integral of the first kind in the real and complex matrix-variate cases

Let $X_j > O, j = 1, 2$ be $p \times p$ real positive definite matrices as defined in Section 5.1 with the associated functions $f_j(X_j), j = 1, 2$ and with the joint function

$f_1(X_1)f_2(X_2)$. Consider the symmetric ratio in (ii) of Section 5.2. Let us take

$$f_1(X_1)=\frac{|X_1|^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\alpha)}|I-X_1|^{\alpha-\frac{p+1}{2}}, O<X_1<I$$

for $\Re(\alpha),\Re(\gamma)>\frac{p-1}{2}$. Let $f_2(X_2)=f(X_2)$ where f is an arbitrary function. Let the function associated with the symmetric ratio U_1 be denoted as $g_1(U_1)$. Then,

$$\begin{aligned} g_1(U_1) &= \frac{1}{\Gamma_p(\alpha)}\int_V |V|^{\frac{p+1}{2}}|U_1|^{-(p+1)}|V^{\frac{1}{2}}U_1^{-1}V^{\frac{1}{2}}|^{\gamma-\frac{p+1}{2}} \\ &\quad\times |I-V^{\frac{1}{2}}U_1^{-1}V^{\frac{1}{2}}|^{\alpha-\frac{p+1}{2}}f(V)\mathrm{d}V \\ &= \frac{|U_1|^{-\alpha-\gamma}}{\Gamma_p(\alpha)}\int_{U_1-V>O}|V|^{\gamma}|U_1-V|^{\alpha-\frac{p+1}{2}}f(V)\mathrm{d}V \\ &= K^{-\alpha}_{1,\gamma,U_1}(f) \end{aligned} \tag{5.3}$$

where for $p=1$, (5.3) is Erdélyi-Kober fractional integral of the first kind of order α and parameter γ. Hence Mathai [11-12] called (5.3) as Erdélyi-Kober fractional integral of the first kind of order α and parameter γ in the real matrix-variate case. The corresponding function of $\tilde{U}_1$ in the complex case, denoted by $g_1(\tilde{U}_1)$, is the following:

$$\begin{aligned} g_1(\tilde{U}_1) &= \frac{|\det(\tilde{U}_1)|^{-\gamma-\alpha}}{\tilde{\Gamma}_p(\alpha)}\int_{\tilde{U}_1-\tilde{V}>O}|\det(\tilde{V})|^{\gamma}|\det(\tilde{U}_1-\tilde{V})|^{\alpha-p}f(\tilde{V})\mathrm{d}\tilde{V} \\ &= \tilde{K}^{-\alpha}_{1,\gamma,\tilde{U}_1}(f) \end{aligned} \tag{5.4}$$

where $\tilde{U}_1$ and $\tilde{V}$ are $p\times p$ Hermitian positive definite matrices such that $\tilde{U}_1-\tilde{V}>O$ (Hermitian positive definite) and $\tilde{K}^{-\alpha}_{1,\gamma,\tilde{U}_1}(f)$ is Erdélyi-Kober fractional integral of the first kind of order α and parameter γ in the complex matrix-variate case. If we start with

$$f_1(\tilde{X}_1)=\frac{\tilde{\Gamma}_p(\alpha+\gamma)}{\tilde{\Gamma}_p(\gamma)\tilde{\Gamma}_p(\alpha)}|\det(\tilde{X}_1)|^{\gamma-p}|\det(I-\tilde{X}_1)|^{\alpha-p}, O<\tilde{X}_1<I$$

for $\Re(\alpha)>p-1,\Re(\gamma)>p-1$, which is a complex matrix-variate type-1 beta density, and let $f_2(\tilde{X}_2)=f(\tilde{X}_2)$ where f is an arbitrary density function, then, $g_1(\tilde{U}_1)$ will be a constant multiple of the fractional integral of the first kind, namely

$$g_1(\tilde{U}_1)=\frac{\tilde{\Gamma}_p(\gamma+\alpha)}{\tilde{\Gamma}_p(\gamma)}\tilde{K}^{-\alpha}_{1,\gamma,\tilde{U}_1}(f).$$

6. Matrix-variate Functions Through Optimization of Mathai Entropy

If $f(x)$ is a statistical density for a real scalar variable x, then Shannon's measure of uncertainly or entropy is defined as

$$S(f) = -c\int_x f(x)\ln f(x)\mathrm{d}x \tag{6.1}$$

where c is a constant. There is a corresponding discrete version also. This (6.1) is generalized in various directions. One α-generalized entropy known as Havrda-Charvat entropy is the following:

$$H_\alpha(f) = c\frac{\int_x[f(x)]^\alpha\mathrm{d}x - 1}{2^{1-\alpha}-1}, \alpha \neq 1. \tag{6.2}$$

It is easy to see that when $\alpha \to 1$, then $H_\alpha(f) \to S(f) =$ Shannon entropy. A variant of (6.2) is Tsallis [22] entropy given by

$$T_\alpha(f) = c\frac{\int_x[f(x)]^\alpha\mathrm{d}x - 1}{1-\alpha}, \alpha \neq 1. \tag{6.3}$$

In (6.1)-(6.3) the density involved is for a real scalar variable. By optimizing (6.3) under the condition that the first moment is prefixed, produced Tsallis statistics of non-extensive statistical mechanics and the whole branch of non-extensive statistical mechanics came into existence. Axiomatic definitions or characterizations of the measures in (6.1) and (6.2), and many items connected with these, were given by Mathai and Rathie [20], which was the first book on axiomatic foundations of information measures.

Mathai had introduced an α-generalized entropy measure on a general framework, which in the real scalar case can be taken as a variant of $H_\alpha(f)$. Let $f(X)$ be a real-valued scalar function of X such that $f(X) \geq 0$ in the domain of X and $\int_X f(X)\mathrm{d}X = 1$ where the differential element $\mathrm{d}X$ is defined in Section 1. Here, X may be a scalar/vector/matrix or a sequence of matrices in the real or complex domain. Then, Mathai entropy, denoted by $M_\alpha(f)$, is the following:

$$M_\alpha(f) = \frac{\int_X[f(X)]^{1+\frac{a-\alpha}{\eta}}\mathrm{d}X - 1}{\alpha - a} = \frac{E[(f(X))^{\frac{a-\alpha}{\eta}}] - 1}{\alpha - a}, \alpha \neq a, \eta > 0 \tag{6.4}$$

where $E[\cdot]$ denotes the expected value of $[\cdot]$. Note that (6.4) covers scalar/ vector/ matrix-variate cases in the real and complex domain and in the real scalar case we can see that $M_\alpha(f) \to S(f)$ when $\alpha \to a$ where a is an anchoring point, α is the parameter of interest and $\eta > 0$ is a measuring unit of deviations of α from a. Note that

$$\lim_{\alpha\to a}\frac{\int_X[f(X)]^{\frac{a-\alpha}{\eta}}-1}{\alpha-a}=-\frac{1}{\eta}\ln f(X)$$

and hence $\frac{[f(X)]^{\frac{a-\alpha}{\eta}}-1}{\alpha-a}$ is an estimate for $-\frac{1}{\eta}\ln f(X)$. Let us optimize (6.4) for the real scalar variable x, under two moment type constraints; (1): $E[x^{\gamma(\frac{a-\alpha}{\eta})}] =$ prefixed, and (2): $E[x^{\gamma(\frac{a-\alpha}{\eta})+\delta}] =$ prefixed. Then, if we use calculus of variation, the Euler equation is the following:

$$\frac{\partial}{\partial f}[f^{1+\frac{a-\alpha}{\eta}}-\lambda_1x^{\gamma(\frac{a-\alpha}{\eta})}f-\lambda_2x^{\gamma(\frac{a-\alpha}{\eta})+\delta}f]=0$$

where λ_1 and λ_2 are Lagrangian multipliers. This leads to the function

$$f=c\ x^{\gamma}[1+\lambda_3x^{\delta}]^{\frac{\eta}{a-\alpha}} \qquad (i)$$

Let us consider the case $\alpha < a$. Then, the exponent $\frac{\eta}{a-\alpha} > 0$ since $\eta > 0$ and hence f can produce a density for all $a, \alpha, \eta, \alpha < a$ if λ_3 is negative. Let $\lambda_3 = -b(a-\alpha), b > 0, \alpha < a$. Then, we have the model

$$f_1(x)=c_1\ x^{\gamma}[1-b(a-\alpha)x^{\delta}]^{\frac{\eta}{a-\alpha}}, \alpha<a, \eta>0, b>0 \qquad (ii)$$

for $1-b(a-\alpha)x^{\delta} > 0$ where c_1 is the corresponding normalizing constant. This is generalized type-1 beta model in the real scalar variable case. When $\alpha > a$, the model in (ii) switches into the model

$$f_2(x)=c_2\ x^{\gamma}[1+b(\alpha-a)x^{\delta}]^{-\frac{\eta}{\alpha-a}}, \alpha>a, b>0, \eta>0 \qquad (iii)$$

where c_2 is the corresponding normalizing constant. When $\alpha \to a$, then both $f_1(x)$ and $f_2(x)$ go to

$$f_3(x)=c_3\ x^{\gamma}\mathrm{e}^{-b\eta x^{\delta}}, b>0, \eta>0. \qquad (iv)$$

The normalizing constant in $f_1(x)$ is evaluated by using a real scalar type-1 beta integral, c_2 is evaluated by using a real scalar type-2 beta integral and c_3 by using a gamma integral. From f_1, one can go to f_2 and f_3 or from f_2 we can go to f_1 and f_3 through the pathway parameter α. This is Mathai's pathway idea [Mathai [10]] and α here is Mathai's pathway parameter.

Our aim in this section is to derive matrix-variate distributions through optimization of Mathai entropy $M_\alpha(f)$. Let X be a $p\times q, p\le q$ matrix of rank p in the real domain. Let $u = \text{tr}(XX')$ where $XX' > O$. Consider the constraints (1): $E[u^{\gamma(\frac{a-\alpha}{\eta})}]$ = prefixed, and (2): $E[u^{\gamma(\frac{a-\alpha}{\eta})+\delta}]$ = prefixed. Then, optimize $M_\alpha(f)$ under these constraints and proceed as in the illustrative example of the real scalar variable case. Then, we will end up with three models for $\alpha < a, \alpha > a, \alpha \to a$, again denoted by $f_1(X), f_2(X), f_3(X)$ respectively, where, for example, $f_2(X)$ is the following:

$$f_2(X) = C_2[\text{tr}(XX')]^\gamma[1 + b(\alpha - a)(\text{tr}(XX'))^\delta]^{-\frac{\eta}{\alpha-a}}, \alpha > a \qquad (v)$$

where C_2 is the corresponding normalizing constant, $b > 0, \eta > 0, \delta > 0, \Re(\gamma) > -\frac{pq}{2}$. Note that for $p = 1$ we have the $1 \times q$ vector case as a special case of (v). Thus, scalar, vector, rectangular matrix cases are all contained in (v) and from (v) one can also go to the corresponding type-1 beta form and the gamma form also, through the parameter α. Thus, three families of densities involving $\text{tr}(XX')$ are there in (v). One can generalize (v) by incorporating a location parameter matrix and two scaling matrices or by replacing XX' by $A(X - M)B(X - M)'$ where $A > O$ is $p \times p$ and $B > O$ is $q \times q$ constant positive definite matrices and M is a $p \times q$ location parameter matrix. Mathematically speaking, the only change will be that the normalizing constants will be multiplied by $|A|^{\frac{q}{2}}|B|^{\frac{p}{2}}$ in f_1, f_2, f_3. The normalizing constants C_1, C_2, C_3 can be evaluated by using Lemma 1.1 and then by using scalar variable type-1 beta, type-2 beta and gamma integrals. Now, let us consider the corresponding complex case. Let $\tilde{X}$ be a $p \times q, p \leq q$ matrix of rank p in the complex domain. Let $\tilde{u} = \text{tr}(\tilde{X}\tilde{X}^*)$. Consider the same constraints as in the real case with u in the real case replaced by $\tilde{u}$ in the present complex case. Then, proceeding as in the real case, we end up with three densities belonging to Mathai's pathway family, again denoted by $f_j(\tilde{X}_j), j = 1, 2, 3$. For example, $f_2(\tilde{X}_2)$ will be the following:

$$f_2(\tilde{X}) = \tilde{C}_2[\text{tr}(\tilde{X}\tilde{X}^*)]^\gamma[1 + b(\alpha - a)(\text{tr}(\tilde{X}\tilde{X}^*))^\delta]^{-\frac{\eta}{\alpha-a}}, \alpha > a \qquad (vi)$$

for $b > 0, \eta > 0, \delta > 0, \Re(\gamma) > -pq$.

Now, let us consider constraints involving a determinant and a trace in the real case. Again, let X be $p \times q, p \leq q$ matrix of rank p in the real domain. Let the constraints be the following: (1): $E[|XX'|^{\gamma(\frac{a-\alpha}{\eta})}(\text{tr}(XX'))^{\rho(\frac{a-\alpha}{\eta})}]$ = fixed, and (2): $E[|XX'|^{\gamma(\frac{a-\alpha}{\eta})}(\text{tr}(XX'))^{\rho(\frac{a-\alpha}{\eta})+\delta}]$ = fixed. Then, optimize $M_\alpha(f)$ under these constraints and follow through the same procedures as before. Then, we will end up with three families of densities. Let us denote them by f_4, f_5, f_6 where for example, f_5 will be the following:

$$f_5(X) = C_5|XX'|^\gamma[\text{tr}(XX')]^\rho[1 + b(\alpha - a)(\text{tr}(XX'))^\delta]^{-\frac{\eta}{\alpha-a}}, \alpha > a \qquad (vii)$$

for $\alpha > a, b > 0, \eta > 0, \delta > 0, \rho > 0, \Re(\gamma) > 0$. Note that from f_5 one can go to f_4 and f_6 through the pathway parameter α. Generalization is available by replacing XX' by $A(X-M)B(X-M)'$ as mentioned before. The normalizing constants C_4, C_5, C_6 can be evaluated by using Lemma 2.2, Lemma 1.1 and then scalar variable integrals. Whenever a determinant enters into the model as a factor and at the same time if the trace has an arbitrary exponent, then the evaluation of the normalizing constant becomes difficult as illustrated in the evaluation of the normalizing constant in Mathai [14]. The density corresponding to $f_5(X)$ in the complex domain, denoted by $f_5(\tilde{X})$, is the following, the constraints are parallel to those in the real case:

$$f_5(\tilde{X}) = \tilde{C}_5|\det(\tilde{X}\tilde{X}^*)|^{\gamma}[\operatorname{tr}(\tilde{X}\tilde{X}^*)]^{\rho}[1+b(\alpha-a)(\operatorname{tr}(\tilde{X}\tilde{X}^*))^{\delta}]^{-\frac{\eta}{\alpha-a}} \qquad (viii)$$

for $\alpha > a, b > 0, \delta > 0, \rho > 0, \eta > 0$ and $\tilde{C}_5$ is the normalizing constant. Here also, for evaluating the normalizing constant one would require Lemma 1.1 and Lemma 2.2, equations (2.1) and (2.2). With the above examples, we will stop the illustration of the optimization of entropy procedures.

Another recent area of development is the topic of singular distributions. What we have discussed so far are nonsingular distributions in the sense the matrices involved were nonsingular when considering square matrices and XX' nonsingular when X is a rectangular matrix. This author has recently given the corresponding singular versions for Gaussian, gamma, type-1 beta and type-2 matrix-variate distributions. In order to limit the size of the paper this aspect will not be discussed in detail here. Interested readers may consult Mathai and Provost [18].

7. Singular Matrix-variate Gamma and Beta Functions

To start with, let us consider the real case. Let X be a $p \times q, p \leq q$ matrix of rank p in the real domain. Let M be a $p \times q$ parameter matrix. Then, the real central rectangular matrix-variate gamma function associated with X can be written in the following form, where we have incorporated the normalizing constant also so that it can be taken as a density:

$$f_1(XX')\mathrm{d}(XX') = \frac{|B|^{\alpha}}{\Gamma_p(\alpha)}|(X-M)(X-M)'|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\operatorname{tr}(B(X-M)(X-M)')}\mathrm{d}(XX') \qquad (7.1)$$

When $M \neq O$ and if we ignore M, then we have the noncentral matrix-variate gamma function or gamma density with shape parameter α and scale parameter matrix B as the following where the second form is written by replacing $\mathrm{d}(XX')$

by dX by using Lemma 1.1:

$$\begin{aligned} f_1(XX')\mathrm{d}(XX') &= \frac{|B|^{\alpha}}{\Gamma_p(\alpha)}|XX'|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(BXX')}\mathrm{d}(XX') \\ &= \frac{|B|^{\alpha}\Gamma_p(\frac{q}{2})}{\Gamma_p(\alpha)\pi^{\frac{pq}{2}}}|XX'|^{\alpha-\frac{q}{2}}\mathrm{e}^{-\mathrm{tr}(BXX')}\mathrm{d}X. \end{aligned} \tag{7.2}$$

When $\alpha = \frac{q}{2}$, then from (7.2) we have

$$f_2(X)\mathrm{d}X = \frac{|B|^{\frac{q}{2}}}{\pi^{\frac{pq}{2}}}\mathrm{e}^{-\mathrm{tr}(BXX')}\mathrm{d}X. \tag{7.3}$$

Note that (7.3) holds whether XX' is nonsingular or singular or for both the cases $p \le q$ and $p > q$ because $XX' = X_1X_1' + ... + X_qX_q'$ where X_j is the j-th column of the $p \times q$ matrix X and $\mathrm{e}^{-\mathrm{tr}(BX_jX_j')} = \mathrm{e}^{-X_j'BX_j}$ is integrable for each $j = 1, ..., q$. We will start with (7.3) to develop singular matrix-variate gamma function and gamma density. When $p > q$ we have two possibilities that the rank of X may be $q < p$ or the rank of X is $r \le q < p$. Let us take the case that the rank of X is q. Then, we may write the $p \times q$ matrix with $p > q$ as the following:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Rightarrow XX' = S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{i}$$

where X_1 and S_{11} are $q \times q$ and of rank q, X_2 is $(p-q) \times q$. Then

$$XX' = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \begin{bmatrix} X_1' & X_2' \end{bmatrix} = \begin{bmatrix} X_1X_1' & X_1X_2' \\ X_2X_1' & X_2X_2' \end{bmatrix} \tag{ii}$$

where X_1 is $q \times q$ and of rank q and hence nonsingular, X_1X_1' is also of rank q and nonsingular but X_1X_1' is symmetric and positive definite also. When X is of rank $r \le q < p$ then, we will take X_1 as $r \times q$ matrix of rank r where also $X_1X_1' > O$. When X_1 is of rank q, then from (i) and (ii), $X_1X_1' = S_{11} > O$ and $q \times q$. Observe that dX = dX' and X and X' both have pq distinct elements. Note that X' is $q \times p$ and its rank is q and hence $X'X > O$. From Lemma 1.1, we have for $S_1 = X'X$,

$$\mathrm{d}X' = \frac{\pi^{\frac{pq}{2}}}{\Gamma_q(\frac{p}{2})}|S_1|^{\frac{p}{2}-\frac{q+1}{2}}\mathrm{d}S_1. \tag{iii}$$

Therefore, from (7.3)

$$\begin{aligned} f_2(X)\mathrm{d}X &= \frac{|B|^{\frac{q}{2}}}{\pi^{\frac{pq}{2}}}\mathrm{e}^{-\mathrm{tr}(BXX')}\mathrm{d}X \\ &= f_2(X)\mathrm{d}X' = \frac{|B|^{\frac{q}{2}}}{\pi^{\frac{pq}{2}}}\frac{\pi^{\frac{pq}{2}}}{\Gamma_q(\frac{p}{2})}|S_1|^{\frac{p}{2}-\frac{q+1}{2}}\mathrm{e}^{-\mathrm{tr}(BS)}\mathrm{d}S_1 \\ &= \frac{|B|^{\frac{q}{2}}}{\Gamma_q(\frac{p}{2})}|S_1|^{\frac{p}{2}-\frac{q+1}{2}}\mathrm{e}^{-\mathrm{tr}(BS)}\mathrm{d}S_1, S = XX', S_1 = X'X \\ &= \frac{1}{\Gamma_q(\frac{p}{2})}|S_1|^{\frac{p}{2}-\frac{q+1}{2}}\mathrm{e}^{-\mathrm{tr}(S_1)}\mathrm{d}S_1 \text{ for } B = I. \end{aligned} \quad \text{(iv)}$$

This (*iv*) is one form of the singular real matrix-variate gamma function or gamma density denoted by $f_3(S_1)\mathrm{d}S_1$. Then, the corresponding singular gamma density in the complex domain, denoted by $f_3(\tilde{S}_1)$, is the following:

$$\begin{aligned} f_3(\tilde{S}_1)\mathrm{d}\tilde{S}_1 &= \frac{|\det(B)|^q}{\tilde{\Gamma}_q(p)}|\det(\tilde{S}_1)|^{p-q}\mathrm{e}^{-\mathrm{tr}(B\tilde{S})}\mathrm{d}\tilde{S}_1, \tilde{S} = \tilde{X}\tilde{X}^*, \tilde{S}_1 = \tilde{X}^*\tilde{X} \\ &= \frac{1}{\tilde{\Gamma}_q(p)}|\det(\tilde{S}_1)|^{p-q}\mathrm{e}^{-\mathrm{tr}(\tilde{S}_1)}\mathrm{d}\tilde{S}_1 \text{ for } B = I. \end{aligned} \quad \text{(v)}$$

From (*i*) and (*ii*), note that S is singular and the rows $[S_{21}, S_{22}]$ are linear functions of the rows in $[S_{11}, S_{12}]$. Hence, a linear combinations of $[S_{11}, S_{12}]$ which makes $S_{21} = O$ (null) should make the corresponding element in S_{22} position also null. Take $-S_{21}S_{11}^{-1}$ times $[S_{11}, S_{12}]$ and add to the second set of rows to make S_{21} position null. Then, the corresponding matrix in the S_{22} position, namely $S_{22} - S_{21}S_{11}^{-1}S_{12} = O \Rightarrow S_{22} = S_{21}S_{11}^{-1}S_{12}$. This is a general result and hence we will write it as a lemma.

Lemma 7.1. *When $X, X_1, X_2, S_{11}, S_{12}, S_{21}, S_{22}$ are as defined in (i) and (ii) above and when the rank of X = rank of X_1 and whether X_1 is $q \times q$ with rank q or X_1 is $r \times q$ and with rank $r \leq q < p$, we have*

$$S_{22} = S_{21}S_{11}^{-1}S_{21}. \quad (7.4)$$

Then, we may take $\mathrm{d}S = \mathrm{d}S_{11} \wedge \mathrm{d}S_{21}$ or $\mathrm{d}S_{11} \wedge \mathrm{d}S_{12}$. Also, $\mathrm{d}X = \mathrm{d}X_1 \wedge \mathrm{d}X_2$ and from Lemma 1.1 when X_1 is $q \times q$ and of rank q, we have

$$\mathrm{d}X_1 = \frac{\pi^{q^2}}{\Gamma_q(\frac{q}{2})}|S_{11}|^{\frac{q}{2}-\frac{q+1}{2}}\mathrm{d}S_{11}. \quad (7.5)$$

Letting $S_{11} = CC'$ where C is $q \times q$ nonsingular matrix and $X_2 = S_{11}C'^{-1}$ and applying Lemma 7.1, we have

$$\mathrm{d}X_2 = |C'|^{-(p-q)}\mathrm{d}S_{21} = |S_{11}|^{\frac{q-p}{2}}\mathrm{d}S_{21}. \qquad (vi)$$

Consider a similar partitioning of the $p \times p$ matrix

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where, for example, B_{11} is $q \times q$. Now,

$$\begin{aligned} \mathrm{tr}(BXX') &= \mathrm{tr}(BS) \\ &= \mathrm{tr}(B_{11}S_{11}) + \mathrm{tr}(B_{12}S_{21}) + \mathrm{tr}(B_{21}S_{12}) + \mathrm{tr}(B_{22}S_{22}) \\ &= \mathrm{tr}(B_{11}X_1X_1') + \mathrm{tr}(B_{21}X_1X_2') + \mathrm{tr}(B_{12}X_2X_1') + \mathrm{tr}(B_{22}X_2X_2'). \quad \text{(vii)} \end{aligned}$$

Now, from (i) to (vii) and (7.1) to (7.5), we have the following result.

Theorem 7.1. *Let X be a $p \times q$ matrix of rank q and with pq distinct real scalar variables as elements. Let $p > q$. Let $X, S, X_j, j = 1, 2, S_{ij}$'s be as defined in (i) and (ii). Let the rank of X_1 and thereby rank of S_{11} be q. Let $\mathrm{d}S$ be interpreted as $\mathrm{d}S = \mathrm{d}S_{11} \wedge \mathrm{d}S_{21}$. Let the singular gamma density of S be denoted as $f_4(S)$ in the real case and $f_4(\tilde{S})$ in the complex domain. Then, we have*

$$f_4(S)\mathrm{d}S_{11} \wedge \mathrm{d}S_{21} = \frac{|B|^{\frac{q}{2}}}{\pi^{\frac{(p-q)q}{2}}\Gamma_q(\frac{q}{2})}|S_{11}|^{\frac{q}{2}-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(BS)}\mathrm{d}S_{11} \wedge \mathrm{d}S_{21} \qquad (7.6)$$

and

$$f_4(\tilde{S})\mathrm{d}\tilde{S}_{11} \wedge \mathrm{d}\tilde{S}_{21} = \frac{|\mathrm{det}(B)|^{q}}{\pi^{(p-q)q}\tilde{\Gamma}_q(q)}|\mathrm{det}(\tilde{S}_{11})|^{q-p}\mathrm{e}^{-\mathrm{tr}(B\tilde{S})}\mathrm{d}\tilde{S}_{11} \wedge \mathrm{d}\tilde{S}_{21}. \qquad (7.7)$$

Form (7.6) and (7.7). one can derive the marginal densities of $S_{11}, \tilde{S}_{11}, S_{21}, \tilde{S}_{21}$ and the conditional densities of S_{21}, given S_{11}, as well as $\tilde{S}_{21}$, given $\tilde{S}_{11}$. Similar procedures will produce the singular matrix-variate type-1 and type-2 beta functions and beta densities. In order to limit the size of the manuscript we stop the discussion here. Some more details may be found in Mathai and Provost [18].

8. Concluding Remarks

Here we have considered introductions to scaling models, Bayesian structures, texture models in communication theory, and distribution of a product of independent variables of the form $U = \sqrt{a}X$ where X may be a scalar, vector or matrix in the real or complex domain. We considered the various structures when $a > 0$ is a real scalar variable only in order to limit the size of the paper. Materials are available when the scaling factor is a matrix, including the ideas of a vector or a rectangular matrix, in the real or complex domain. Some combinations of the distributions of a and X in Mathai's pathway family [Mathai [10]] are considered for illustrative purposes. One can look into other combinations involving densities from outside Mathai's pathway family. Mainly the Bayesian structures are considered for illustrative purposes because the other situations mentioned can be converted to Bayesian structures and hence more details on other situations are not discussed in order to limit the size of the manuscript. The main purpose of this paper is to introduce one family of problems which are being currently pursued in the literature so that interested students and researchers can get into the areas. All the models introduced in this paper are nonsingular models in the sense when square matrices are involved the matrices are nonsingular and when a rectangular matrix X is involved then XX' is nonsingular. One real matrix-variate singular gamma function or singular gamma density is introduced in Section 7. One can consider singular versions of all the models discussed in this paper. Most of this aspect is open, some singular situations are tackled in Mathai and Provost [19].

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