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CLASSES OF L^1-CONVERGENCE OF FOURIER SERIES

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Abstract: In this paper, wider classes of Fourier cosine series are introduced and found that $a_n \log n = o(1),\ n \to \infty$ is a necessary and sufficient condition for L^1-convergence. Our results generalize the results obtained by A.N. Kolmogorov as well as R. Bala and B. Ram for cosine series while our new classes $\mathcal{JS}$ quasi convex and $\mathcal{JS}$ semi convex are the extensions of the classes quasi convex null sequence and semi convex respectively.

Keywords and Phrases: Dirichlet kernel, conjugate Dirichlet kernel, Fejer kernel, conjugate Fejer kernel, L^1- convergence.

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1. Introduction

Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

be cosine trigonometric series with partial sum denoted by $S_n(x) = \frac{a_0}{2} + \sum\limits_{k=1}^{n} a_k \cos kx$ and let $f(x) = \lim\limits_{n\to\infty} S_n(x)$.

The problem of L^1-convergence of Fourier cosine series (1.1) has been settled for various special classes of coefficients. *"It is well known that the condition $a_n \log n = o(1),\ n \to \infty$ is both necessary and sufficient condition for L^1-convergence for some classes of Fourier cosine series".* In 1913, W. H. Young [30] found that $a_n \log n = o(1),\ n \to \infty$ is both necessary and sufficient condition for integrability and L^1-convergence of cosine trigonometric series with convex coefficients ($\Delta^2 a_n = \Delta(\Delta a_n) = \Delta a_n - \Delta a_{n+1} = a_n - 2a_{n+1} + a_{n+2} \geq 0,\ \ \forall n$) and A. N. Kolmogorov [16] extended the Young's result for cosine trigonometric series with quasi convex coefficients $\left(\sum_{n=1}^{\infty} n|\Delta^2 a_n| < \infty\right)$.

Theorem 1.1. [16] *If $\{a_n\}$ is quasi convex null sequence, then for the L^1-convergence of the cosine series (1.1) it is necessary and sufficient that $\lim_{n\to\infty} a_n \log n = 0$.*

S. A. Telyakovskiĭ [25] extended the classical result of A. N. Kolmogorov [16] with class S (introduced by Sidon [20]) of Fourier coefficients that contains the class of quasi convex coefficient and proved that if $\{a_n\} \in S$ then (1.1) is a Fourier series of some $f \in L^1(0, \pi)$ and that

$$||S_n - f|| = o(1), \quad n \to \infty, \tag{1.2}$$

if and only if

$$a_n \log n = o(1), \quad n \to \infty, \tag{1.3}$$

$||.||$ is the $L^1(0, \pi)$-norm.

T. Kano [13] defined the class of semi convex sequence as follows:
A sequence $\{a_n\}$ is said to be semi convex sequence if $a_n \to 0$ as $n \to \infty$ and

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, \qquad (a_0 = 0)$$

R. Bala and B. Ram [1] have proved the theorem for cosine series with semi convex sequence in the following form:

Theorem 1.2. [1] *If $\{a_n\}$ is a semi convex null sequence then for the convergence of the cosine series (1.1) in the metric space L it is necessary and sufficient that $a_n \log n = o(1)$, $n \to \infty$.*

The results obtained by these authors were further generalized and extended by many authors such as J. W. Garrett and Č. V. Stanojević ([9], [10], [11]), B. Ram [18], N. Singh and K. M. Sharma ([21], [22], [23]), R. Bojanic and Č. V. Stanojević [4], C. P. Chen [5], R. Bala and B. Ram [1], F. Móricz [17], S. S. Bhatia and B.

Ram [3], Ž. Tomovskii ([26], [27], [28], [29]), N. Hooda and B. Ram [12], K. Kaur *et al.* [15], J. Kaur and S. S. Bhatia [14], S. K. Chouhan, J. Kaur and S. S. Bhatia ([6], [7]) and others by considering various generalizations of classes of sequences.

In this paper, the L^1-convergence of Fourier cosine series with newly defined classes are studied. The paper is organized in four sections as follows: In the section 2, some classes of L^1-convergence and the concerned results are given. Also, the new classes of L^1-convergence are given in this section which are the extensions of previous well known classes. The section 3 presents main results which are the generalization of some existing results.

2. Classes of L^1-convergence

In this section, firstly we recall some classes of L^1-convergence and their concerned results:

G. A. Fomin [8] introduced the class $\mathfrak{F}_p$ which is the extension of class S.

Definition 2.1. [8] *A null sequence $\{a_n\}$ belongs to the class $\mathfrak{F}_p$ if for some $1 < p \leq 2$,*

$$\sum_{n=1}^{\infty}\left(\frac{\sum\limits_{k=n}^{\infty}|\Delta a_k|^p}{n}\right)^{1/p} < \infty,$$

The class $\mathfrak{F}_p$ is wider when p is closer to 1. He proved the following result concerning L^1-convergence of Fourier series.

Theorem 2.2. [8] *For some $1 < p \leq 2$. Let $\{a_n\} \in \mathfrak{F}_p$. Then (1.1) is a Fourier series of some $f \in L^1(0,\pi)$ and (1.2) $\Leftrightarrow$ (1.3).*

In 1981, Č. V. Stanojević [24] introduced new classes such as class $\mathcal{C} \cap BV$, $\mathcal{C}_p$, $\mathcal{C}_p^*$, P which are the extensions of previous well known classes. Also, a necessary and sufficient conditions for L^1-convergence for certain larger classes of Fourier series have been proved in [24].

The Class P of Č. V. Stanojević [24] which is the natural extension of class BV reads as follow:

Definition 2.3. [24] *A null sequence $\{a_n\}$ belongs to the class P if*

$$\frac{1}{n}\sum_{k=1}^{n} k|\Delta a_k| = o(1), \quad n \to \infty.$$

The results concerning L^1-convergence of Fourier cosine series under the class P proved by him are given as follows:

Theorem 2.4. [24] *Let (1.1) be a Fourier series with* $\{a_n\} \in P$ *and* $n\Delta a_n = o(1), \quad n \to \infty$ *then (1.2)* $\Leftrightarrow$ *(1.3).*

In the present paper, classes of quasi convex sequences and semi convex sequences are extended respectively in the following manner:

Class $\mathcal{JS}$ quasi convex. A null sequence $\{a_n\}$ is said to be belong to class $\mathcal{JS}$ quasi convex if

$$\frac{1}{n+1}\sum_{k=1}^{n} k(k+1)|\Delta^2 a_k| = o(1), \quad n \to \infty. \tag{2.1}$$

Remark. *The class $\mathcal{JS}$ quasi convex extends the class of quasi convex null sequence.*

If $\{a_n\}$ is a quasi convex null sequence then $\lim\limits_{n\to\infty} a_n = 0$ and $\sum\limits_{n=1}^{\infty} n|\Delta^2 a_n| < \infty$ which implies that $n|\Delta^2 a_n| = o(1), \ n \to \infty$.

Consider,

$$\frac{1}{n+1}\sum_{k=1}^{n} k(k+1)|\Delta^2 a_k| = \frac{2|\Delta^2 a_1|}{n+1} + \frac{6|\Delta^2 a_2|}{n+1} + ... + \frac{n(n+1)|\Delta^2 a_n|}{n+1}.$$

By given hypothesis, each term of above equation approaches to zero as $n \to \infty$. Therefore,

$$\frac{1}{n+1}\sum_{k=1}^{n} k(k+1)|\Delta^2 a_k| = o(1), \quad n \to \infty.$$

Hence $\{a_n\}$ belongs to the class $\mathcal{JS}$ quasi convex.
But the converse need not be true as the term $n|\Delta^2 a_n| = o(1), \quad n \to \infty$ does not ensure the convergence of the series $\sum\limits_{n=1}^{\infty} n|\Delta^2 a_n|$.

Class $\mathcal{JS}$ semi convex. A null sequence $\{a_n\}$ is said to be belong to class $\mathcal{JS}$ semi convex if

$$\frac{1}{n+1}\sum_{k=1}^{n} k(k+1)|\Delta^2 a_{k-1} + \Delta^2 a_k| = o(1), \quad n \to \infty \quad (\text{where} \ \ a_0 = a_1 = 0) \tag{2.2}$$

Remark. *The class $\mathcal{JS}$ semi convex is natural extension of class semi convex.*

3. Main Results

The first main result of this section reads as follow:

Theorem 3.1. *Let (1.1) be a Fourier series of some* $f \in L^1(0, \pi)$ *and let* $\{a_n\}$

belongs to class $\mathcal{JS}$ quasi convex and $n\Delta a_n = o(1), \quad n \to \infty$. Then $g_n(x)$ (where $g_n(x) = S_n(x) - a_{n+1}D_n(x), \quad D_n(x)$ is a Dirichlet kernel) converges in L^1 norm.

Proof. Consider

$$||g_n - f|| \;=\; ||g_n - \sigma_n + \sigma_n - f|| = ||g_n - \sigma_n|| + ||\sigma_n - f||$$

where σ_n is the Fejer sum of S_n $\left(\sigma_n(f) = \frac{1}{n+1}\sum\limits_{k=0}^{n} S_k(x)\right)$

As we know that, if $f \in L^1(0,\pi)$ then $||\sigma_n - f|| = o(1) \quad n \to \infty$.

So, it is sufficient to show that $||g_n - \sigma_n|| = o(1), \quad \text{as} \quad n \to \infty$

Consider

$$\begin{aligned} ||g_n - \sigma_n(x)|| &= \int_0^\pi |g_n(x) - \sigma_n(x)|dx \\ &= \int_0^\pi |S_n(x) - \sigma_n(x) - a_{n+1}D_n(x)|dx \\ &= \int_0^\pi \left| \frac{1}{n+1}\sum_{k=1}^{n} ka_k \cos kx - a_{n+1}D_n(x)\right| dx \end{aligned}$$

Apply Abel's transformation on the first term of R.H.S.,

$$||g_n - \sigma_n(x)|| = \int_0^\pi \left| \frac{1}{n+1}\sum_{k=1}^{n-1} \Delta(ka_k)D_k(x) + \frac{n}{n+1}a_nD_n(x) - a_{n+1}D_n(x)\right| dx$$

$$= \int_0^\pi \left| \frac{1}{n+1}\sum_{k=1}^{n} \Delta(ka_k)D_k(x) - \frac{1}{n+1}\Delta(na_n)D_n(x) + \frac{n}{n+1}a_nD_n(x) - a_{n+1}D_n(x)\right| dx$$

$$= \int_0^\pi \left| \frac{1}{n+1}\sum_{k=1}^{n} k\Delta a_kD_k(x) - \frac{1}{n+1}\sum_{k=1}^{n} a_{k+1}D_k(x)\right| dx$$

Again using Abel's transformation on both the series of R.H.S.,

$$\begin{aligned} ||g_n - \sigma_n(x)|| &= \int_0^\pi \left| \frac{1}{n+1}\sum_{k=1}^{n-1} \Delta(k\Delta a_k)(k+1)F_k(x) + n\Delta a_nF_n(x) \right. \\ &\quad \left. - \frac{1}{n+1}\sum_{k=1}^{n-1}(k+1)\Delta a_{k+1}F_k(x) - a_{n+1}F_n(x)\right| dx \end{aligned}$$

where $F_n(x)$ represents Fejer kernel.

$$\begin{aligned} ||g_n-\sigma_n(x)|| &= \int_0^\pi \left| \frac{1}{n+1}\sum_{k=1}^{n}\Delta(k\Delta a_k)(k+1)F_k(x) - \Delta(n\Delta a_n)F_n(x) + n\Delta a_n F_n(x) \right. \\ &\qquad \left. -\frac{1}{n+1}\sum_{k=1}^{n}(k+1)\Delta a_{k+1}F_k(x) + \Delta a_{n+1}F_n(x) - a_{n+1}F_n(x) \right| dx \end{aligned}$$

$$\begin{aligned} ||g_n-\sigma_n(x)|| = &\int_0^\pi \left| \frac{1}{n+1}\sum_{k=1}^{n}k(k+1)\Delta^2 a_k F_k(x) - \frac{2}{n+1}\sum_{k=1}^{n}(k+1)\Delta a_{k+1}F_k(x) \right. \\ &\qquad \left. +(n+1)\Delta a_{n+1}F_n(x) - a_{n+2}F_n(x) \right| dx \\ \leq &\frac{1}{n+1}\sum_{k=1}^{n}k(k+1)|\Delta^2 a_k|\int_0^\pi |F_k(x)|dx + \frac{2}{n+1}\sum_{k=1}^{n}(k+1)|\Delta a_{k+1}|\int_0^\pi |F_k(x)|dx \\ &+ (n+1)|\Delta a_{n+1}|\int_0^\pi |F_n(x)|dx + |a_{n+2}|\int_0^\pi |F_n(x)|dx \\ \leq &\frac{\pi}{n+1}\sum_{k=1}^{n}k(k+1)|\Delta^2 a_k| + \frac{2\pi}{n+1}\sum_{k=1}^{n}(k+1)|\Delta a_{k+1}| + \pi(n+1)|\Delta a_{n+1}| + \pi|a_{n+2}| \end{aligned} \tag{3.1}$$

The second term of above inequality after applying Abel's transformation becomes

$$\frac{1}{n+1}\sum_{k=1}^{n}(k+1)|\Delta a_{k+1}| \leq \frac{1}{n+1}\sum_{k=1}^{n-1}\frac{(k+1)(k+2)}{2}|\Delta^2 a_{k+1}| + \frac{1}{2}(n+2)|\Delta a_{n+1}|$$

So, by given hypothesis, $||g_n-\sigma_n(x)|| = o(1), \quad n\to\infty$, as all terms on the R.H.S of inequality (3.1) tends to zero as $n\to\infty$.

Therefore, the conclusion of main result holds.

Corollary 3.2. *Let (1.1) be a Fourier series of some $f\in L^1(0,\pi)$ and $\{a_k\}$ belongs to the class $\mathcal{JS}$ quasi convex and $n\Delta a_n = o(1), \quad n\to\infty$ then the necessary and sufficient condition for L^1-convergence of the cosine series is $\lim_{n\to\infty} a_n\log n = 0$.*

Proof. We have $||S_n - f(x)|| \leq ||S_n - g_n|| + ||g_n - f|| = ||g_n - f|| + o(a_n\log n)$
Since $||g_n - f|| = o(1)$ as $n\to\infty$ by Theorem 3.1
Therefore it follows that

$$\lim_{n\to\infty}\int_0^\pi |S_n(x)-f(x)|dx = 0 \text{ if and only if } \lim_{n\to\infty} a_n\log n = o(1)$$

The second main result of this section reads as follows:

Theorem 3.3. *Let (1.1) be a Fourier series of some $f \in L^1(o,\pi)$ and a monotonically decreasing sequence $\{a_n\}$ belongs to the class $\mathcal{JS}$ semi convex with $n\Delta a_n = o(1),\ n \to \infty$. Then $g_n(x)$ converges in L^1 if and only if $a_n \log n = o(1),\ n \to \infty$.*

Proof. Consider

$$\begin{aligned} ||g_n(x) - f(x)|| &= \int_0^\pi |g_n(x) - f(x)|dx \\ &\le \int_0^\pi |g_n(x) - \sigma_n(x)|dx + \int_0^\pi |\sigma_n(x) - f(x)|dx \\ &= I_1 + I_2 \\ I_1 = \int_0^\pi |g_n(x) - \sigma_n(x)|dx &= \int_0^\pi |S_n(x) - \sigma_n(x) - a_{n+1}D_n(x)|dx \end{aligned}$$

Consider

$$\begin{aligned} S_n(x) - \sigma_n(x) &= \frac{1}{n+1}\sum_{k=1}^{n} ka_k \cos kx \\ &= \frac{1}{2\sin x}\left[\frac{1}{n+1}\sum_{k=1}^{n} ka_k(2\cos kx \sin kx)\right] \\ &= \frac{1}{2\sin x}\left[\frac{1}{n+1}\sum_{k=1}^{n} ka_k(\sin(k+1)x - \sin(k-1)x)\right] \\ &= \frac{1}{2\sin x}\left[\frac{1}{n+1}\sum_{k=1}^{n} [(k-1)a_{k-1} - (k+1)a_{k+1}]\sin kx \right. \\ &\qquad \left. + a_{n+1}\sin nx + \frac{n}{n+1}a_n \sin(n+1)x\right] \\ S_n(x) - \sigma_n(x) &= \frac{1}{2\sin x}\left[\frac{1}{n+1}\sum_{k=1}^{n} k(\Delta a_{k-1} + \Delta a_k)\sin kx - \frac{1}{n+1}\sum_{k=1}^{n}(a_{k-1} + a_{k+1})\sin kx \right. \\ &\qquad \left. + a_{n+1}\sin nx + \frac{n}{n+1}a_n \sin nx\right] \end{aligned}$$

Since $\{a_k\}$ is a monotonically decreasing sequence, therefore second term of above equality changes to the following inequality:

$$\begin{aligned} S_n(x) - \sigma_n(x) &\le \frac{1}{2\sin x}\left[\frac{1}{n+1}\sum_{k=1}^{n} k(\Delta a_{k-1} + \Delta a_k)\sin kx - \frac{1}{n+1}\sum_{k=1}^{n}(a_k + a_{k+1})\sin kx \right. \\ &\qquad \left. + a_{n+1}\sin nx + \frac{n}{n+1}a_n \sin(n+1)x\right] \end{aligned}$$

Apply Abel's transformation on first two terms of above inequality, we get

$$\begin{aligned} S_n(x)-\sigma_n(x) \quad &\leq \quad \frac{1}{2\sin x}\left[\frac{1}{n+1}\sum_{k=1}^{n-1}(\Delta(k\Delta a_{k-1})+\Delta(k\Delta a_k))\tilde{D}_k(x)\right.\\ &-\frac{1}{n+1}\sum_{k=1}^{n-1}(\Delta a_k+\Delta a_{k+1})\tilde{D}_k(x)+\frac{n}{n+1}\Delta a_{n-1}\tilde{D}_n(x)+\frac{n}{n+1}\Delta a_n\tilde{D}_n(x)\\ &\left.-\frac{1}{n+1}a_n\tilde{D}_n(x)-\frac{1}{n+1}a_{n+1}\tilde{D}_n(x)+a_{n+1}\sin nx+\frac{n}{n+1}a_n\sin(n+1)x\right] \end{aligned}$$

where $\tilde{D}_n(x)$ represents conjugate Dirichlet kernel.

$$\begin{aligned} S_n(x)-\sigma_n(x) \quad &\leq \quad \frac{1}{2\sin x}\left[\frac{1}{n+1}\sum_{k=1}^{n}(\Delta(k\Delta a_{k-1})+\Delta(k\Delta a_k))\tilde{D}_k(x)+\Delta a_n\tilde{D}_n(x)+\Delta a_{n+1}\tilde{D}_n(x)\right.\\ &-\frac{1}{n+1}\sum_{k=1}^{n}(\Delta a_k+\Delta a_{k+1})\tilde{D}_k(x)-\frac{1}{n+1}a_{n+1}\tilde{D}_n(x)-\frac{1}{n+1}a_{n+2}\tilde{D}_n(x)\\ &\left.+a_{n+1}\sin nx+\frac{n}{n+1}a_n\sin(n+1)x\right] \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2\sin x}\left[\frac{1}{n+1}\sum_{k=1}^{n}k(\Delta^2 a_{k-1}+\Delta^2 a_k)\tilde{D}_k(x)-\frac{2}{n+1}\sum_{k=1}^{n}(\Delta a_k+\Delta a_{k+1})\tilde{D}_k(x)+\Delta a_n\tilde{D}_n(x)\right.\\ &\left.+\Delta a_{n+1}\tilde{D}_n(x)-\frac{1}{n+1}a_{n+1}\tilde{D}_n(x)-\frac{1}{n+1}a_{n+2}\tilde{D}_n(x)+a_{n+1}\sin nx+\frac{n}{n+1}a_n\sin(n+1)x\right] \end{aligned}$$

Apply Abel's transformation on the second term of above equation, we have

$$\begin{aligned} S_n(x)-\sigma_n(x) \quad &\leq \quad \frac{1}{2\sin x}\left[\frac{1}{n+1}\sum_{k=1}^{n}k(\Delta^2 a_{k-1}+\Delta^2 a_k)\tilde{D}_k(x)-\frac{2}{n+1}\sum_{k=1}^{n-1}(\Delta^2 a_k+\Delta^2 a_{k+1})\right.\\ &(k+1)\tilde{F}_k(x)-\Delta a_n\tilde{F}_n(x)-\Delta a_{n+1}\tilde{F}_n(x)+\Delta a_n\tilde{D}_n(x)+\Delta a_{n+1}\tilde{D}_n(x)\\ &\left.-\frac{1}{n+1}a_{n+1}\tilde{D}_n(x)-\frac{1}{n+1}a_{n+2}\tilde{D}_n(x)+a_{n+1}\sin nx+\frac{n}{n+1}a_n\sin(n+1)x\right] \end{aligned}$$

Here $\tilde{F}_n(x)$ represents conjugate Fejer kernel.

$$\begin{aligned} I_1 \quad = \quad &\int_0^{\pi}\left|\frac{1}{2\sin x}\left[\sum_{k=1}^{n}k(\Delta^2 a_{k-1}+\Delta^2 a_k)\tilde{D}_k(x)-\frac{2}{n+1}\sum_{k=1}^{n}(\Delta^2 a_k+\Delta^2 a_{k+1})(k+1)\tilde{F}_k(x)\right.\right.\\ &-\Delta a_{n+1}\tilde{F}_n(x)-\Delta a_{n+2}\tilde{F}_n(x)+\Delta a_n\tilde{D}_n(x)+\Delta a_{n+1}\tilde{D}_n(x)\\ &\left.\left.-\frac{1}{n+1}(a_{n+1}+a_{n+2})\tilde{D}_n(x)+a_{n+1}\sin nx+\frac{n}{n+1}a_n\sin(n+1)x\right]-a_nD_n(x)\right|dx \end{aligned}$$

$$\leq \frac{1}{n+1}\sum_{k=1}^{n} k|\Delta^2 a_{k-1}+\Delta^2 a_k| \int_0^{\pi} \left|\frac{\tilde{D}_n(x)}{2\sin x}\right| dx + |\Delta a_n + \Delta a_{n+1}| \int_0^{\pi} \left|\frac{\tilde{D}_n(x)}{2\sin x}\right| dx$$
$$+ \left|\frac{a_{n+1}+a_{n+2}}{n+1}\right| \int_0^{\pi} \left|\frac{\tilde{D}_n(x)}{2\sin x}\right| dx + C\left[\frac{2}{n+1}\sum_{k=1}^{n}(k+1)|\Delta^2 a_k + \Delta^2 a_{k+1}|\right.$$
$$\left.\int_0^{\pi} |\tilde{F}_k(x)|dx + |\Delta a_{n+1} + \Delta a_{n+2}| \int_0^{\pi} |\tilde{F}_n(x)|dx\right] + a_{n+1}\int_0^{\pi}\left|\frac{\sin nx}{2\sin x}\right| dx$$
$$+ \frac{n}{n+1} a_n \int_0^{\pi} \left|\frac{\sin(n+1)x}{2\sin x}\right| dx + a_n \int_0^{\pi} |\tilde{D}_n(x)|dx$$
$$\leq \frac{1}{n+1}\sum_{k=1}^{n} k(k+1)|\Delta^2 a_{k-1}+\Delta^2 a_k| + |\Delta a_n + \Delta a_{n+1}| n + \left|\frac{a_{n+1}+a_{n+2}}{n+1}\right| n$$
$$+ C\pi\left[\frac{2}{n+1}\sum_{k=1}^{n}(k+1)|\Delta^2 a_k + \Delta^2 a_{k+1}| + |\Delta a_{n+1}+\Delta a_{n+2}|\right]$$
$$+ a_{n+1}\log n + \frac{n}{n+1} a_n \log n + a_n \log n \tag{3.2}$$

Further, by the given hypothesis all the terms on the right side of inequality (3.2) tends to zero as $n \to \infty$.
Therefore $||g_n - \sigma_n(x)|| = o(1), \quad n \to \infty$.
Since $f \in L^1(0,\pi)$ implies $||\sigma_n(x) - f||$ tends to zero as $n \to \infty$.
Hence, the conclusion holds.

Corollary 3.4. *Let (1.1) be a Fourier series of some $f \in L^1(o,\pi)$ and a monotonically decreasing sequence $\{a_n\}$ belongs to the class $\mathcal{JS}$ semi convex with $n\Delta a_n = o(1), \;\; n \to \infty$ then the necessary and sufficient condition for L^1-convergence of the cosine series is* $\lim\limits_{n\to\infty} a_n \log n = 0$.

Remark. The Theorem 1.1 proved by A. N. Kolmogorov [16] and the Theorem 1.2 proved by R. Bala and B. Ram [1] have been deduced as corollaries of our main results.

4. Conclusion

The results presented in this paper generalize the work of A. N. Kolmogorov [16] and R. Bala and B. Ram [1] for Fourier cosine series.

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