South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 1 (2024), pp. 445-456

DOI: 10.56827/SEAJMMS.2024.2001.34 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

HARDY SPACES OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS

S. Sunil Varma and Jocelyn Johnson

Department of Mathematics,
Madras Christian College,
Tambaram, Chennai - 600059, Tamil Nadu, INDIA

E-mail : sunilvarma@mcc.edu.in, jocelynjohnson1010@gmail.com

(**Received:** *Nov. 29, 2023* **Accepted:** *Apr. 27, 2024* **Published:** *Apr. 30, 2024*)

Abstract: In this paper we consider various subclasses of normalized, analytic functions defined in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and study the Hardy space of the functions in these subclasses. This study provides an analysis of the growth of these functions near the boundary of the open unit disk and the Taylor's coefficients of them. The study is carried out using the methods of integral means and subordination of analytic functions. Determination of explicit indices of the Hardy space and order of the growth rate of the Taylor coefficient of these functions are important results here. The novelty of the work here is an attempt to extend the study of the above mentioned features for functions in standard subclasses of analytic univalent functions which were not considered by researchers in the past.

Keywords and Phrases: Analytic function, Univalent function, Hardy space, Subordination.

2020 Mathematics Subject Classification: 30C45, 30C80, 30H05, 30H10.

1. Introduction

The study on function spaces of analytic functions is of recent interest to researchers working in the field of Geometric Function Theory. There are quite a few different kinds of function spaces of analytic functions whose extremal problems, coefficient inequalities, integral formulae and other geometric properties have been studied by the researchers in the recent past (see [2], [4], [8], [10] and [20]). One such function space of analytic functions defined in the unit disk is the Hardy Space - named by F.Riesz in the year 1923 in honour of a research paper by the great English Mathematician G. H. Hardy published in the year 1915 [11]. We now recall the definition of Hardy space.

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane. For functions f analytic in the open unit disk, the integral means are defined by

$$M_p(r, f) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}, & \text{if } 0 < p < \infty \\ max_{0\le\theta<2\pi}|f(re^{i\theta})|, & \text{if } p = \infty. \end{cases}$$

The Hardy spaces H^p consists of analytic functions f defined in the open unit disk for which the integral means $M_p(r, f)$ remains bounded as $r \to 1$. Thus H^∞ consists of all bounded functions in the unit disk whereas the Hardy space H^2 consists of all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined in the unit disk for which $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. It is evident that for $0 < p < q \le \infty$, $H^p \supset H^q$. For more details on Hardy spaces, one may refer to [5].

Let $\mathcal{H}$ be the class of all analytic functions in Δ and $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$ with normalization $f(0) = 0$ and $f'(0) = 1$. Such functions can be expressed in Taylor series as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

Call $\mathcal{S}$ to be the class of univalent functions in the class $\mathcal{A}$.

In pursuit of proving the famous Bieberbach conjecture researchers have introduced and studied numerous subclasses of class $\mathcal{S}$ in different contexts (see [6], [21]. Recently, the researchers in Univalent Function Theory started exploring the Hardy spaces of analytic functions and their derivatives in certain standard geometric subclasses like starlike and convex functions, close-to-convex functions [7], spiral-like functions [1] the class $MV[\alpha, k]$ [15], Bazilevic functions [14] and functions of bounded boundary rotation [18] are some of the works done in the field.

Motivated by the works of above researchers, in this paper we have considered certain subclasses of normalized analytic functions introduced and studied by various authors and studied the Hardy spaces of functions and their derivatives in these classes together with the growth estimate of the Taylor's coefficients of the functions in these subclasses.

2. Some Subclasses of Normalized Analytic Functions in the Unit Disk

In this section, we recall the definition of subordination and compile the definitions of previously introduced subclass of $\mathcal{S}$ whose Hardy spaces and growth of coefficients will be determined in the subsequent sections.

Definition 2.1. [6] *Let f and g be analytic functions defined in the unit disk Δ with $f(0) = g(0)$. f is said to be subordinate to g, written as $f \prec g$, if there exists a function ω, analytic in Δ, with $\omega(0) = 0$, $|\omega(z)| < 1$ such that*

$$f(z) = g(\omega(z)) \tag{2}$$

for each $z \in \Delta$.

Definition 2.2. [17, 9] *A function $f \in \mathcal{A}$ is said to be in class $M(\beta)$ if*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, \text{ for } z \in \Delta \text{ and } \beta > 1\cdot \tag{3}$$

Definition 2.3. [17, 9] *A function $f \in \mathcal{A}$ is said to be in class $N(\beta)$ if*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta, \text{ for } z \in \Delta \text{ and } \beta > 1. \tag{4}$$

Remark 2.1. [9] $f \in N(\beta) \Leftrightarrow zf' \in M(\beta)$.

Definition 2.4. [19] *An analytic function f defined in Δ is said to be in class SQ if*

$$Re(\sqrt{f'(z)}) > \frac{1}{2}, z \in \Delta.$$

Functions in class SQ satisfies the subordination

$$f'(z) \prec \frac{1}{(1-z)^2}\,, \; z \in \Delta. \tag{5}$$

Definition 2.5. [3] *A function $f \in \mathcal{S}$ is said to be strongly starlike of order β $(0 < \beta \leq 1)$ if*

$$\left|arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\beta, \;\; z \in \Delta.$$

The class of all strongly starlike functions is denoted by $\tilde{S}^*(\beta)$, for some β.

Remark 2.2. [3] *A function f is in $\tilde{S}^*(\beta)$ $(0 < \beta \leq 1)$ iff*

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \; z \in \Delta. \tag{6}$$

Definition 2.6. [16] *A function $f \in \mathcal{S}$ is said to be strongly convex of order β if*

$$\left| arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2}\beta, \quad z \in \Delta$$

for some β $(0 < \beta \leq 1)$ denoted by $\tilde{K}^*(\beta)$.

3. Preliminary Lemmas

We now recall certain standard facts about the Hardy space H^p of analytic functions f defined in the open unit disk.

Lemma 3.1. [5] *If $f' \in H^p$ and $p < 1$ then $f \in H^{\frac{p}{1-p}}$.*

Lemma 3.2. [5] *If $f' \in H^p$ and $p \geq 1$ then $f \in H^{\infty}$.*

Lemma 3.3. [13] *If $f \in \mathcal{A}$ satisfies $z^{\gamma} f(z) \in H^p$ where $0 < p < \infty$ and for some $\gamma \in \mathbb{R}$, then $f \in H^p$.*

Lemma 3.4. [12] *If $f(z) \in H^p$ where $0 < p < 1$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n$ then $a_n = o(n^{\frac{1}{p}-1})$.*

Lemma 3.5. [5] *(Hardy-Littlewood's subordination theorem) Let $f(z)$ and $F(z)$ be analytic in Δ and suppose $f \prec F$. Then $M_p(r, f) \leq M_p(r, F)$, $0 < p \leq \infty$.*

Lemma 3.6. [5] *If $f(z)$ is analytic and univalent in Δ, then $f \in H^p$ for all $p < \frac{1}{2}$.*

Lemma 3.7. [5] *Every analytic function $f(z)$ with positive real part in Δ is of class H^p for all $p < 1$.*

4. Main Results

Theorem 4.1. *If $f \in M(\beta)$, then*

1. $f \in H^p$, *for all* $p < \dfrac{1}{4(\beta - 1)}$ *and for all* $\beta > 1$.
2. $f' \in H^p$, *for all* $p < \dfrac{1}{4\beta - 3}$ *and for all* $\beta > 1$.
3. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, *then* $a_n = o(n^{\frac{1}{p}-1})$ *for* $p < \dfrac{1}{4(\beta - 1)}$, *for all* $\beta > 1$.

Proof. Let $f \in M(\beta)$.

By (3),

$$
\begin{aligned}
&Re\left(\frac{\beta - \frac{zf'}{f}}{\beta - 1}\right) > 0 \\
&\implies \left(\frac{\beta - \frac{zf'}{f}}{\beta - 1}\right) = p(z), \text{ where } Re(p(z)) > 0 \\
&\implies \frac{zf'}{f} = \beta - (\beta - 1)p(z) \\
&\implies f' = \beta\left(\frac{f(z)}{z}\right) - (\beta - 1)p(z)\left(\frac{f(z)}{z}\right).
\end{aligned}
$$

Taking modulus on both sides,

$$
|f'(re^{i\theta})| \leq \beta\left|\frac{f(re^{i\theta})}{re^{i\theta}}\right| + (\beta - 1)|p(re^{i\theta})|\left|\frac{f(re^{i\theta})}{re^{i\theta}}\right|.
$$

Using the definition of Integral Means,

$$
\begin{aligned}
M_\lambda^\lambda(r, f') &= \frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|^\lambda d\theta \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\left[\beta\left|\frac{f(re^{i\theta})}{re^{i\theta}}\right| + (\beta - 1)|p(re^{i\theta})|\left|\frac{f(re^{i\theta})}{re^{i\theta}}\right|\right]^\lambda d\theta
\end{aligned}
$$

Since $(a+b)^p \leq 2^p(a^p + b^p),\ a, b \geq 0$ and $0 < p < \infty$. For $z = re^{i\theta}$,

$$
\begin{aligned}
M_\lambda^\lambda(r, f') &\leq \frac{2^\lambda}{2\pi}\int_0^{2\pi}\left(\left|\beta\left(\frac{f(z)}{z}\right)\right|^\lambda + \left|(\beta - 1)p(z)\left(\frac{f(z)}{z}\right)\right|^\lambda\right) d\theta, \\
&= \frac{(2\beta)^\lambda}{2\pi}\int_0^{2\pi}\left|\frac{f(z)}{z}\right|^\lambda d\theta + \frac{(2(\beta - 1))^\lambda}{2\pi}\int_0^{2\pi}\left|p(z).\frac{f(z)}{z}\right|^\lambda d\theta \\
&\leq (2\beta)^\lambda\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(z)}{z}\right|^\lambda d\theta\right) + \\
&\quad (2(\beta - 1))^\lambda\left[\left(\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^{\lambda m} d\theta\right)^{\frac{1}{m}}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(z)}{z}\right|^{\lambda n} d\theta\right)^{\frac{1}{n}}\right]
\end{aligned}
$$

where $\dfrac{1}{m} + \dfrac{1}{n} = 1$ and $m, n > 1$.

Therefore,

$$
\begin{aligned}
M_\lambda^\lambda(r, f') \leq (2\beta)^\lambda M_\lambda^\lambda\left(r, \frac{f(z)}{z}\right) \\
+ (2(\beta-1))^\lambda \left[M_{\lambda m}^\lambda(r, p(z)).M_{\lambda n}^\lambda\left(r, \frac{f(z)}{z}\right)\right].
\end{aligned}
\tag{7}
$$

For $f \in M(\beta)$ where $\beta > 1$, by [9]

$$
\frac{f(z)}{z} \prec (1-z)^{2(\beta-1)}.
$$

For $|z| < 1$,

$$
\begin{aligned}
M_\lambda^\lambda(r, (1-z)^{2(\beta-1)}) &= \frac{1}{2\pi}\int_0^{2\pi} \left|(1-z)^{2(\beta-1)}\right|^\lambda d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |(1-z)|^{2(\beta-1)\lambda} d\theta.
\end{aligned}
$$

Employing Lemma 3.6, $2(\beta-1)\lambda < \frac{1}{2} \implies \lambda < \frac{1}{4(\beta-1)}$.
Hence,

$$
(1-z)^{2(\beta-1)} \in H^\lambda, \text{ for all } \lambda < \frac{1}{4(\beta-1)}.
$$

Therefore, applying Hardy-Littlewood subordination theorem, we find that in RHS of (7)

$$
M_\lambda\left(r, \frac{f(z)}{z}\right) \text{ is bounded for all } \lambda < \frac{1}{4(\beta-1)}.
$$

This indicates that,

$$
\frac{f(z)}{z} \in H^\lambda, \text{ for all } \lambda < \frac{1}{4(\beta-1)}. \tag{8}
$$

Lemma 3.3 leads to the conclusion that

$$
f(z) \in H^\lambda, \text{ for all } \lambda < \frac{1}{4(\beta-1)}.
$$

By Lemma 3.7, $\lim_{r\to 1^-} M_{\lambda m}^\lambda(r, p(z))$ exists if $\lambda m < 1$ and by (8), $\lim_{r\to 1^-} M_{\lambda n}^\lambda\left(r, \frac{f(z)}{z}\right)$ exists if $\lambda n < \frac{1}{4(\beta-1)}$.
It becomes evident in (7) that,

$$
\lim_{r\to 1^-} M_\lambda^\lambda(r, f') \tag{9}
$$

exists under the conditions $\lambda < \frac{1}{4(\beta-1)}$, $\lambda < \frac{1}{m}$ and $4\lambda(\beta-1) < \frac{1}{n}$.
Following a series of calculations, we obtain the limit in (9) to exist for all $\lambda < \frac{1}{4\beta-3}$.
The growth condition on the Taylor coefficients, a_n of $f(z)$ follows from Lemma 3.4.

Theorem 4.2. *Let $f \in N(\beta)$, where $\beta > 1$.*

1 *For $\beta > 1, f' \in H^p$, for all $p < \dfrac{1}{4(\beta-1)}$.*

2. *If $\beta > \dfrac{5}{4}$, then $f \in H^p$, for all $p < \dfrac{1}{4\beta-5}$.*

3. *If $\beta \leq \dfrac{5}{4}$, then $f \in H^\infty$.*

4. *If $f(z) = z + \displaystyle\sum_{n=2}^{\infty} a_n z^n$ for $\beta > \dfrac{5}{4}$, then $a_n = o(n^{\frac{1}{p}-1})$ for $p < \dfrac{1}{4\beta-5}$.*

Proof. For $\beta > 1$, let $f \in N(\beta)$, then by Remark 2.1, $zf' \in M(\beta)$.
By Theorem 4.1, $zf' \in H^p$, for all $p < \frac{1}{4(\beta-1)}$.
Consequently, it can be inferred from Lemma 3.3 and Lemma 3.1 that

$$f \in H^q, \text{ where } q = \frac{p}{1-p}, \; p < \frac{1}{4(\beta-1)} \text{ and } \beta > \frac{5}{4}.$$

Using Lemma 3.2, for $\beta \leq \frac{5}{4}$, we get, $f \in H^\infty$.
As a consequence of Lemma 3.4, we get a bound for the coefficients of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ as $a_n = o(n^{\frac{1}{p}-1})$ for $p < \frac{1}{4\beta-5}$.

Theorem 4.3. *Let $f \in SQ$, then*

1. *$f' \in H^p$, for all $p < \dfrac{1}{2}$.*
2. *$f \in H^p$, for all $p < 1$.*
3. *If $f(z) = z + \displaystyle\sum_{n=2}^{\infty} a_n z^n$, then $a_n = o(n^{\frac{1}{p}-1})$ for $p < 1$.*

Proof. Consider $f \in SQ$.
Using (5) and Lemma 3.6, we get,

$$M_p(r, f') \leq M_p(r, (1-z)^{-2}).$$

$M_p(r, (1-z)^{-2})$ is bounded for all $p < \frac{1}{2}$.
This leads to the result that $f' \in H^p$ for all $p < \frac{1}{2}$.
Since $p < \frac{1}{2} < 1$, Lemma 3.1 gives $f \in H^q$ where $q = \frac{p}{1-p}$ and for all $q < 1$.
Alternate Proof. For a function $f \in SQ$,

$$Re(\sqrt{f'(z)}) > \frac{1}{2}.$$

$$\implies \left(\frac{\sqrt{f'(z)} - 1/2}{1/2}\right) = p(z), \text{ where } Re(p(z)) > 0$$
$$\implies (f'(z))^{\frac{1}{2}} = \frac{p(z)}{2} + \frac{1}{2}.$$

Let $g(z) = (f'(z))^{\frac{1}{2}}$ then,

$$\begin{aligned} |g(z)|^\lambda &\leq \left(\left|\frac{p(z)}{z}\right| + \frac{1}{2}\right)^\lambda \\ &\leq 2^\lambda\left(\left|\frac{p(z)}{z}\right|^\lambda + \left(\frac{1}{2}\right)^\lambda\right) \\ &= |p(z)|^\lambda + 1. \end{aligned}$$

Now,

$$M_\lambda^\lambda(r, g) \leq \frac{1}{2\pi}\int_0^{2\pi} |p(re^{i\theta})|^\lambda d\theta + 1.$$

Using Lemma 3.7, $p(z) \in H^\lambda$, for all $\lambda < 1$.
Therefore, $g \in H^\lambda$, for all $\lambda < 1$. This leads us to the conclusion that $f'(z) \in H^\lambda$ for all $\lambda < \frac{1}{2}$.
Lemma 3.1 implies $f \in H^\lambda$ for all $\lambda < 1$.
Applying Lemma 3.4 in determining the growth of Taylor coefficients of f proves the last part.

Theorem 4.4. *Consider the collection of analytic functions*

$$SQ_{AT} = \left\{f \middle| Re\left(\sqrt{\frac{f(z)}{z}}\right) > \frac{1}{2}\right\} \text{ for } |z| < 1.$$

If $f \in SQ_{AT}$, then

1. *$f \in H^p$ for all $p < \dfrac{1}{2}$.*
2. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $a_n = o(n^{\frac{1}{p}-1})$ for $p < \dfrac{1}{2}$.*

Proof. Let $g(z) = \int_0^z \frac{f(t)}{t} dt$.
Consider $f \in SQ_{AT}\cdot$

$$\implies Re\left(\sqrt{\frac{f(z)}{z}}\right) > \frac{1}{2}$$
$$\implies Re(\sqrt{g'(z)}) > \frac{1}{2}$$
$$\implies g \in SQ\cdot$$

Now that implies $g'(z) = \frac{f(z)}{z} \in H^p$ for all $p < \frac{1}{2}\cdot$
Therefore, by Lemma 3.3, we get $f \in H^p$ for all $p < \frac{1}{2}\cdot$

Theorem 4.5. *Let $f \in \tilde{S}^*(\beta)$ $(0 < \beta \leq 1)$, then*

$$1.\quad f' \in H^p,\ \textit{for all } p < \frac{1}{2+\beta}\ \textit{and}$$
$$2.\quad f \in H^p,\ \textit{for all } p < \frac{1}{1+\beta}\cdot$$

Further if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $a_n = o(n^{\frac{1}{p}-1})$ for $p < \frac{1}{1+\beta}\cdot$

Proof. Let $f \in \tilde{S}^*(\beta)$, then by (6),

$$\frac{zf'(z)}{f(z)} \leq \left(\frac{1+\omega(z)}{1-\omega(z)}\right)^{\beta}$$
$$= (p(z))^{\beta}$$
$$\implies f'(z) \leq (p(z))^{\beta}\left(\frac{f(z)}{z}\right)\cdot$$

For $z = re^{i\theta}, z \in \Delta$,

$$M_{\lambda}^{\lambda}(r, f'(z)) = \frac{1}{2\pi}\int_0^{2\pi} |f'(z)|^{\lambda} d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \left|(p(z))^{\beta}\left(\frac{f(z)}{z}\right)\right|^{\lambda} d\theta$$
$$\leq \left(\frac{1}{2\pi}\int_0^{2\pi} |p(z)|^{\beta\lambda m} d\theta\right)\left(\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f(z)}{z}\right|^{\lambda n} d\theta\right)$$
$$\text{where } \frac{1}{m} + \frac{1}{n} = 1 \text{ and } m, n > 1$$
$$= I_1(z).I_2(z)\cdot$$

The integral $I_1(z)$ is bounded for $\beta\lambda m < 1$ and the integral $I_2(z)$ is bounded for $\lambda n < \frac{1}{2}$ as $r \to 1^-$, using Lemma 3.7 and Lemma 3.6, respectively.
Therefore,

$$\lim_{r\to 1^-} M_\lambda^\lambda(r, f')$$

exists for all $\lambda < \frac{1}{\beta+2}$.
Hence the Hardy spaces of f follows from Lemma 3.1.

Theorem 4.6. *Let $f \in \tilde{K}^*(\beta)$ $(0 < \beta \leq 1)$, then*

$$1. \quad f' \in H^p, \text{ for all } p < \frac{1}{1+\beta} \text{ and}$$

$$2. \quad f \in H^p, \text{ for all } p < \frac{1}{\beta}.$$

Proof. Consider $f \in \tilde{K}^*(\beta)$, then $zf' \in \tilde{S}^*(\beta)$.
Now, $zf' \in H^\lambda$, for all $\lambda < \frac{1}{1+\beta}$.
Therefore, $f' \in H^\lambda$, for all $\lambda < \frac{1}{1+\beta}$.
Lemma 3.1 implies $f \in H^p$, for all $p < \frac{1}{\beta}$.

5. Conclusion

In this paper, we obtained the Hardy spaces H^p of functions in certain geometric subclasses of analytic functions defined on the open unit disk. The growth of the Taylor's coefficients $|a_n|$ of the functions in these subclassses were found to be of the order of power of n involving the index of the Hardy space to which the functions belonged to. It will be interesting to obtain similar results for various other geometric subclasses of analytic functions defined in the open unit disk in the complex plane.

Acknowledgement

The authors thank the referees for their valuable suggestions which resulted in improving the paper.

References

[1] Basgöze T. and Keogh F. R., The Hardy class of a spiral-like function and its derivative, Proc. Amer. Math. Soc., 26(2) (1970), 266-269.

[2] Bhatter S., Jangid K., Meena S. and Purohit S. D., Certain Integral Formulae Involving incomplete I-functions, Sci. and Tech, 26(4) (2021).

[3] Brannan D. A. and Kirwan W. E., On some classes of bounded univalent functions, J. Lond. Math. Soc., 2(1) (1969), 431–443.

[4] Catherine B´en´eteau and Dmitry Khavinson, Selected problems in classical function theory, Invariant subspaces of the shift operator, Contemp. Math., Vol. 638, Amer. Math. Soc., Providence, RI, (2015), 255–265.

[5] Duren P. L., Theory of H^p Spaces, A Series of Monographs and Textbooks in Pure and Applied Mathematics, 38, Academic Press, New York and London, 1970.

[6] Duren P. L., Univalent Functions, Grundlehren Math. Wisschaften 259, Springer, Berlin, 1983.

[7] Eenigenburg P. J. and Keogh F. R., The Hardy class of some univalent functions and their derivatives, Michigan Math J., 17 (1970), 335-346.

[8] Elodie P., Hardy spaces of generalized analytic functions and composition operator, Concrete Operators, 5(1) (2018), 9-23.

[9] Md. Firoz Ali and Vasudevarao A., Coefficient Inequalities and Yamashita's Conjecture For Some Classes of Analytic Functions, J. Aust. Math. Soc., 100 (2016), 1-20.

[10] Gour Murli Manohar, Joshi Sunil and Purohit Sunil Dutt, Certain coefficient inequalities for the classes of q-starlike and q-convex functions, South East Asian J. of Mathematics and Mathematical Sciences, 18(3) (2022), 43-54.

[11] Hardy G. H., The Mean Value of the Modulus of an Analytic Function, Proc. Lond. Math. Soc., 14 (1915), 269-277.

[12] Hardy G. H. and Littlewood J. E., Some Properties of Fractional Integrals II, Math. Z., 34 (1932), 403-439.

[13] Kim Y. C., Lee K. S. and Srivastava H. M., Certain Classes of Integral Operators Associated with the Hardy Space of Analytic Functions, Complex Var. Elli. Equ., 20 (1992), 1-12.

[14] Kim Y. C. and Srivastava H. M., The Hardy Space for a certain subclass of Bazilevic functions, Appl. Math. and Comp., 183 (2006), 1201-1207.

[15] Miller S. S., Mocanu P. T. and Reade M. O., The Hardy Classes for Functions in the class $MV[\alpha, k]$, J. Math. Anal. and Appl., 51 (1975), 33-42.

[16] Mocanu P. T., On strongly-starlike and strongly-convex functions, Studia Univ. Babes-Bolyai Math., 31(4) (1986), 16–21.

[17] Owa S. and Nishiwaki J., Coefficient Estimates For Certain Classes of Analytic Functions, J. of Ineq. in Pure and Appl. Math., 3(5) (2002), 1-6.

[18] Pinchuk B., The Hardy class of functions of bounded boundary rotation, Proc. Amer. Math. Soc., 3(2) (1973), 355-360.

[19] Răducanu D., On Coefficient Estimates for a Certain Class of Analytic Functions, Mathematics, 11(1) (2022), 1-8.

[20] Selvakumaran K. A., Rajaguru P., Purohit S. D. and Suthar D. L., Certain geometric properties of the canonical Weierstrass product of an entire function associated with conic domains, J. Funct. Spaces, 2022 (2022), Art. ID 2876673.

[21] Thomas D. K., Tuneski N. and Vasudevarao A., Univalent Functions: A Primer, De Gruyter Studies in Mathematics, 69, Walter de Gruyter GmbH & Co KG, 2018.