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CERTAIN NEW IDENTITIES OF BASIC BILATERAL HYPERGEOMETRIC SERIES

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Abstract: In the present work, we have applied Cauchy's method to establish some basic bilateral hypergeometric series identities, using the known identities of terminating unilateral series. We also have discussed some important special cases of our results.

Keywords and Phrases: Basic Hypergeometric Series, Basic Bilateral Hypergeometric Series, q-Series, Summations, Transformations, Cauchy's method.

2020 Mathematics Subject Classification: 33D15.

1. Introduction

The basic hypergeometric functions have been an important object of study in the theory of special functions because of their wide range of applications spanning across various fields such as mathematical physics, number theory, orthogonal polynomials and combinatorics. Among these, basic bilateral hypergeometric functions, which generalize the classical hypergeometric functions by allowing both series to extend indefinitely in both directions, are particularly significant. The transformations and summation formulae of these functions are important from the point of view that they reveal deep structural insights and connections between different mathematical entities [9].

The theory of transformations and summations of basic bilateral hypergeometric functions has been enriched by the works of Bailey [3], Slater [16], Shukla [15], Andrews [2], Verma & Jain [17] and many others. One of the powerful technique for deriving such transformations is Cauchy's method. Cauchy [6] introduced the method to extend an unilateral series to a bilateral series. The Cauchy's method has been used to develop the transformations of basic bilateral hypergeometric series by Bailey [5], Slater [16], Jouhet and Schlosser [10, 11], Chen and Fu [7], Jouhet [12], Zhang [19], Zhang & Zhang [18], Zhang and Hu [20] and Ali and Rizvi [1].

In the present paper, we have exploited the Cauchy's method to obtain the identities of basic bilateral hypergeometric series from certain known terminating unilateral series identities.

2. Notations and Definitions

Here we follow the notations and definitions of [9]. We define a basic hypergeometric series by

$$ {}_r\phi_s\left(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n}{(q, b_1, b_2, ..., b_s; q)_n} [(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r} z^n, \quad (2.1) $$

where

$$ (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}. $$

If $0 < |q| < 1$ then the series is absolutely convergent for all z where $r \leq s$ and for $|z| < 1$ if $r = s + 1$. If $|q| > 1$ then the series is absolutely convergent for $|z| < \dfrac{|b_1 b_2 ... b_s q|}{|a_1 a_2 ... a_r|}$.

A basic bilateral hypergeometric series is defined as

$$ {}_r\psi_s\left(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q, z\right) = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n}{(b_1, b_2, ..., b_s; q)_n} [(-1)^n q^{\frac{n(n-1)}{2}}]^{s-r} z^n. \quad (2.2) $$

In the next section of the paper we use the following identities.

$$ {}_{12}\phi_{11}\Big(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}, -q\sqrt{a}, k^2q^{n-1}/a, q^{-n}; \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k\sqrt{q/a}, -k\sqrt{q/a}, k/\sqrt{a}, -k/\sqrt{a}, aq^{2-n}/k, kq^{n+1}; q, q\Big) $$
$$ = \frac{(kq; q)_n (k^2/a^2q; q)_n}{(k/aq; q)_n (k^2/a; q)_{n-1}} \frac{1}{(1 - k^2q^{2n-1}/a)} \times $$

$$ {}_7\phi_6\left(a, q\sqrt{a}, -q\sqrt{a}, b, c, d, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, a^2q^{2-n}/k^2; q, q\right), \quad (2.3) $$

where $k = a^2q/bcd$.

[Bailey W. N. [4], Equ. 2]

$$ {}_{12}\phi_{11}\Big(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, \sqrt{aq}, -\sqrt{aq}, \sqrt{a}, -\sqrt{a}, k^2q^{n+1}/a, q^{-n}; \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k\sqrt{q/a}, -k\sqrt{q/a}, kq/\sqrt{a}, -kq/\sqrt{a}, aq^{-n}/k, kq^{n+1}; q, q\Big) = \frac{(kq;q)_n(k^2q/a^2;q)_n}{(kq/a;q)_n(k^2q/a;q)_n}\times {}_5\phi_4\left(a, b, c, d, q^{-n}; aq/b, aq/c, aq/d, a^2q^{-n}/k^2; q, q\right), \quad (2.4) $$

where $k = a^2q/bcd$.

[Bailey W. N. [4], Equ. 1]

$$ {}_{10}\phi_9\Big(a, q\sqrt{a}, -q\sqrt{a}, a\sqrt{q/k}, -a\sqrt{q/k}, q\sqrt{kq}, -q\sqrt{kq}, a/kq^2, kq^n, q^{-n}; \sqrt{a}, -\sqrt{a}, \sqrt{q/k}, -\sqrt{q/k}, a/\sqrt{kq}, -a/\sqrt{kq}, kq^3, aq^{1-n}/k, aq^{n+1}; q, q\Big) = \frac{(k;q^2)_n}{(kq^2;q^2)_n}\frac{(aq, k^2q/a^2;q)_n}{(k, k/a;q)_n}\times {}_4\phi_3\Big(a\sqrt{q/k}, -a\sqrt{q/k}, a^2/k^2q^3, q^{-n}; a/\sqrt{kq}, -a/\sqrt{kq}, a^2q^{-n}/k^2; q, q\Big). \quad (2.5) $$

[Laughlin J. MC and Zimmer P. [11], Equ. 4.5]

$$ {}_{12}\phi_{11}\Big(k, q\sqrt{k}, -q\sqrt{k}, y, z, aq/k, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a}, -k\sqrt{q/a}, kaq^{n+1}/yz, q^{-n}; \sqrt{k}, -\sqrt{k}, kq/y, kq/z, k^2/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, yzq^{-n}/a, kq^{n+1}; q, q\Big) = \frac{(kq, kq/yz, aq/y, aq/z;q)_n}{(kq/y, kq/z, aq/yz, aq;q)_n}\times {}_5\phi_4\left(y, z, a^2q/k^2, kaq^{n+1}/yz, q^{-n}; aq/y, aq/z, aq^{n+1}, yzq^{-n}/a; q, q\right). \quad (2.6) $$

[Laughlin J. MC and Zimmer P. [12], Cor.10]

$$
\begin{aligned}
&{}_{r+1}W_r\left(b;q^{-n},a_5,...,a_{r+1};q,z\right)\\
&\quad= q^{-\frac{n(n-1)}{2}}\left(\frac{-z}{q}\right)^n\frac{(\pm q\sqrt{b},b,a_5,...,a_{r+1};q)_n}{(\pm\sqrt{b},bq^{n+1},bq/a_5,...,bq/a_{r+1};q)_n}\times\\
&\qquad {}_{r+1}W_r\left(\frac{q^{-2n}}{b};q^{-n},\frac{q^{-n}}{a_5},...,\frac{q^{-n}}{a_{r+1}};q,\frac{q^{2n+r-3}b^{r-3}}{(a_5...a_{r+1})^2z}\right). \quad (2.7)
\end{aligned}
$$

[Cohl H. S., Costas Santos R. S., Ge L. [8]]

$$
\begin{aligned}
&{}_{r+2}\phi_{r+1}\left(a,b,b_1q^{m_1},...,b_rq^{m_r};bq,b_1,...,b_r;q,\frac{q^{1-(m_1+...+m_r)}}{a}\right)\\
&\quad=\frac{(q,bq/a;q)_\infty}{(bq,q/a;q)_\infty}\frac{(b_1/b;q)_{m_1}...(b_r/b;q)_{m_r}}{(b_1;q)_{m_1}...(b_r;q)_{m_r}}b^{m_1+...+m_r}. \quad (2.8)
\end{aligned}
$$

[Gasper G., Rahman M. [9], Equ. II.26]

3. Identities of Basic Bilateral Hypergeometric Series

In this section, we have established the following identities of basic bilateral hypergeometric series.

$$
\begin{aligned}
&{}_{12}\psi_{12}\left(kq^{-n},q\sqrt{k},-q\sqrt{k},kb/a,kc/a,kd/a,\sqrt{aq},-\sqrt{aq},q\sqrt{a},\right.\\
&\quad -q\sqrt{a},q^{-n},k^2q^{n-1}/a;q^{n+1},\sqrt{k},-\sqrt{k},aq/b,aq/c,aq/d,\\
&\quad \left.k\sqrt{q/a},-k\sqrt{q/a},k/\sqrt{a},-k/\sqrt{a},kq^{n+1},aq^{2-n}/k;q,q\right)\\
&=\frac{(k^2/aq;q^2)_n}{(k^2q/a;q^2)_n}\frac{(kq,q/b,q/c,q/d,q/k,k^2/a^2q;q)_n}{(k^2/aq,aq/kb,aq/kc,aq/kd,k/aq,k^2/aq;q)_n}\times\\
&\quad\left(\frac{a}{k}\right)^n{}_7\psi_7\left(aq^{-n},q\sqrt{a},-q\sqrt{a},b,c,d,q^{-n};q^{n+1},\sqrt{a},-\sqrt{a},\right.\\
&\quad \left.aq/b,aq/c,aq/d,a^2q^{2-n}/k^2;q,q\right), \quad (3.1)
\end{aligned}
$$

where $k=a^2q^{1+n}/bcd$.

$$
\begin{aligned}
&{}_{12}\psi_{12}\left(kq^{-n},q\sqrt{k},-q\sqrt{k},kb/a,kc/a,kd/a,\sqrt{aq},-\sqrt{aq},\sqrt{a},\right.\\
&\quad -\sqrt{a},k^2q^{n+1}/a,q^{-n};q^{n+1},\sqrt{k},-\sqrt{k},aq/b,aq/c,aq/d,\\
&\quad \left.k\sqrt{q/a},-k\sqrt{q/a},kq/\sqrt{a},-kq/\sqrt{a},aq^{-n}/k,kq^{n+1};q,q\right)
\end{aligned}
$$

$$= \frac{(k^2q/a^2, kq, q/b, q/c, q/d, q/k; q)_n}{(kq/a, k^2q/a, aq/kb, aq/kc, aq/kd, q/a; q)_n} \left(\frac{a}{k}\right)^n \times$$
$${}_5\psi_5 \left(aq^{-n}, b, c, d, q^{-n}; q^{n+1}, aq/b, aq/c, aq/d, a^2q^{-n}/k^2; q, q\right), \quad (3.2)$$

where $k = a^2q^{1+n}/bcd$.

$${}_{10}\psi_{10} \Big(aq^{-n}, q\sqrt{a}, -q\sqrt{a}, a\sqrt{q/k}, -a\sqrt{q/k}, q\sqrt{kq}, -q\sqrt{kq},$$
$$aq^n/kq^2, kq^n, q^{-n}; q^{n+1}, \sqrt{a}, -\sqrt{a}, \sqrt{q/k}, -\sqrt{q/k},$$
$$a/\sqrt{kq}, -a/\sqrt{kq}, kq^{3-n}, q^{1-n}/k, aq^{n+1}; q, q\Big)$$
$$= \frac{(1/kq^2, k; q^2)_n}{(1/k, kq^2; q^2)_n} \frac{(aq, a^2/k^2q^3, k^2q/a^2, q/a; q)_n}{(k, a/kq^2, 1/kq^2, k/a; q)_n} \times$$
$${}_4\psi_4 \Big(a\sqrt{q/k}, -a\sqrt{q/k}, a^2q^n/k^2q^3, q^{-n}; q^{n+1}, a/\sqrt{kq},$$
$$-a/\sqrt{kq}, a^2q^{-n}/k^2; q, q\Big). \quad (3.3)$$

$${}_{12}\psi_{12} \Big(kq^{-n}, q\sqrt{k}, -q\sqrt{k}, y, z, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a},$$
$$-k\sqrt{q/a}, aq/k, kaq^{n+1}/yz, q^{-n}; q^{n+1}, \sqrt{k}, -\sqrt{k}, kq/y,$$
$$kq/z, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, k^2q^{-n}/a, kq^{n+1},$$
$$yzq^{-n}/a; q, q\Big)$$
$$= \frac{(kq, a^2q/k^2, q/k, aq/y, aq/z; q)_n}{(aq, aq/k, aq/k^2, kq/y, kq/z; q)_n} \frac{(kq/yz; q)_{2n}}{(aq/yz; q)_{2n}} \times$$
$$\left(\frac{a}{k}\right)^n {}_5\psi_5 \left(y, z, a^2q^{n+1}/k^2, kaq^{n+1}/yz, q^{-n}; q^{n+1}, aq/y,\right.$$
$$\left. aq/z, aq^{n+1}, yzq^{-n}/a; q, q\right). \quad (3.4)$$

$${}_{r+1}\psi_{r+1} \Big(bq^{-n}, q\sqrt{b}, -q\sqrt{b}, q^{-n}, a_5, ..., a_{r+1}; q^{n+1}, \sqrt{b}, -\sqrt{b},$$
$$bq^{n+1}, bq/a_5, ..., bq/a_{r+1}; q, z\Big)$$
$$= \left(\frac{q^{r-5}b^{r-3}}{(a_5...a_{r+1})^2}\right)^n \times$$
$${}_{r+1}\psi_{r+1} \Big(q^{-n}/b, q/\sqrt{b}, -q/\sqrt{b}, q^{-n}, a_5/b, ..., a_{r+1}/b; q^{n+1},$$
$$1/\sqrt{b}, -1/\sqrt{b}, q^{1+n}/b, q/a_5, ..., q/a_{r+1}; q, \frac{q^{4n+r-3}b^{r-3}}{(a_5...a_{r+1})^2 z}\Big). \quad (3.5)$$

We also proved the following summation formula.

$$
{}_{r+2}\psi_{r+2}\left(q^{-n}, b, b_1q^{m_1}, ..., b_rq^{m_r}; q^{n+1}, bq, b_1, ..., b_r; q, q^{1+2n-(m_1+...+m_r)}\right) = \frac{(q;q)_n(q;q)_n}{(1/b;q)_n(bq;q)_n}\frac{(b_1/b;q)_{m_1}...(b_r/b;q)_{m_r}}{(b_1;q)_{m_1}...(b_r;q)_{m_r}}b^{(m_1+...+m_r-n)}. \quad (3.6)
$$

Proof of (3.1). From (2.3), we have

$$
{}_{12}\phi_{11}\Big(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}, -q\sqrt{a}, k^2q^{n-1}/a, q^{-n}; \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k\sqrt{q/a}, -k\sqrt{q/a}, k/\sqrt{a}, -k/\sqrt{a}, aq^{2-n}/k, kq^{n+1}; q, q\Big)
$$

$$
= \frac{(kq;q)_n(k^2/a^2q;q)_n}{(k/aq;q)_n(k^2/a;q)_{2n}}(k^2q^n/aq;q)_n\times {}_7\phi_6\left(a, q\sqrt{a}, -q\sqrt{a}, b, c, d, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, a^2q^{2-n}/k^2; q, q\right)
$$

where $k = a^2q/bcd$.
Taking $n \to 2n$ and then replacing $p = s + n$, after some simplification we get

$$
\frac{(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}; q)_n}{(q, \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k\sqrt{q/a}, -k\sqrt{q/a}, k/\sqrt{a}; q)_n}\times \frac{(-q\sqrt{a}, k^2q^{2n-1}/a, q^{-2n}; q)_n}{(-k/\sqrt{a}, aq^{2-2n}/k, kq^{2n+1}; q)_n}q^n\times
$$

$$
\sum_{s=-n}^{n}\frac{(kq^n, q^{n+1}\sqrt{k}, -q^{n+1}\sqrt{k}, kbq^n/a, kcq^n/a, kdq^n/a; q)_s}{(q^{n+1}, q^n\sqrt{k}, -q^n\sqrt{k}, aq^{n+1}/b, aq^{n+1}/c, aq^{n+1}/d; q)_s}\times \frac{(q^n\sqrt{aq}, -q^n\sqrt{aq}, q^{n+1}\sqrt{a}, -q^{n+1}\sqrt{a}, q^{-n}, k^2q^{3n-1}/a; q)_s}{(kq^n\sqrt{q/a}, -kq^n\sqrt{q/a}, kq^n/\sqrt{a}, -kq^n/\sqrt{a}, kq^{3n+1}, aq^{2-n}/k; q)_s}q^s
$$

$$
= \frac{(kq;q)_{2n}(k^2/a^2q;q)_{2n}}{(k/aq;q)_{2n}(k^2/a;q)_{4n}}(k^2q^{2n}/aq;q)_{2n}\times \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d, q^{-2n}; q)_n}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, a^2q^{2-2n}/k^2; q)_n}q^n\times
$$

$$\sum_{s=-n}^{n} \frac{(aq^n, q^{n+1}\sqrt{a}, -q^{n+1}\sqrt{a}, bq^n, cq^n, dq^n, q^{-n}; q)_s}{(q^{n+1}, q^n\sqrt{a}, -q^n\sqrt{a}, aq^{n+1}/b, aq^{n+1}/c, aq^{n+1}/d, a^2q^{2-n}/k^2; q)_s} q^s.$$

If we put $k = kq^{-2n}, a = aq^{-2n}, b = bq^{-n}, c = cq^{-n}, d = dq^{-n}$, we obtain (3.1).

Proof of (3.2). Taking $n \to 2n$ in (2.4) and then replacing $p = s + n$, after some simplification we get

$$\frac{(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, \sqrt{aq}, -\sqrt{aq}; q)_n}{(q, \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k\sqrt{q/a}, -k\sqrt{q/a}; q)_n} \times$$
$$\frac{(\sqrt{a}, -\sqrt{a}, k^2q^{2n+1}/a, q^{-2n}; q)_n}{(kq/\sqrt{a}, -kq/\sqrt{a}, aq^{-2n}/k, kq^{2n+1}; q)_n} q^n \times$$
$$\sum_{s=-n}^{n} \frac{(kq^n, q^{n+1}\sqrt{k}, -q^{n+1}\sqrt{k}, kbq^n/a, kcq^n/a; q)_s}{(q^{n+1}, q^n\sqrt{k}, -q^n\sqrt{k}, aq^{n+1}/b, aq^{n+1}/c, aq^{n+1}/d; q)_s} \times$$
$$\frac{(kdq^n/a, q^n\sqrt{aq}, -q^n\sqrt{aq}, q^n\sqrt{a}, -q^n\sqrt{a}; q)_s}{(kq^n\sqrt{q/a}, -kq^n\sqrt{q/a}, kq^{n+1}/\sqrt{a}, -kq^{n+1}/\sqrt{a}; q)_s} \times$$
$$\frac{(k^2q^{3n+1}/a, q^{-n}; q)_s}{(aq^{-n}/k, kq^{3n+1}; q)_s} q^s$$
$$= \frac{(kq; q)_{2n}(k^2q/a^2; q)_{2n}}{(kq/a; q)_{2n}(k^2q/a; q)_{2n}} \times$$
$$\frac{(a, b, c, d, q^{-2n}; q)_n}{(q, aq/b, aq/c, aq/d, a^2q^{-2n}/k^2; q)_n} q^n \times$$
$$\sum_{s=-n}^{n} \frac{(aq^n, bq^n, cq^n, dq^n, q^{-n}; q)_s}{(q^{n+1}, aq^{n+1}/b, aq^{n+1}/c, aq^{n+1}/d, a^2q^{-n}/k^2; q)_s} q^s.$$

For $k = kq^{-2n}, a = aq^{-2n}, b = bq^{-n}, c = cq^{-n}, d = dq^{-n}$ in above equation, we get (3.2).

Proof of (3.3). Taking $n \to 2n$ in (2.5) and then replacing $p = s + n$, after some simplification we get

$$\frac{(a, q\sqrt{a}, -q\sqrt{a}, a\sqrt{q/k}, -a\sqrt{q/k}, q\sqrt{kq}, -q\sqrt{kq}; q)_n}{(q, \sqrt{a}, -\sqrt{a}, \sqrt{q/k}, -\sqrt{q/k}, a/\sqrt{kq}, -a/\sqrt{kq}; q)_n} \times$$
$$\frac{(a/kq^2, kq^{2n}, q^{-2n}; q)_n}{(kq^3, aq^{1-2n}/k, aq^{2n+1}; q)_n} q^n \times$$

$$\sum_{s=-n}^{n} \frac{(aq^n, q^{n+1}\sqrt{a}, -q^{n+1}\sqrt{a}, aq^n\sqrt{q/k}, -aq^n\sqrt{q/k}; q)_s}{(q^{n+1}, q^n\sqrt{a}, -q^n\sqrt{a}, q^n\sqrt{q/k}, -q^n\sqrt{q/k}; q)_s} \times$$
$$\frac{(q^{n+1}\sqrt{kq}, -q^{n+1}\sqrt{kq}, aq^n/kq^2, kq^{3n}, q^{-n}; q)_s}{(aq^n/\sqrt{kq}, -aq^n/\sqrt{kq}, kq^{3+n}, aq^{1-n}/k, aq^{3n+1}; q)_s} q^s$$
$$= \frac{(k; q^2)_{2n}}{(kq^2; q^2)_{2n}} \frac{(aq, k^2q/a^2; q)_{2n}}{(k, k/a; q)_{2n}} \times$$
$$\frac{(a\sqrt{q/k}, -a\sqrt{q/k}, a^2/k^2q^3, q^{-2n}; q)_n}{(q, a/\sqrt{kq}, -a/\sqrt{kq}, a^2q^{-2n}/k^2; q)_n} q^n \times$$
$$\sum_{s=-n}^{n} \frac{(aq^n\sqrt{q/k}, -aq^n\sqrt{q/k}, a^2q^n/k^2q^3, q^{-n}; q)_s}{(q^{n+1}, aq^n/\sqrt{kq}, -aq^n/\sqrt{kq}, a^2q^{-n}/k^2; q)_s} q^s.$$

The above equation gives (3.3) on taking $k = kq^{-2n}, a = aq^{-2n}$.

Proof of (3.4). Taking $n \to 2n$ in (2.6) and then replacing $p = s + n$, after some simplification we get

$$\frac{(k, q\sqrt{k}, -q\sqrt{k}, y, z, aq/k, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a}, ; q)_n}{(q, \sqrt{k}, -\sqrt{k}, kq/y, kq/z, k^2/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, ; q)_n)} \times$$
$$\frac{(-k\sqrt{q/a}, kaq^{2n+1}/yz, q^{-2n}; q)_n}{(kq^{2n+1}, yzq^{-2n}/a, -\sqrt{aq}; q)_n} q^n \times$$
$$\sum_{s=-n}^{n} \frac{(kq^n, q^{n+1}\sqrt{k}, -q^{n+1}\sqrt{k}, yq^n, zq^n, aq^{n+1}/k; q)_s}{(q^{n+1}, q^n\sqrt{k}, -q^n\sqrt{k}, kq^{n+1}/y, kq^{n+1}/z, k^2q^n/a; q)_s} \times$$
$$\frac{(kq^n/\sqrt{a}, -kq^n/\sqrt{a}, kq^n\sqrt{q/a}, -kq^n\sqrt{q/a}, kaq^{3n+1}/yz; q)_s}{(q^{n+1}\sqrt{a}, -q^{n+1}\sqrt{a}, q^n\sqrt{aq}, -q^n\sqrt{aq}, kq^{3n+1}; q)_s} \times$$
$$\frac{(q^{-n}; q)_s}{(yzq^{-n}/a; q)_s} q^s$$
$$= \frac{(kq, kq/yz, aq/y, aq/z; q)_{2n}}{(kq/y, kq/z, aq/yz, aq; q)_{2n}} \times$$
$$\frac{(y, z, a^2q/k^2, kaq^{2n+1}/yz, q^{-2n}; q)_n}{(q, aq/y, aq/z, aq^{2n+1}, yzq^{-2n}/a; q)_n} q^n \times$$
$$\sum_{s=-n}^{n} \frac{(yq^n, zq^n, a^2q^{n+1}/k^2, kaq^{3n+1}/yz, q^{-n}; q)_s}{(q^{n+1}, aq^{n+1}/y, aq^{n+1}/z, aq^{3n+1}, yzq^{-n}/a; q)_s} q^s.$$

Now on taking $k = kq^{-2n}, a = aq^{-2n}, y = yq^{-n}, z = zq^{-n}$, we obtain (3.4).

Proof of (3.5). Taking $n \to 2n$ in (2.7) and then replacing $p = s + n$, after some simplification we get

$$\frac{(b, q\sqrt{b}, -q\sqrt{b}, q^{-2n}, a_5, ..., a_{r+1}; q)_n}{(q, \sqrt{b}, -\sqrt{b}, bq^{2n+1}, bq/a_5, ..., bq/a_{r+1}; q)_n} z^n \times$$

$$\sum_{s=-n}^{n} \frac{(bq^n, q^{n+1}\sqrt{b}, -q^{n+1}\sqrt{b}, q^{-n}, a_5q^n, ..., a_{r+1}q^n; q)_{s+n}}{(q^{n+1}, q^n\sqrt{b}, -q^n\sqrt{b}, bq^{3n+1}, bq^{n+1}/a_5, ..., bq^{n+1}/a_{r+1}; q)_s} z^s$$

$$= q^{-n(2n-1)} \left(\frac{-z}{q}\right)^{2n} \frac{(q\sqrt{b}, -q\sqrt{b}, b, a_5, ..., a_{r+1}; q)_{2n}}{(\sqrt{b}, -\sqrt{b}, bq^{2n+1}, bq/a_5, ..., bq/a_{r+1}; q)_{2n}} \times$$

$$\frac{(q^{-4n}/b, q^{1-2n}/\sqrt{b}, -q^{1-2n}/\sqrt{b}, q^{-2n}, q^{-2n}a_5/b, ..., q^{-2n}a_{r+1}/b; q)_n}{(q, q^{-2n}/\sqrt{b}, -q^{-2n}/\sqrt{b}, q^{1-2n}/b, q^{1-2n}/a_5, ..., q^{1-2n}/a_{r+1}; q)_n}$$

$$\left(\frac{q^{4n+r-3}b^{r-3}}{(a_5...a_{r+1})^2 z}\right)^n \times$$

$$\sum_{s=-n}^{n} \frac{(q^{-3n}/b, q^{1-n}/\sqrt{b}, -q^{1-n}/\sqrt{b}, q^{-n}, q^{-n}a_5/b, ..., q^{-n}a_{r+1}/b; q)_s}{(q^{n+1}, q^{-n}/\sqrt{b}, -q^{-n}/\sqrt{b}, q^{1-n}/b, q^{1-n}/a_5, ..., q^{1-n}/a_{r+1}; q)_s}$$

$$\left(\frac{q^{4n+r-3}b^{r-3}}{(a_5...a_{r+1})^2 z}\right)^s.$$

Putting $a_5 = a_5q^{-n}, ..., a_{r+1} = a_{r+1}q^{-n}$ & $b = bq^{-2n}$ we get (3.5).

Proof of (3.6). Taking $n \to 2n$ in (2.8), if we put $a = q^{-n}$ and then replacing $p = s + n$, after some simplification we get

$$\frac{(q^{-2n}, b, b_1q^{m_1}, ..., b_rq^{m_r}; q)_n}{(q, bq, b_1, ..., b_r; q)_n} \left(q^{1+2n-(m_1+...+m_r)}\right)^n \times$$

$$\sum_{s=-n}^{n} \frac{(q^{-n}, bq^n, b_1q^{m_1+n}, ..., b_rq^{m_r+n}; q)_s}{(q^{n+1}, bq^{n+1}, b_1q^n, ..., b_rq^n; q)_s} \left(q^{1+2n-(m_1+...+m_r)}\right)^s$$

$$= \frac{(q;q)_{2n}}{(bq;q)_{2n}} \frac{(b_1/b;q)_{m_1}...(b_r/b;q)_{m_r}}{(b_1;q)_{m_1}...(b_r;q)_{m_r}} b^{m_1+...+m_r}.$$

Now if we put $b = bq^{-n}, b_1 = b_1q^{-n}, ..., b_r = b_rq^{-n}$, we get (3.6).

4. Special Cases

For $d = aq^{n+1}$ in (3.1), we get

$$
{}_{10}\psi_{10}\Big(kq^{-n}, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}, -q\sqrt{a},
k^2q^{n-1}/a; q^{n+1}, \sqrt{k}, -\sqrt{k}, aq/b, aq/c, k\sqrt{q/a}, -k\sqrt{q/a},
k/\sqrt{a}, -k/\sqrt{a}, aq^{2-n}/k; q, q\Big)
= \frac{(k^2/aq;q^2)_n}{(k^2q/a;q^2)_n}\frac{(aq, q/b, q/c, q/k, k^2/a^2q;q)_n}{(k^2/aq, aq/kb, aq/kc, q/a, k/aq;q)_n}\times
{}_6\psi_6\left(aq^{-n}, q\sqrt{a}, -q\sqrt{a}, b, c, aq^{n+1}; q^{n+1}, \sqrt{a}, -\sqrt{a}, aq/b,
aq/c, a^2q^{2-n}/k^2; q, q\right), \quad (4.1)
$$

where $k = a/bcq^n$.
If we take $c = a$ in (3.1), we get

$$
{}_{12}\phi_{11}\Big(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kd/a, \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}, -q\sqrt{a},
kq^{-n}, q^{-n}, k^2q^{n-1}/a; q^{n+1}, \sqrt{k}, -\sqrt{k}, aq/b, aq/d,
k\sqrt{q/a}, -k\sqrt{q/a}, k/\sqrt{a}, -k/\sqrt{a}, kq^{n+1}, aq^{2-n}/k; q, q\Big)
= \frac{(k^2/aq;q^2)_n}{(k^2q/a;q^2)_n}\frac{(kq, q/b, q/d, k^2/a^2q;q)_n}{(k^2/aq, aq/kb, aq/kd, k/aq;q)_n}\left(\frac{a}{k}\right)^n\times
{}_7\phi_6\left(a, q\sqrt{a}, -q\sqrt{a}, b, d, aq^{-n}, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/d,
q^{n+1}, a^2q^{2-n}/k^2; q, q\right), \quad (4.2)
$$

where $k = aq/bd$.
Taking $d = aq^{n+1}$ in (3.2), we get

$$
{}_{10}\psi_{10}\Big(kq^{-n}, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, \sqrt{aq}, -\sqrt{aq}, \sqrt{a}, -\sqrt{a},
k^2q^{n+1}/a; q^{n+1}, \sqrt{k}, -\sqrt{k}, aq/b, aq/c, k\sqrt{q/a}, -k\sqrt{q/a},
kq/\sqrt{a}, -kq/\sqrt{a}, aq^{-n}/k; q, q\Big)
= \frac{(k^2q/a^2, aq, q/b, q/c, q/k;q)_n}{(kq/a, k^2q/a, aq/kb, aq/kc, q/a;q)_n}\times
{}_4\psi_4\left(aq^{-n}, b, c, aq^{n+1}; q^{n+1}, aq/b, aq/c, a^2q^{-n}/k^2; q, q\right), \quad (4.3)
$$

where $k = a/bcq^n$.
Further, for $c = a$ in (4.3), we get the following identity.

$$\begin{aligned}
&{}_{10}\phi_9\Big(k, q\sqrt{k}, -q\sqrt{k}, kb/a, \sqrt{aq}, -\sqrt{aq}, \sqrt{a}, -\sqrt{a}, kq^{-n}, \\
&\qquad k^2q^{n+1}/a; \sqrt{k}, -\sqrt{k}, aq/b, k\sqrt{q/a}, -k\sqrt{q/a}, kq/\sqrt{a}, \\
&\qquad\qquad -kq/\sqrt{a}, q^{n+1}, aq^{-n}/k; q, q\Big) \\
&\qquad = \frac{(k^2q/a^2, aq, q/b; q)_n}{(kq/a, k^2q/a, aq/kb; q)_n}\times \\
&\qquad\qquad {}_4\phi_3\left(aq^{-n}, b, a, aq^{n+1}; q^{n+1}, aq/b, a^2q^{-n}/k^2; q, q\right), \qquad (4.4)
\end{aligned}$$

where $k = 1/bq^n$.

For $k = a^2/q^3$ in (3.3), we get

$$\begin{aligned}
&{}_{10}\phi_9\left(a, q\sqrt{a}, -q\sqrt{a}, q^2, -q^2, -a, aq^{-n}, q^{n-1}/a, a^2q^{n-3}, q^{-n};\right. \\
&\qquad \left.\sqrt{a}, -\sqrt{a}, q^2/a, -q^2/a, -q, q^{n+1}, a^2q^{-n}, q^{4-n}/a^2, aq^{n+1}; q, q\right) \\
&\qquad = \frac{(q/a^2, a^2/q^3; q^2)_n}{(a^2/q, q^3/a^2; q^2)_n}\frac{(aq, q^3/a^2, a^2/q^5; q)_n}{(a^2/q^3, q/a^2, a/q^3; q)_n}\times \\
&\qquad\qquad {}_4\phi_3\left(q^2, -q^2, q^{n+3}/a^2, q^{-n}; -q, q^{n+1}, q^{6-n}/a^2; q, q\right). \qquad (4.5)
\end{aligned}$$

Taking $y = a$ and $z = k$ in (3.4), we get

$$\begin{aligned}
&{}_{10}\phi_9\Big(k, q\sqrt{k}, -q\sqrt{k}, aq/k, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a}, -k\sqrt{q/a}, \\
&\qquad a, q^{-n}; \sqrt{k}, -\sqrt{k}, kq/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, \\
&\qquad\qquad k^2q^{-n}/a, kq^{n+1}; q, q\Big) \\
&\qquad = \frac{(kq, a^2q/k^2, q/k; q)_n}{(aq, aq/k^2, kq/a; q)_n}\frac{(q/a; q)_{2n}}{(q/k; q)_{2n}}\left(\frac{a}{k}\right)^n\times \\
&\qquad\qquad {}_4\phi_3\left(a, k, a^2q^{n+1}/k^2, q^{-n}; aq/k, aq^{n+1}, kq^{-n}; q, q\right), \qquad (4.6)
\end{aligned}$$

which for $k = \sqrt{a}$ gives

$$\begin{aligned}
&{}_3\phi_2\left(a, \sqrt{a}, q^{-n}; q\sqrt{a}, q^{-n}\sqrt{a}; q, q\right) \\
&\qquad = \frac{(q/\sqrt{a}; q)_{2n}}{(q/a; q)_{2n}}\frac{(q, q/\sqrt{a}; q)_n}{(q\sqrt{a}, q/\sqrt{a}; q)_n}\left(\frac{1}{\sqrt{a}}\right)^n. \qquad (4.7)
\end{aligned}$$

For $b_1 = q$ in (3.6), we get

$$ {}_{r+2}\phi_{r+1}\left(q^{-n}, b, q^{m_1+1}, b_2q^{m_2}, ..., b_rq^{m_r}; q^{n+1}, bq, b_2, ..., b_r; \; q, q^{1+2n-(m_1+...+m_r)}\right) = \frac{(q;q)_n(q;q)_n}{(1/b;q)_n(bq;q)_n}\frac{(q/b;q)_{m_1}}{(q;q)_{m_1}}\frac{(b_2/b;q)_{m_2}...(b_r/b;q)_{m_r}}{(b_2;q)_{m_2}...(b_r;q)_{m_r}}\times b^{(m_1+...+m_r-n)}. \quad (4.8) $$

5. Conclusion

In our work we have used a classical method known as Cauchy's method to obtain some identities of basic bilateral hypergeometric series. The work suggests further investigations to bilateralize the known identities of unilateral series of various natures available in the literature.

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