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# CERTAIN RESULTS INVOLVING RAMANUJAN'S THETA FUNCTIONS AND CONTINUED FRACTIONS

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**Abstract:** In this paper certain continued fractions associated with Ramanujan's theta functions have been discussed.

**Keywords and Phrases:** Continued fraction, theta functions, Jacobi's theta functions, Jacobi's triple product identity.

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## 1. Introduction, Notation and Definitions

The Ramanujan theta function is a fundamental theory in basic hypergeometric series that extends the structure of the Jacobi theta functions [1, 3, 7], while retaining their essential characteristics. Specifically, the Jacobi triple product exhibits a very sophisticated structure when expressed using the Ramanujan theta functions [2, 10]. The Ramanujan theta function is typically employed to identify the critical dimensions in theories such as bosonic string theory, superstring theory, and M-theory.

On the other hand, continued fractions play an important role in various branches of mathematics. They naturally arise in long division and in the theory of approximating real numbers with rational numbers [4, 5]. A continued fraction is an expression obtained through an iterative process of representing a number as the sum of its integer part and the reciprocal of another number [9], then writing this other number as the sum of its integer part and another reciprocal, and so on. It is related to number theory and helps us find good approximations for real-life constants. Results [6, 8] are require to establish certain continued fractions related to theta functions.

Throughout the paper, for any complex number $a$ and $q$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1,$$

$$(a_1,a_2,...,a_r;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty...(a_r;q)_\infty$$

Four Jacobi's theta functions are defined as

$$\theta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin(2n+1)z, \tag{1.1}$$

$$\theta_2(z,q) = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}\cos(2n+1)z, \tag{1.2}$$

$$\theta_3(z,q) = 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz, \tag{1.3}$$

$$\theta_4(z,q) = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz. \tag{1.4}$$

If we put $z=0$ in (1.1)-(1.4) we find,

$$\theta_1(q) = 0, \tag{1.5}$$

$$\theta_2(q) = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} = 2q^{\frac{1}{4}}\sum_{n=0}^{\infty} q^{n^2+n}, \tag{1.6}$$

$$\theta_3(q) = 1+2\sum_{n=1}^{\infty} q^{n^2}, \tag{1.7}$$

$$\theta_4(q) = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}. \tag{1.8}$$

There are large number of beautiful identities related to $\theta_2(q)$, $\theta_3(q)$ and $\theta_4(q)$. It is saying that no topic in mathematics is more replete with beautiful formulas than that related to $\theta_2(q)$, $\theta_3(q)$ and $\theta_4(q)$. One of the most beautiful identities involving $\theta_2(q)$, $\theta_3(q)$ and $\theta_4(q)$ is

$$\theta_3^4(q) = \theta_2^4(q) + \theta_4^4(q). \tag{1.9}$$

Motivated with interesting identities involving Jacobi's theta functions, Ramanujan defined a very general theta function as,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{1.10}$$

Two useful cases of $f(a,b)$ are

$$\Phi(q) = f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty} \tag{1.11}$$

and

$$\Psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.12}$$

If we apply Jacobi's triple product identity [1; Entry 19, p. 35] we have,

$$f(a,b) = (-a,-b,ab;ab)_\infty, \quad |ab| < 1. \tag{1.13}$$

From [1; Entry 30 (II) & (III), p. 46] we have,

$$f(a,b) = f(a^3b, ab^3) + af(b/a, a^5b^3). \tag{1.14}$$

From [1; Entry 30 (IV), p. 46] we note that,

$$f(a,b)f(-a,-b) = f(-a^2,-b^2)\Phi(-ab). \tag{1.15}$$

From [2; entry 29, p. 45], we note that for $ab = cd$,

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af(b/c, ac^2d)f(c/d, acd^2). \tag{1.16}$$

From [2; lemma (1.2.1) p. 13] we have,

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab^n), b(ab)^{-n}), \qquad n \text{ is an integer.} \tag{1.17}$$

Ramanujan recorded many continued fractions in his notebooks. Most famous is Rogers-Ramanujan continued fraction defined by,

$$\frac{f(-q,-q^4)}{f(-q^2,-q^5)} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\ldots}\quad . \tag{1.18}$$

(1.17) is the generalization of a very old result,

$$\frac{\sqrt{5}-1}{2} = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+...} \quad . \tag{1.19}$$

## 2. Main Results

In this section, certain interesting results on continued fractions have been established.

**(i).** Taking $a = -q$, $b = -q^5$ in (1.13) we get,

$$f(-q, -q^5) = (q, q^5, q^6; q^6)_\infty. \tag{2.1}$$

Again, taking $a = -q^3 = b$ in (1.13) we find,

$$f(-q^3, -q^3) = (q^3; q^6)^2_\infty (q^6; q^6)_\infty. \tag{2.2}$$

Now, taking the ratio of (2.1) and (2.2) and comparing with [1 ;(6.2.37) p. 154] we get

$$\frac{f(-q, -q^5)}{f(-q^3, -q^3)} = \frac{(q, q^5; q^6)_\infty}{(q^3; q^6)^2_\infty} = \frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+...} \quad . \tag{2.3}$$

**(ii).** Taking $a = -q$, $b = -q^7$ in (1.13) we have

$$f(-q, -q^7) = (q, q^7, q^8; q^8)_\infty. \tag{2.4}$$

Again, taking $a = -q^3$, $b = -q^5$ in (1.13) we get,

$$f(-q^3, -q^5) = (q^3, q^5, q^8; q^8)_\infty. \tag{2.5}$$

Now, taking the ratio of (2.4) and (2.5) and comparing with [1 ;(6.2.38) p. 154] we obtain

$$\frac{f(-q, -q^7)}{f(-q^3, -q^5)} = \frac{(q, q^7; q^8)_\infty}{(q^3, q^5; q^8)_\infty} = \frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+...} \quad . \tag{2.6}$$

**(iii).** Taking $a = -q = c$, $b = d = -q^4$ in (1.16) we find

$$f^2(-q, -q^4) = f(q^2, q^8) f(q^5, q^5) - q f(q^3, q^7) f(1, q^{10}), \tag{2.7}$$

Again, taking $a = c = -q^2$ and $b = d = -q^3$ in (1.16) we get,

$$f^2(-q^2, -q^3) = f(q^4, q^6) f(q^5, q^5) - q^2 f(q, q^9) f(1, q^{10}). \tag{2.8}$$

Taking the ratio of (2.7) and (2.8) and using (1.8) we have

$$\frac{f(q^2,q^8)f(q^5,q^5)-qf(q^3,q^7)f(1,q^{10})}{f(q^4,q^6)f(q^5,q^5)-q^2f(q,q^9)f(1,q^{10})}=\left\{\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+...}\right\}^2. \tag{2.9}$$

**(iv).** Taking $a == c = -q$, $b = d = -q^7$ in (1.16) we find

$$f^2(-q,-q^7)=f(q^2,q^{14})f(q^8,q^8)-qf(q^6,q^{10})f(1,q^{16}), \tag{2.10}$$

Again, taking $a = c = -q^3$ and $b = d = -q^5$ in (1.16) we get,

$$f^2(-q^3,-q^5)=f(q^6,q^{10})f(q^8,q^8)-q^3f(q^2,q^{14})f(1,q^{16}). \tag{2.11}$$

Taking the ratio of (2.10) and (2.11) and using (2.6) we find

$$\frac{f(q^2,q^{14})f(q^8,q^8)-qf(q^6,q^{10})f(1,q^{16})}{f(q^6,q^{10})f(q^8,q^8)-q^3f(q^2,q^{14})f(1,q^{16})}=\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+...}\right\}^2. \tag{2.12}$$

**(v).** Taking $a = c = -q$, $b = d = -q^5$ in (1.16) we get

$$f^2(-q,-q^5)=f(q^2,q^{10})f(q^6,q^6)-qf(q^4,q^8)f(1,q^{12}), \tag{2.13}$$

Again, taking $a = c = -q^3$ and $b = d = -q^3$ in (1.16) we get,

$$f^2(-q^3,-q^3)=f(q^6,q^6)f(q^6,q^6)-q^3f(1,q^{12})f(1,q^{12}). \tag{2.14}$$

Taking the ratio of (2.13) and (2.14) and using (2.3) we find

$$\frac{f(q^2,q^{10})f(q^6,q^6)-qf(q^4,q^8)f(1,q^{12})}{f(q^6,q^6)f(q^6,q^6)-q^3f(1,q^{12})f(1,q^{12})}=\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+...}\right\}^2. \tag{2.15}$$

**(vi).** Taking $a = -q$, $b = -q^4$, $c = -q$, $d = -q^7$ in (1.16) we get

$$f(-q,-q^4)f(-q,-q^7)=f(q^2,q^{11})f(q^8,q^5)-qf(q^3,q^{10})f(-q^3,q^{16}). \tag{2.16}$$

Again, taking $a = -q^2$, $b = -q^3$, $c = -q^3$ and $d = -q^5$ in (1.16) we get,

$$f(-q^2,-q^3)f(-q^3,-q^5)=f(q^5,q^8)f(q^7,q^6)-q^2f(1,q^{13})f(q^{-2},q^{15}). \tag{2.17}$$

Taking the ratio of (2.16) and (2.17) and making use of (1.18) and (2.6) we find

$$\frac{f(q^2,q^{11})f(q^8,q^5)-qf(q^3,q^{10})f(-q^3,q^{16})}{f(q^5,q^8)f(q^7,q^6)-q^2f(1,q^{13})f(q^{-2},q^{15})}$$
$$=\left\{\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+...}\right\}\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+...}\right\}. \tag{2.18}$$

Similar other results can also be scored.

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