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SOME MATCHING COEFFICIENTS OF $q-$PRODUCTS

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Abstract: We find some results on matching coefficients for certain q-products. Some of the results are associated with Rogers−Ramanujan continued fraction

$$R(q) = \frac{(q, q^4; q^5)_\infty}{(q^2, q^3; q^5)_\infty},$$

while some are associated with analogous of Rogers−Ramanujan functions. The techniques used for proving the results involves Ramanujan's theta functions, identities for Rogers−Ramanujan type functions, and q-series manipulations.

Keywords and Phrases: Matching coefficient, q-product, Rogers−Ramanujan continued fraction, Rogers−Ramanujan type functions.

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1. Introduction

Recently, Baruah and Das [7] have found some interesting results on the series expansion of certain q-products having matching coefficients with their reciprocals. For example, consider

$$S_1(q) = \sum_{n=0}^{\infty} s_1(n)q^n,$$

and

$$\frac{1}{S_1(q)} = \sum_{n=0}^{\infty} s_1'(n)q^n.$$

For some positive integers a, b, c, d and x, we have $s_1(an+b) = \pm x s_1'(cn+d)$, for all $n \geq 0$, then the power series $S_1(q)$ is said to have matching coefficients with their reciprocals $1/S_1(q)$. They also presented some conjectures based on the numerical evidence in their paper. And later on, in 2023, these conjectures were proved by Du and Tang [11] using algorithmic approach.

In this paper, we have found some matching coefficients for two power series $S_2(q) = \sum_{n=0}^{\infty} s_2(n)q^n$ and $S_2'(q) = \sum_{n=0}^{\infty} s_2'(n)q^n$ as $s_2(an+b) = \pm x s_2'(cn+d)$, where $S_2'(q)$ is not necessarily be the reciprocal of $S_2(q)$. The results mainly arise from identities belonging to Rogers−Ramanujan type functions and some q-series manipulations. Before proceeding further, we record the definition for q-Pochhammer symbol, which is given by:

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \qquad \text{and} \qquad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),$$

where a, q are complex numbers with $|q| < 1$. For convenience, we set

$$(a_1, a_2, q, \ldots, a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty (q;q)_\infty.$$

Also, for a positive integer l, we use

$$f_l = (q^l;q^l)_\infty.$$

Let $G(q)$ and $H(q)$ be the Rogers–Ramanujan functions defined, respectively, by

$$G(q) := \frac{1}{(q,q^4;q^5)_\infty}, \qquad H(q) := \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{1.1}$$

The Rogers–Ramanujan continued fraction, $R(q)$ can be represented as the quotient of $H(q)$ and $G(q)$ as:

$$R(q) = \frac{H(q)}{G(q)}.$$

Let us consider Ramanujan's parameter [6, p. 33], [8, p. 523], [15, p. 362]

$$k(q) := R(q)R(q^2)^2.$$

Ramanujan also introduced another two parameters in his lost notebook

$$\mu(q) := R(q)R(q^4), \qquad \nu(q) := \frac{R(q^{1/2})^2 R(q)}{R(q^2)}.$$

In this paper, we present the matching coefficients for the terms associated with Ramanujan's parameters and some of Rogers−Ramanujan type functions.

Now we consider $S_2(q)$ as $q\mu(q)$ and $S_2'(q)$ as the reciprocal of $q\mu(q)$ (as defined earlier) and let

$$\sum_{n=0}^{\infty}\tau_1'(n)q^n=\frac{1}{q\mu(q)},\qquad \sum_{n=0}^{\infty}\tau_1(n)q^n=q\mu(q),$$

and we give results for the matching coefficients $\tau_1(n)$ and $\tau_1'(n)$, for certain values of n. Similarly, we provide the matching coefficients for the following:

$$\sum_{n=0}^{\infty}\tau_2'(n)q^n=\frac{1}{\nu(q^2)},\qquad \sum_{n=0}^{\infty}\tau_2(n)q^n=\nu(q^2),$$

and

$$\sum_{n=0}^{\infty}\tau_3'(n)q^n=\frac{k(q)}{qk(q^2)},\qquad \sum_{n=0}^{\infty}\tau_3(n)q^n=\frac{qk(q^2)}{k(q)}.$$

The following theorem represents the matching coefficients for $\tau_i(n)$ and $\tau_i'(n)$ for $1\le i\le 4$.

Theorem 1.1. *For $n\ge 0$, we have*

$$\tau_2'(2n)+\tau_2(2n)=-2\tau_1'(2n)-2\tau_1(2n)+4, \tag{1.2}$$

$$\tau_3'(5n+r)=\tau_3(5n+r),\quad \text{for } r\in\{2,3\}, \tag{1.3}$$

$$\tau_3'(2n+1)-\tau_3(2n+1)=\tau_1'(2n+1)-\tau_1(2n+1), \tag{1.4}$$

$$\tau_3'(2n)-\tau_3(2n)=-\tau_1'(2n)+\tau_1(2n). \tag{1.5}$$

Consider

$$\sum_{n=0}^{\infty}\Upsilon_1'(n)q^n=G(q)H(q),\qquad \sum_{n=0}^{\infty}\Upsilon_1(n)q^n=G(-q)H(-q),$$

$$\sum_{n=0}^{\infty}\Upsilon_2'(n)q^n=\frac{G(q)J(-q)}{H(q)},\qquad \sum_{n=0}^{\infty}\Upsilon_2(n)q^n=\frac{H(q)K(-q)}{G(q)},$$

$$\sum_{n=0}^{\infty}\Upsilon_{3,i,j}'(n)q^n=\frac{J(q^i)H(q^2)}{G^j(q)H^i(q)},\qquad \sum_{n=0}^{\infty}\Upsilon_{3,i,j}(n)q^n=\frac{K(q^i)G(q^2)}{H^j(q)G^i(q)},$$

$$\sum_{n=0}^{\infty}\Upsilon_4'(n)q^n=K(q)J(-q),\qquad \sum_{n=0}^{\infty}\Upsilon_4(n)q^n=K(-q)J(q),$$

where $J(q)$ and $K(q)$ are q-products shown as:

$$J(q) = \frac{f_2(q^3, q^7, q^{10}; q^{10})_\infty}{f_1^2}, \qquad K(q) = \frac{f_2(q, q^9, q^{10}; q^{10})_\infty}{f_1^2}. \tag{1.6}$$

Theorem 1.2. *For $n \geq 0$, we have*

$$\Upsilon'_1(2n) = \Upsilon_1(2n), \tag{1.7}$$
$$\Upsilon'_1(2n+1) = -\Upsilon_1(2n+1), \tag{1.8}$$
$$\Upsilon'_2(n+1) = -\Upsilon_2(n), \tag{1.9}$$
$$\Upsilon'_{3,2,1}(5n+r) = -\Upsilon_{3,2,1}(5n+r), \quad \textit{for } r \in \{1,2,3,4\}, \tag{1.10}$$
$$\Upsilon'_{3,1,2}(5n+r) = -\Upsilon_{3,1,2}(5n+r-1), \quad \textit{for } r \in \{1,4\}, \tag{1.11}$$
$$\Upsilon'_4(2n+1) = -\Upsilon_4(2n+1), \tag{1.12}$$
$$\Upsilon'_4(2n) = \Upsilon_4(2n). \tag{1.13}$$

Next, we consider the following analogous of the Rogers−Ramanujan functions (1.1) as:

$$S(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n} = \frac{(q, q^5, q^6; q^6)_\infty f_2}{f_1 f_4} = \frac{f_6^2}{f_3 f_4}, \tag{1.14}$$

$$T(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} = \frac{(q^2, q^4, q^6; q^6)_\infty f_2}{f_1 f_4} = \frac{f_2^2}{f_1 f_4}, \tag{1.15}$$

$$N(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^4;q^4)_n} = \frac{(q^3, q^3, q^6; q^6)_\infty f_2}{f_1 f_4} = \frac{f_2 f_3^2}{f_1 f_4 f_6}, \tag{1.16}$$

and the following continued fraction that was established by Naika et al. in [14]:

$$U(q) = q\frac{L(q)}{M(q)}, \tag{1.17}$$

where

$$L(q) = \frac{(q, q^{11}, q^{12}; q^{12})_\infty}{f_4}, \qquad M(q) = \frac{(q^5, q^7, q^{12}; q^{12})_\infty}{f_4}.$$

Consider $S'_2(q)$ and $S_2(q)$ as $T(q^i)/S(q^i)$ and $S(q^j)/T(q^j)$, respectively, for some positive integers i and j, where $S'_2(q)$ not necessarily always be the reciprocal of $S_2(q)$. Let

$$\sum_{n=0}^{\infty} \omega'_{1,i}(n) q^n = \frac{T(q^i)}{S(q^i)}, \qquad \sum_{n=0}^{\infty} \omega_{1,i}(n) q^n = \frac{S(q^i)}{T(q^i)}.$$

Also, let

$$\sum_{n=0}^{\infty} \omega_2'(n)q^n = S(-q)N(q), \qquad \sum_{n=0}^{\infty} \omega_2(n)q^n = N(-q)S(q),$$

and
$$\sum_{n=0}^{\infty} \omega_3'(n)q^n = \frac{1}{U(q)}, \qquad \sum_{n=0}^{\infty} \omega_3(n)q^n = U(q).$$

Theorem 1.3. *For $n \geq 0$, we have*

$$\omega_{1,2}'(6n+4) = -\omega_{1,2}(6n+3), \tag{1.18}$$
$$\omega_{1,2}'(6n+2) + \omega_{1,2}(6n+1) = \omega_{1,2}'(2n+1) + \omega_{1,2}(2n), \tag{1.19}$$
$$\omega_{1,8}'(4n+r) = \omega_{1,4}(4n+r-3), \quad \textit{for } r \in \{1,2\}, \tag{1.20}$$
$$\omega_{1,8}'(8n+4) = \omega_{1,4}(8n+1), \tag{1.21}$$
$$\omega_{1,8}'(24n+16) = \omega_{1,4}(24n+13), \tag{1.22}$$
$$\omega_2'(2n) = \omega_2(2n), \tag{1.23}$$
$$\omega_2'(2n+1) = -\omega_2(2n+1), \tag{1.24}$$
$$\omega_3'(2n+2) = -\omega_3(2n+2), \tag{1.25}$$
$$\omega_3'(3n+1) = \omega_3(3n+1), \tag{1.26}$$
$$\omega_3'(12n+r) = \omega_3(12n+r), \quad \textit{for } r \in \{5,9\}. \tag{1.27}$$

Let us suppose

$$\sum_{n=0}^{\infty} \upsilon_{1,i}(n)q^n = \frac{f_i^3}{f_{3i}}, \qquad \sum_{n=0}^{\infty} \upsilon_{2,i}(n)q^n = \frac{f_{3i}^3}{f_i},$$
$$\sum_{n=0}^{\infty} \upsilon_{3,i}(n)q^n = \frac{f_{2i}^2}{f_i}, \qquad \sum_{n=0}^{\infty} \upsilon_{4,i}(n)q^n = \frac{f_i^2}{f_{2i}},$$
$$\sum_{n=0}^{\infty} \upsilon_5'(n)q^n = f(-q^5)f(q), \qquad \sum_{n=0}^{\infty} \upsilon_5(n)q^n = f(q^5)f(-q),$$

where

$$f(-q) = (q;q)_\infty.$$

Theorem 1.4. *For $n \geq 0$ and $\alpha > 0$, we have*

$$\upsilon_{1,4}(3n+2) = \upsilon_{1,1}(3n+2), \tag{1.28}$$
$$\upsilon_{1,4}(6n+r) = \upsilon_{1,1}(6n+r), \quad \text{for } r \in \{0,4\} \tag{1.29}$$
$$\upsilon_{1,4}(3n) - \upsilon_{1,1}(3n) = \upsilon_{1,4}(3^{\alpha}n) - \upsilon_{1,1}(3^{\alpha}n), \tag{1.30}$$
$$\upsilon_{1,4}(9n+3) - \upsilon_{1,1}(9n+3) = \upsilon_{1,4}(3^{\alpha+1}n+3^{\alpha}) - \upsilon_{1,1}(3^{\alpha+1}n+3^{\alpha}), \tag{1.31}$$
$$\upsilon_{1,4}(3^{\alpha+1}n+2\cdot 3^{\alpha}) = \upsilon_{1,1}(3^{\alpha+1}n+2\cdot 3^{\alpha}), \tag{1.32}$$
$$\upsilon_{2,4}(2n) = \upsilon_{1,1}(2n+1), \tag{1.33}$$
$$\upsilon_{3,9}(3n+r) = \upsilon_{3,1}(3n+r+1), \quad \text{for } r \in \{0,1\} \tag{1.34}$$
$$\upsilon_{4,9}(3n+r) = \upsilon_{1,9}(3n+r), \quad \text{for } r \in \{0,2\} \tag{1.35}$$
$$\upsilon_5'(2n) = \upsilon_5(2n), \tag{1.36}$$
$$\upsilon_5'(2n+1) = -\upsilon_5(2n+1), \tag{1.37}$$
$$\upsilon_5'(10n+r) = \upsilon_5(10n+r) = 0, \quad \text{for } r \in \{4,8\}. \tag{1.38}$$

The paper is organized as follows. Section 2 contain some preliminary results that will be used to prove the main results. Section 3 includes the proof for Theorems 1.1−1.4.

2. Preliminaries

For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ is given by:

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

The two special cases of $f(a,b)$ are

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2},$$
$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}.$$

Also,

$$\varphi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2}.$$

Jacobi's triple product identity is given by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

A very useful consequence of Jacobi triple identity is the infinite product identity:

$$(q;q)_\infty^3 = \sum_{n=-\infty}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$

Euler's pentagonal number theorem is given by

$$f_1 = (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.$$

Lemma 2.1. *We have*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8}, \tag{2.1}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}, \tag{2.2}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{2.3}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{2.4}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q \frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{2.5}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q \frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{2.6}$$

$$f_1 f_5^3 = 2q^2 f_4 f_{20}^3 + f_2^3 f_{10} - 2q^3 \frac{f_4^4 f_{40}^2 f_{10}}{f_2 f_8^2} - q \frac{f_2^2 f_{10}^2 f_{20}}{f_4}, \tag{2.7}$$

$$f_1^3 f_5 = 2q^2 \frac{f_4^6 f_{40}^2 f_{10}}{f_2 f_8^2 f_{20}^2} + \frac{f_4 f_{10}^2 f_2^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3, \tag{2.8}$$

$$\frac{1}{f_1 f_5^3} = 2q^2 \frac{f_4^2 f_{20}^6}{f_2^3 f_{10}^9} + \frac{f_4 f_{20}^3}{f_{10}^8} + 2q^3 \frac{f_4^5 f_{20}^3 f_{40}^2}{f_2^4 f_{10}^8 f_8^2} + q \frac{f_{20}^4}{f_2 f_{10}^7}, \tag{2.9}$$

$$\frac{1}{f_1^3 f_5} = 2q^2 \frac{f_4^9 f_{40}^2}{f_2^{10} f_8^2 f_{10}^2 f_{20}} + \frac{f_4^4}{f_2^7 f_{10}} - 2q \frac{f_4^6 f_{20}^2}{f_2^9 f_{10}^3} + 5q \frac{f_4^3 f_{20}}{f_2^8}. \tag{2.10}$$

Proof. Identities (2.1)−(2.6) comes from [12]. (2.7) and (2.8) are the identities from [13]. Identities (2.9) and (2.10) obtained by replacing q by $-q$ in identities (2.7) and (2.8), respectively and then using

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}. \tag{2.11}$$

Lemma 2.2. [12] *We have*

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}, \tag{2.12}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{2.13}$$

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}, \tag{2.14}$$

Lemma 2.3. [9] *We have*

$$\frac{1}{q\mu(q)} - q\mu(q) = \frac{f_2^3 f_{10}^5}{q f_1 f_4 f_5^3 f_{20}^3}, \tag{2.15}$$

$$\frac{1}{\sqrt{q\mu(q)}} + \sqrt{q\mu(q)} = \sqrt{\frac{f_2^8}{q f_1^3 f_4^3 f_5 f_{20}}}, \tag{2.16}$$

$$\frac{1}{\sqrt{\nu(q^2)}} + \sqrt{\nu(q^2)} = 2\sqrt{\frac{f_4^3 f_{20}}{f_2 f_{10}^3}}, \tag{2.17}$$

$$\frac{k(q)}{qk(q^2)} - \frac{qk(q^2)}{k(q)} = \frac{f_1 f_5^3}{q f_{10}^4}. \tag{2.18}$$

Lemma 2.4. [10] *We have*

$$G(q)H(q) - G(-q)H(-q) = 2q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}, \tag{2.19}$$

$$G(q)H(q) + G(-q)H(-q) = 2\frac{f_8 f_{20}^2}{f_2^2 f_{40}}, \tag{2.20}$$

$$f(-q^5)f(q) - f(q^5)f(-q) = 2q\frac{f_4^2 f_{10} f_{40}}{f_8 f_{20}}, \tag{2.21}$$

$$f(-q^5)f(q) + f(q^5)f(-q) = 2\frac{f_2 f_8 f_{20}^2}{f_4 f_{40}}. \tag{2.22}$$

Lemma 2.5. [2] *We have*

$$G^2(q)J(-q) + qH^2(q)K(-q) = \frac{f_5}{f_1}, \tag{2.23}$$

$$J(q)H(q)H(q^2) + K(q)G(q)G(q^2) = 2\frac{f_{10}^2}{f_1^2}, \tag{2.24}$$

$$J(q^2)G(q)H(q^2) + qK(q^2)H(q)G(q^2) = \frac{f_2^2 f_{20}}{f_1^2 f_4}. \tag{2.25}$$

Lemma 2.6. [5] *We have*

$$K(q)J(-q)+K(-q)J(q)=2\frac{f_4^3f_{20}}{f_2^4}, \tag{2.26}$$

$$K(q)J(-q)-K(-q)J(q)=-2q\frac{f_4f_{20}^3}{f_2^3f_{10}}. \tag{2.27}$$

Lemma 2.7. [1] *We have*

$$T^2(q^2)+qS^2(q^2)=\frac{f_3^3f_4f_{12}}{f_1f_6^2f_8^2}. \tag{2.28}$$

Lemma 2.8. [4] *We have*

$$T(q^4)T(q^8)-q^3S(q^4)S(q^8)=\frac{1}{2qf_4f_{32}}\left(\frac{f_{12}^2f_{24}}{f_{48}}-\frac{f_1^2f_8^2}{f_2f_{16}}\right). \tag{2.29}$$

Lemma 2.9. [16] *We have*

$$S(-q)N(q)-N(-q)S(q)=2q\frac{f_2f_{12}^4}{f_4^3f_6^2}, \tag{2.30}$$

$$S(-q)N(q)+N(-q)S(q)=2\frac{f_4}{f_2}. \tag{2.31}$$

Lemma 2.10. [3] *We have*

$$\frac{1}{U(q)}+U(q)=\frac{f_3^3f_4}{qf_1f_{12}^3}, \tag{2.32}$$

$$\frac{1}{U(q)}-U(q)=\frac{f_2^2f_6^4}{qf_1f_3f_{12}^4}. \tag{2.33}$$

3. Proof of Theorems 1.1-1.4.

This section is devoted to prove the theorems shown in Section 1.

Proof of Theorem 1.1. Squaring both sides (2.16) and (2.17), we find that

$$\sum_{n=0}^{\infty}\tau_1'(n)q^{n+1}+\sum_{n=0}^{\infty}\tau_1(n)q^{n+1}+2q=\frac{f_2^8}{f_1^3f_4^3f_5f_{20}}, \tag{3.1}$$

$$\sum_{n=0}^{\infty}\tau_2'(n)q^{n}+\sum_{n=0}^{\infty}\tau_2(n)q^{n}+2=4\frac{f_4^3f_{20}}{f_2f_{10}^3}. \tag{3.2}$$

Substituting (2.10) in (3.1), then extracting the odd terms

$$\sum_{n=0}^{\infty}\tau_1'(2n)q^n+\sum_{n=0}^{\infty}\tau_1(2n)q^n=-2\frac{f_2^3f_{10}}{f_1f_5^3}+3, \tag{3.3}$$

and bringing out the even terms from (3.2), we have

$$\sum_{n=0}^{\infty}\tau_2'(2n)q^n+\sum_{n=0}^{\infty}\tau_2(2n)q^n+2=4\frac{f_2^3f_{10}}{f_1f_5^3}. \tag{3.4}$$

Comparing (3.3) and (3.4), we get (1.2). Now consider (2.18),

$$\sum_{n=0}^{\infty}\tau_3'(n)q^{n+1}+\sum_{n=0}^{\infty}\tau_3(n)q^{n+1}=\frac{f_1f_5^3}{f_{10}^4}.$$

With the help of Euler's pentagonal number theorem, we obtain

$$\sum_{n=0}^{\infty}\tau_3'(n)q^{n+1}+\sum_{n=0}^{\infty}\tau_3(n)q^{n+1}=\frac{f_5^3}{f_{10}^4}\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}.$$

Here, $n(3n-1)/2\not\equiv 3,4 \pmod 5$, which gives (1.3). Similarly, consider (2.15) and (2.18),

$$\sum_{n=0}^{\infty}\tau_1'(n)q^{n+1}-\sum_{n=0}^{\infty}\tau_1(n)q^{n+1}=\frac{f_2^3f_{10}^5}{f_1f_4f_5^3f_{20}^3}, \tag{3.5}$$

$$\sum_{n=0}^{\infty}\tau_3'(n)q^{n+1}-\sum_{n=0}^{\infty}\tau_3(n)q^{n+1}=\frac{f_1f_5^3}{f_{10}^4}. \tag{3.6}$$

Using (2.7) and (2.9), then extracting the even and odd terms, we obtain (1.4) and (1.5), respectively.

Proof of Theorem 1.2. Consider (2.19), we have

$$\sum_{n=0}^{\infty}\Upsilon_1'(n)q^n-\sum_{n=0}^{\infty}\Upsilon_1(n)q^n=2q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}.$$

Extracting the terms involving the even powers of q, we get (1.7). Similarly, considering (2.20), we have

$$\sum_{n=0}^{\infty}\Upsilon_1'(n)q^n+\sum_{n=0}^{\infty}\Upsilon_1(n)q^n=2\frac{f_8f_{20}^2}{f_2^2f_{40}}.$$

Then extracting the terms involving the odd powers of q and we arrive at (1.8). From (2.23), dividing both sides by $G(q)H(q)$, we obtain

$$\sum_{n=0}^{\infty}\Upsilon_2'(n)q^n+\sum_{n=0}^{\infty}\Upsilon_2(n)q^{n+1}=1,$$

and we readily arrive at (1.9). From (2.24), dividing both sides by $G^2(q)H^2(q)$, we obtain

$$\sum_{n=0}^{\infty}\Upsilon_{3,2,1}'(n)q^n+\sum_{n=0}^{\infty}\Upsilon_{3,2,1}(n)q^n=2\frac{f_{10}^2}{f_5^2}.$$

Bringing out the terms involving $q^{5n+1},q^{5n+2},q^{5n+3},q^{5n+4}$, we have (1.10). Also, from (2.25), dividing both sides by $G^2(q)H^2(q)$, we obtain

$$\sum_{n=0}^{\infty}\Upsilon_{3,1,2}'(n)q^n+\sum_{n=0}^{\infty}\Upsilon_{3,1,2}(n)q^{n+1}=\frac{f_2^2f_{20}}{f_4f_5^2}=\frac{f_{20}}{f_5^2}\sum_{n=-\infty}^{\infty}(-1)^nq^{2n^2}.$$

Here, $2n^2\not\equiv 1,4 \pmod 5$, therefore we get (1.11). Consider (2.26),

$$\sum_{n=0}^{\infty}\Upsilon_4'(n)q^n+\sum_{n=0}^{\infty}\Upsilon_4(n)q^n=2\frac{f_4^3f_{20}}{f_2^4}.$$

Extracting the terms containing the odd powers of q to get (1.12). Similarly, considering (2.27) and extracting the terms involving even powers of q, we obtain (1.13).

Proof of Theorem 1.3. Consider

$$\sum_{n=0}^{\infty}\omega_{1,2}'(n)q^n+q\sum_{n=0}^{\infty}\omega_{1,2}(n)q^n=\frac{T^2(q^2)+qS^2(q^2)}{T(q^2)S(q^2)}.$$

Using (1.14), (1.15), and (2.28), we have

$$\sum_{n=0}^{\infty}\omega_{1,2}'(n)q^n+\sum_{n=0}^{\infty}\omega_{1,2}(n)q^{n+1}=\frac{f_3^3f_2}{f_1f_4f_6f_{12}}.$$

Substituting (2.2), we get

$$\sum_{n=0}^{\infty}\omega_{1,2}'(n)q^n+\sum_{n=0}^{\infty}\omega_{1,2}(n)q^{n+1}=\frac{f_4^2f_6}{f_2f_{12}^2}+q\frac{f_2f_{12}^2}{f_4^2f_6}.$$

Extracting the terms containing the even and odd powers of q from above, we have

$$\sum_{n=0}^{\infty}\omega'_{1,2}(2n)q^n+\sum_{n=0}^{\infty}\omega_{1,2}(2n+1)q^{n+1}=\frac{f_2^2f_3}{f_1f_6^2}, \tag{3.7}$$

$$\sum_{n=0}^{\infty}\omega'_{1,2}(2n+1)q^n+\sum_{n=0}^{\infty}\omega_{1,2}(2n)q^n=\frac{f_1f_6^2}{f_2^2f_3}. \tag{3.8}$$

Substituting (2.12) in (3.7)

$$\sum_{n=0}^{\infty}\omega'_{1,2}(2n)q^n+\sum_{n=0}^{\infty}\omega_{1,2}(2n+1)q^{n+1}=\frac{f_9^2}{f_6f_{18}}+q\frac{f_3f_{18}^2}{f_6^2f_9}. \tag{3.9}$$

Bringing out the terms involving q^{3n+1}, q^{3n+2}, we get

$$\sum_{n=0}^{\infty}\omega'_{1,2}(6n+2)q^n+\sum_{n=0}^{\infty}\omega_{1,2}(6n+1)q^n=\frac{f_1f_6^2}{f_2^2f_3}, \tag{3.10}$$

$$\sum_{n=0}^{\infty}\omega'_{1,2}(6n+4)q^n+\sum_{n=0}^{\infty}\omega_{1,2}(6n+3)q^n=0. \tag{3.11}$$

From (3.11), we get (1.18). Similarly, comparing (3.10) and (3.8), we arrive at (1.19). Consider

$$\sum_{n=0}^{\infty}\omega'_{1,8}(n)q^n-q^3\sum_{n=0}^{\infty}\omega_{1,4}(n)q^n=\frac{T(q^8)T(q^4)-q^3S(q^8)S(q^4)}{T(q^4)S(q^8)}.$$

Using (1.14), (1.15), and (2.29), we have

$$\sum_{n=0}^{\infty}\omega'_{1,8}(n)q^n-\sum_{n=0}^{\infty}\omega_{1,4}(n)q^{n+3}=\frac{1}{2q}\left(\frac{f_{12}^2f_{16}f_{24}^2}{f_8^2f_{48}^3}-\frac{f_1^2f_{24}}{f_2f_{48}^2}\right).$$

Using (2.1) in above, we get

$$2\sum_{n=0}^{\infty}\omega'_{1,8}(n)q^{n+1}-2\sum_{n=0}^{\infty}\omega_{1,4}(n)q^{n+4}=\frac{f_{12}^2f_{16}f_{24}^2}{f_8^2f_{48}^3}-\frac{f_8^5f_{24}}{f_4^2f_{16}^2f_{48}^2}+2q\frac{f_{16}^2f_{24}}{f_8f_{48}^2}.$$

Extracting the terms involving $q^{4n}, q^{4n+1}, q^{4n+2}, q^{4n+3}$ from above, we obtain

$$2\sum_{n=0}^{\infty}\omega'_{1,8}(4n+3)q^{n+1} - 2\sum_{n=0}^{\infty}\omega_{1,4}(4n)q^{n+1} = \frac{f_3^2 f_4 f_6^2}{f_2^2 f_{12}^3} - \frac{f_2^5 f_6}{f_1^2 f_4^2 f_{12}^2}, \tag{3.12}$$

$$2\sum_{n=0}^{\infty}\omega'_{1,8}(4n)q^{n} - 2\sum_{n=0}^{\infty}\omega_{1,4}(4n-3)q^{n} = 2\frac{f_4^2 f_6}{f_2 f_{12}^2}, \tag{3.13}$$

$$2\sum_{n=0}^{\infty}\omega'_{1,8}(4n+1)q^{n} - 2\sum_{n=0}^{\infty}\omega_{1,4}(4n-2)q^{n} = 0, \tag{3.14}$$

$$2\sum_{n=0}^{\infty}\omega'_{1,8}(4n+2)q^{n} - 2\sum_{n=0}^{\infty}\omega_{1,4}(4n-1)q^{n} = 0. \tag{3.15}$$

The last two equations prove (1.20). Extracting the terms containing even and odd powers of q from (3.13), we get

$$\sum_{n=0}^{\infty}\omega'_{1,8}(8n)q^{n} - \sum_{n=0}^{\infty}\omega_{1,4}(8n-3)q^{n} = \frac{f_2^2 f_3}{f_1 f_6^2}, \tag{3.16}$$

$$\sum_{n=0}^{\infty}\omega'_{1,8}(8n+4)q^{n} - \sum_{n=0}^{\infty}\omega_{1,4}(8n+1)q^{n} = 0. \tag{3.17}$$

From (3.17), we prove (1.21). Substituting (2.12) in (3.16), we have

$$\sum_{n=0}^{\infty}\omega'_{1,8}(8n)q^{n} - \sum_{n=0}^{\infty}\omega_{1,4}(8n-3)q^{n} = \frac{f_9^2}{f_6 f_{18}} + q\frac{f_3 f_{18}^2}{f_6^2 f_9}.$$

Bringing out the terms containing q^{3n+2}, we arrive at (1.22). Consider

$$\begin{aligned}\sum_{n=0}^{\infty}\omega'_2(n)q^{n} - \sum_{n=0}^{\infty}\omega_2(n)q^{n} &= S(-q)N(q) - N(-q)S(q),\\ &= 2q\frac{f_2 f_{12}^4}{f_4^3 f_6^2}. \qquad \text{(From 2.30)}\end{aligned}$$

Extracting the even terms, we obtain (1.23). Now, consider

$$\begin{aligned}\sum_{n=0}^{\infty}\omega'_2(n)q^{n} + \sum_{n=0}^{\infty}\omega_2(n)q^{n} &= S(-q)N(q) + N(-q)S(q),\\ &= 2\frac{f_4}{f_2}, \qquad \text{(From 2.31)}\end{aligned}$$

and extracting the odd terms to get (1.24). Consider (2.32), we have

$$\sum_{n=0}^{\infty}\omega_3'(n)q^{n+1}+\sum_{n=0}^{\infty}\omega_3(n)q^{n+1}=\frac{f_3^3f_4}{f_1f_{12}^3} \tag{3.18}$$

Substituting the values from (2.2), we obtain

$$\sum_{n=0}^{\infty}\omega_3'(n)q^{n+1}+\sum_{n=0}^{\infty}\omega_3(n)q^{n+1}=\frac{f_4^4f_6^2}{f_2^2f_{12}^4}+q.$$

Extracting the terms containing odd powers of q to get (1.25). Consider (2.33), we have

$$\sum_{n=0}^{\infty}\omega_3'(n)q^{n+1}-\sum_{n=0}^{\infty}\omega_3(n)q^{n+1}=\frac{f_2^2f_6^4}{f_1f_3f_{12}^4}.$$

Substituting (2.12), we get

$$\sum_{n=0}^{\infty}\omega_3'(n)q^{n+1}-\sum_{n=0}^{\infty}\omega_3(n)q^{n+1}=\frac{f_6^5f_9^2}{f_3^2f_{12}^4f_{18}}+q\frac{f_6^4f_{18}^2}{f_3f_9f_{12}^4}.$$

Extracting the terms involving $q^{3n+1}, q^{3n+2}, q^{3n+3}$ from above equation, we obtain

$$\sum_{n=0}^{\infty}\omega_3'(3n)q^{n}-\sum_{n=0}^{\infty}\omega_3(3n)q^{n}=\frac{f_2^4f_6^2}{f_1f_3f_4^4}, \tag{3.19}$$

$$\sum_{n=0}^{\infty}\omega_3'(3n+1)q^{n}-\sum_{n=0}^{\infty}\omega_3(3n+1)q^{n}=0, \tag{3.20}$$

$$\sum_{n=0}^{\infty}\omega_3'(3n+2)q^{n+1}-\sum_{n=0}^{\infty}\omega_3(3n+2)q^{n+1}=\frac{f_2^5f_3^2}{f_1^2f_4^4f_6}. \tag{3.21}$$

We can easily arrive at (1.26) from (3.20). Using (2.6) in (3.19), we have

$$\sum_{n=0}^{\infty}\omega_3'(3n)q^{n}-\sum_{n=0}^{\infty}\omega_3(3n)q^{n}=\frac{f_2^2f_8^2f_{12}^5}{f_4^5f_6^2f_{24}^2}+q\frac{f_4f_{24}^2}{f_8^2f_{12}}.$$

Extracting the terms containing odd powers of q, we have

$$\sum_{n=0}^{\infty}\omega_3'(6n+3)q^{n}-\sum_{n=0}^{\infty}\omega_3(6n+3)q^{n}=\frac{f_2f_{12}^2}{f_4^2f_6}.$$

Bringing out the terms containing odd powers of q, we obtain (1.27) for $r = 9$. Similarly, consider (3.21) and using (2.4),

$$\sum_{n=0}^{\infty} \omega_3'(3n+2)q^{n+1} - \sum_{n=0}^{\infty} \omega_3(3n+2)q^{n+1} = \frac{f_{12}^2}{f_8 f_{24}} + 2q\frac{f_2 f_6 f_8 f_{24}}{f_4^3 f_{12}}.$$

Extracting the terms containing even powers of q, we have

$$\sum_{n=0}^{\infty} \omega_3'(6n+5)q^{n+1} - \sum_{n=0}^{\infty} \omega_3(6n+5)q^{n+1} = \frac{f_6^2}{f_4 f_{12}}.$$

Bringing out the terms containing odd powers of q, we get (1.27).

Proof of Theorem 1.4. Consider

$$\sum_{n=0}^{\infty} \upsilon_{1,4}(n)q^n - \sum_{n=0}^{\infty} \upsilon_{1,1}(n)q^n = \frac{f_4^3}{f_{12}} - \frac{f_1^3}{f_3}.$$

From (2.5), we have

$$\sum_{n=0}^{\infty} \upsilon_{1,4}(n)q^n - \sum_{n=0}^{\infty} \upsilon_{1,1}(n)q^n = 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}.$$

Substituting (2.13), we have

$$\sum_{n=0}^{\infty} \upsilon_{1,4}(n)q^n - \sum_{n=0}^{\infty} \upsilon_{1,1}(n)q^n = 3q\frac{f_{12}^3 f_{18}^2}{f_6^2 f_{36}} - 6q^3\frac{f_{12}^2 f_{36}^2}{f_6 f_{18}}.$$

Extracting the terms involving $q^{3n}, q^{3n+1}, q^{3n+2}$, we have

$$\sum_{n=0}^{\infty} \upsilon_{1,4}(3n)q^n - \sum_{n=0}^{\infty} \upsilon_{1,1}(3n)q^n = -6q\frac{f_4^2 f_{12}^2}{f_2 f_6}, \tag{3.22}$$

$$\sum_{n=0}^{\infty} \upsilon_{1,4}(3n+1)q^n - \sum_{n=0}^{\infty} \upsilon_{1,1}(3n+1)q^n = 3\frac{f_4^3 f_6^2}{f_2^2 f_{12}}, \tag{3.23}$$

$$\sum_{n=0}^{\infty} \upsilon_{1,4}(3n+2)q^n - \sum_{n=0}^{\infty} \upsilon_{1,1}(3n+2)q^n = 0. \tag{3.24}$$

From (3.24), we have (1.28). Extracting the terms involving q^{2n} and q^{2n+1} from (3.22), we get (1.29) (for $r=0$) and

$$\sum_{n=0}^{\infty} \upsilon_{1,4}(6n+3)q^n - \sum_{n=0}^{\infty} \upsilon_{1,1}(6n+3)q^n = -6\frac{f_2^2 f_6^2}{f_1 f_3}, \tag{3.25}$$

respectively. Similarly, on bringing out the terms involving q^{2n+1}, we obtain (1.29) (for $r=4$). Also, using (2.12), we get

$$\sum_{n=0}^{\infty} \upsilon_{1,4}(3n)q^n - \sum_{n=0}^{\infty} \upsilon_{1,1}(3n)q^n = -6q\frac{f_{12}^3 f_{18}^2}{f_6^2 f_{36}} - 6q^3\frac{f_{12}^2 f_{36}^2}{f_6 f_{18}}.$$

Extracting the terms involving $q^{3n}, q^{3n+1}, q^{3n+2}$ from above which completes the proof of (1.30), (1.31), (1.32), respectively, for $\alpha = 1$. Rest of the proof part can be proved using induction. Consider

$$\sum_{n=0}^{\infty} \upsilon_{2,1}(n)q^n - \sum_{n=0}^{\infty} \upsilon_{2,4}(n)q^{n+1} = \frac{f_3^3}{f_1} - q\frac{f_{12}^3}{f_4}.$$

From (2.2), we obtain

$$\sum_{n=0}^{\infty} \upsilon_{2,1}(n)q^n - \sum_{n=0}^{\infty} \upsilon_{2,4}(n)q^{n+1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}}.$$

Bringing out the terms containing odd powers of q, we have (1.33). Next, we consider

$$\sum_{n=0}^{\infty} \upsilon_{3,1}(n)q^n - \sum_{n=0}^{\infty} \upsilon_{3,9}(n)q^{n+1} = \frac{f_2^2}{f_1} - q\frac{f_{18}^2}{f_9}.$$

Using (2.12), we have

$$\sum_{n=0}^{\infty} \upsilon_{3,1}(n)q^n - \sum_{n=0}^{\infty} \upsilon_{3,9}(n)q^{n+1} = \frac{f_6 f_9^2}{f_3 f_{18}}.$$

Bringing out the terms involving q^{3n+1}, q^{3n+2}, we get (1.34). Similarly, considering

$$\sum_{n=0}^{\infty} \upsilon_{4,9}(n)q^n - \sum_{n=0}^{\infty} \upsilon_{4,1}(n)q^n = \frac{f_9^2}{f_{18}} - \frac{f_1^2}{f_2}.$$

Using (2.13), we have

$$\sum_{n=0}^{\infty} \upsilon_{4,9}(n)q^n - \sum_{n=0}^{\infty} \upsilon_{4,1}(n)q^n = 2q\frac{f_3 f_{18}^2}{f_6 f_9}.$$

Extracting the terms involving q^{3n}, q^{3n+2}, we arrive at (1.35). Considering

$$\sum_{n=0}^{\infty} \upsilon_5'(n)q^n + \sum_{n=0}^{\infty} \upsilon_5(n)q^n = 2\frac{f_2 f_8 f_{20}^2}{f_4 f_{40}}.$$

Extracting the terms involving q^{2n+1}, we have $\upsilon_5'(2n+1) = -\upsilon_5(2n+1)$. Also, from (2.22),

$$\sum_{n=0}^{\infty} \upsilon_5'(n)q^n + \sum_{n=0}^{\infty} \upsilon_5(n)q^n = 2q\frac{f_4^2 f_{10} f_{40}}{f_8 f_{20}}.$$

Bringing out the even and odd powers of q, we have (1.36) and

$$\sum_{n=0}^{\infty} \upsilon_5'(2n+1)q^n + \sum_{n=0}^{\infty} \upsilon_5(2n+1)q^n = 2\frac{f_5 f_{20}}{f_{10}} \cdot \frac{f_2^2}{f_4},$$

respectively.

$$\sum_{n=0}^{\infty} \upsilon_5'(2n+1)q^n + \sum_{n=0}^{\infty} \upsilon_5(2n+1)q^n = 2\frac{f_5 f_{20}}{f_{10}} \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}.$$

As $q^{2n^2} \not\equiv 1, 4 \pmod 5$, we have $\upsilon_5'(10n+r) = -\upsilon_5(10n+r)$ for $r \in \{4, 8\}$. Then using (1.36), we arrive at (1.38).

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