South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 1 (2024), pp. 339-350

DOI: 10.56827/SEAJMMS.2024.2001.26 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

STUDY ON DIFFERENT TYPES OF CONNECTIONS ON CHAKI-PSEUDO PARALLEL INVARIANT SUBMANIFOLDS OF LORENTZIAN PARA-KENMOTSU MANIFOLD

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(**Received:** *Mar. 06, 2024* **Accepted:** *Apr. 26, 2024* **Published:** *Apr. 30, 2024)*

Abstract: The focus of this research is to investigate Chaki-pseudo parallel submanifolds in Lorentzian para-Kenmotsu manifolds. This study examines the properties of these submanifolds, including their totally geodesic nature under different connections such as the semisymmetric connection, Schouten-van Kampen connection, and Tanaka Webstar connections.

Keywords and Phrases: Lorentzian para-Kenmotsu manifold, invariant submanifold, totally geodesic submanifold, semisymmetric metric connection, Schouten-van Kampen connection and Tanaka Webstar connections.

2020 Mathematics Subject Classification: 53B15, 53C05, 53C17, 53C40.

1. Introduction

In the field of mathematics, invariant submanifolds play a crucial role in understanding the behavior of dynamical systems. An invariant submanifold is a subset of a manifold that remains invariant under the flow of a particular dynamical system. The study of invariant submanifolds can reveal important properties of the system, such as the existence of periodic orbits and the emergence of chaotic behavior. In addition, invariant submanifolds have applications in various areas of science and engineering, including celestial mechanics, fluid dynamics, and control theory ([6], [18]). This topic has a rich history dating back to the pioneering work of Henri Poincare in the late 19^{th} century, and it continues to be an active area of research today. Through its many applications and deep theoretical connections, the study of invariant submanifolds offers insights into the complex behavior of dynamical systems and continues to be of great interest to mathematicians and scientists alike.

Many studies have been carried out by many researchers on invariant submanifolds of different ambient manifolds such as ([1], [22]). Invariant submanifolds are an essential tool when analyzing the behavior of nonlinear autonomous systems, as demonstrated in various studies [11]. Similarly, geodesics is the fundamental concept in the theory of relativity and are frequently used to study various aspects of this field [19]. By applying the theory of invariant submanifolds to these areas, researchers are able to gain a deeper understanding of the underlying mathematical structures influencing the system's behavior. In 2018, a specific class of Lorentzian manifolds known as Lorentzian almost paracontact metric manifolds, or Lorentzian para-Kenmotsu manifolds, was introduced by [12]. The study of invariant submanifolds of these manifolds was further explored by M. Atceken in 2022, where he provided the necessary and sufficient conditions for a Lorentzian para-Kenmotsu manifold to be totally geodesic [2].

Semisymmetric linear connections were originally introduced by Friedmann and Schouten in 1924 [9], and since then, many researchers have studied semisymmetric metric connections in various contexts. The Schouten-van Kampen connection, initially introduced for the study of non-holomorphic manifolds, has also been investigated by several authors [21], [24]. In 2006, Bejancu studied the Schouten-van Kampen connection on foliated manifolds [4], while Olszak focused on an almost contact metric structure [20]. Additionally, in 2018, the concept of Chaki-pseudo parallel invariant submanifolds of Sasakian manifolds was studied by [13]. Motivated by these previously mentioned research papers, we aim to investigate the geodesic properties of Semisymmetric metric connections, the Schouten-van Kampen connection, and the Tanaka-Webster connection on Lorentzian para-Kenmotsu manifolds.

In Section 2, we provide an overview of Lorentzian para-Kenmotsu manifolds and its submanifolds, highlighting their key definitions and properties. In section 3, we examined some results on Chaki-pseudo parallel submanifold of Lorentzian para-Kenmotsu manifold with respect to the Semisymmetric metric connection. In section 4, we studied the results on Chaki-pseudo parallel submanifold of Lorentzian para-Kenmotsu manifold with respect to the Schouten-van Kampen connection. In section 5, we investigated some results on Chaki-pseudo parallel invariant submanifold of Lorentzian para-Kenmotsu manifold with respect to the Tanaka Webster connection. In section 6, we concluded our result.

2. Preliminaries

The study of almost paracontact metric manifolds, which is a natural extension of the study of almost para-Hermitian manifolds, was initiated in [16]. The manifold $\overset{\star}{M}^n(\phi, \xi, \eta, g)$ is considered a Lorentzian almost paracontact manifold [23] if the following conditions hold for all vector fields Ω_1 and Ω_2 on $\overset{\star}{M}^n$:

$$\phi^2\Omega_1 = \Omega_1 + \eta(\Omega_1)\xi, \quad \phi\xi = 0, \tag{2.1}$$

$$g(\phi\Omega_1, \phi\Omega_2) = g(\Omega_1, \Omega_2) + \eta(\Omega_1)\eta(\Omega_2), \tag{2.2}$$

$$\eta(\xi) = -1, \tag{2.3}$$

$$\eta(\Omega_1) = g(\Omega_1, \xi). \tag{2.4}$$

A Lorentzian almost paracontact manifold $\overset{\star}{M}^n(\phi, \xi, \eta, g)$ is classified as a Lorentzian para Kenmotsu manifold($\mathcal{LPKM}$) if the following condition is satisfied for all Ω_1 and Ω_2 in the set of differentiable vector fields $\Gamma(T\overset{\star}{M}^n)$. Here, $\bar{\nabla}$ denotes the Levi-Civita connection($\mathcal{LCC}$),

$$(\bar{\nabla}_{\Omega_1}\phi)\Omega_2 = -g(\phi\Omega_1, \Omega_2)\xi - \eta(\Omega_2)\phi\Omega_1. \tag{2.5}$$

Furthermore, The contact structure is referred to as a K-contact (or paracontact) structure, if ξ is a killing vector field. In this case, the following relationship holds:

$$\nabla_{\Omega_1}\xi = \phi\Omega_1. \tag{2.6}$$

In a $\mathcal{LPKM}$ $\overset{\star}{M}^n(\phi, \xi, \eta, g)$, we have

$$\bar{\nabla}_{\Omega_1}\xi = -\phi^2\Omega_1 = -\Omega_1 - \eta(\Omega_1)\xi, \tag{2.7}$$

$$(\bar{\nabla}_{\Omega_1}\eta)\Omega_2 = -g(\Omega_1, \Omega_2) - \eta(\Omega_1)\eta(\Omega_2). \tag{2.8}$$

Now for the $\mathcal{LPKM}$ $\overset{\star}{M}^n$, M be the immersed submanifold. The tangent and normal subspaces of M in $\overset{\star}{M}$ is represented by $\Gamma(TM)$ and $\Gamma(T^\perp M)$. Then formulae for Gauss and Weingarten are as follows,

$$\bar{\nabla}_{\Omega_1}\Omega_2 = \nabla_{\Omega_1}\Omega_2 + \pi(\Omega_1, \Omega_2), \tag{2.9}$$

$$\bar{\nabla}_{\Omega_1}V = -A_V\Omega_1 + \nabla^\perp_{\Omega_1}V, \tag{2.10}$$

for all $\Omega_1, \Omega_2 \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, the connections on M are denoted by ∇ and $\nabla^\perp$ and $\Gamma(T^\perp M)$. Then we denote second fundamental form($\mathcal{SFF}$) as π and shape opertor as A of M which are related by,

$$g(A_V \Omega_1, \Omega_2) = g(\pi(\Omega_1, \Omega_2), V). \tag{2.11}$$

The covariant derivative π is defined by

$$(\bar{\nabla}_{\Omega_1}\pi)(\Omega_2, \Omega_3) = \nabla^\perp_{\Omega_1}\pi(\Omega_2, \Omega_3) - \pi(\nabla_{\Omega_1}\Omega_2, \Omega_3) - \pi(\Omega_2, \nabla_{\Omega_1}\Omega_3), \tag{2.12}$$

for all $\Omega_1, \Omega_2, \Omega_3 \in \Gamma(TM)$. The $\mathcal{SFF}$ of the submanifolds is said to be parallel if it satisfies $\bar{\nabla}\pi = 0$ [16].

Let M^m, $m < n$, be an invariant submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}{}^n(\phi, \xi, \eta, g)$.

Let $\bar{\nabla}$ (Levi-Civita connection), $\bar{\tilde{\nabla}}$ (Semisymmetric metric connection), $\bar{\hat{\nabla}}$ (Schouten-van Kampen connection) and $\bar{\overset{\star}{\nabla}}$ (Tanaka-Webster connection) are generated by the induced Levi-civita connections $\nabla, \tilde{\nabla}, \hat{\nabla}$ and $\overset{\star}{\nabla}$ on $\mathcal{LPKM}$ $\overset{\star}{M}{}^n$ respectively. For these connections the Gauss formula are [17] given by,

$$\bar{\nabla}_{\Omega_1}\Omega_2 = \nabla_{\Omega_1}\Omega_2 + \pi(\Omega_1, \Omega_2), \tag{2.13}$$

$$\bar{\tilde{\nabla}}_{\Omega_1}\Omega_2 = \tilde{\nabla}_{\Omega_1}\Omega_2 + \tilde{\pi}(\Omega_1, \Omega_2), \tag{2.14}$$

$$\bar{\hat{\nabla}}_{\Omega_1}\Omega_2 = \hat{\nabla}_{\Omega_1}\Omega_2 + \hat{\pi}(\Omega_1, \Omega_2), \tag{2.15}$$

$$\bar{\overset{\star}{\nabla}}_{\Omega_1}\Omega_2 = \overset{\star}{\nabla}_{\Omega_1}\Omega_2 + \overset{\star}{\pi}(\Omega_1, \Omega_2), \tag{2.16}$$

for all $\Omega_1, \Omega_2 \in \Gamma(TM)$ and for the connections $\nabla, \tilde{\nabla}, \hat{\nabla}$ and $\overset{\star}{\nabla}$, the $\mathcal{SFF}$ of M are denoted by $\pi, \tilde{\pi}, \hat{\pi}$, and $\overset{\star}{\pi}$ respectively. Based on the behavior of the almost contact metric structure the classification of invariant and anti-invariant submanifolds are defined. At every point on M if the structure vector field ξ is tangent to M and $\phi\Omega_1$ is tangent to M for any vector field, then the submanifold M of an almost contact metric manifold is considered as invariant. This can be expressed as $\phi(TM) \subset TM$ at every point in M. Now, for invariant submanifolds of a $\mathcal{LPKM}$, the following relation holds (as stated in reference [4]). A submanifold is called totally geodesic($\mathcal{TG}$) if $\pi(\Omega_1, \Omega_2) = 0$, for all Ω_1, $\Omega_2 \in$ TM.

Now, for the invariant submanifolds of Lorentzian para-Kenmotsu manifold following relation hold:

$$\pi(\Omega_1, \xi) = 0, \tag{2.17}$$

A submanifold M of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\bar{\nabla}$ (respectively $\tilde{\bar{\bar{\nabla}}}$, $\hat{\bar{\nabla}}$ and $\overset{\star}{\bar{\bar{\nabla}}}$) is defined as Chaki-pseudo parallel if its $\mathcal{SFF}$ π (respectively $\tilde{\pi}, \hat{\pi}, \overset{\star}{\pi}$) satisfies,

$$(\nabla_{\Omega_1}\pi)(\Omega_2,\Omega_3) = 2\alpha(\Omega_1)\pi(\Omega_2,\Omega_3) + \alpha(\Omega_2)\pi(\Omega_1,\Omega_3) + \alpha(\Omega_3)\pi(\Omega_1,\Omega_2), \quad (2.18)$$

respectively

$$(\tilde{\nabla}_{\Omega_1}\tilde{\pi})(\Omega_2,\Omega_3) = 2\alpha(\Omega_1)\tilde{\pi}(\Omega_2,\Omega_3) + \alpha(\Omega_2)\tilde{\pi}(\Omega_1,\Omega_3) + \alpha(\Omega_3)\tilde{\pi}(\Omega_1,\Omega_2), \quad (2.19)$$

$$(\hat{\nabla}_{\Omega_1}\hat{\pi})(\Omega_2,\Omega_3) = 2\alpha(\Omega_1)\hat{\pi}(\Omega_2,\Omega_3) + \alpha(\Omega_2)\hat{\pi}(\Omega_1,\Omega_3) + \alpha(\Omega_3)\hat{\pi}(\Omega_1,\Omega_2), \quad (2.20)$$

$$(\overset{\star}{\nabla}_{\Omega_1}\overset{\star}{\pi})(\Omega_2,\Omega_3) = 2\alpha(\Omega_1)\overset{\star}{\pi}(\Omega_2,\Omega_3) + \alpha(\Omega_2)\overset{\star}{\pi}(\Omega_1,\Omega_3) + \alpha(\Omega_3)\overset{\star}{\pi}(\Omega_1,\Omega_2), \quad (2.21)$$

for all $\Omega_1, \Omega_2, \Omega_3$ on M, where nowhere vanishing 1-form is denoted by α. π is said to be parallel and M is said to be parallel submanifold of $\overset{\star}{M}$ if $\alpha(\Omega_1) = 0$.

3. Result on Chaki-pseudo parallel invariant submanifold of Lorentzian para-Kenmotsu manifold with semisymmetric metric connection ($\mathcal{SSMC}$)

The semisymmetric metric connection($\mathcal{SSMC}$) $\tilde{\bar{\bar{\nabla}}}$ and $\bar{\nabla}$ on $\overset{\star}{M}^n$ are given by [25],

$$\tilde{\bar{\bar{\nabla}}}_{\Omega_1}\Omega_2 = \bar{\nabla}_{\Omega_1}\Omega_2 + \eta(\Omega_2)\Omega_1 - g(\Omega_1,\Omega_2)\xi. \quad (3.1)$$

Theorem 3.1. *If M is an invariant submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$, then we can establish that M is $\mathcal{TG}$ if and only if M is Chaki-pseudo parallel with respect to the connection $\bar{\nabla}$.*

Proof. If M is Chaki-pseudo parallel invariant submanifold of $\overset{\star}{M}^n$, then by considering the equations (2.12) and (2.18) we get,

$$\begin{aligned} &\nabla^{\perp}_{\Omega_1}(\pi(\Omega_2,\Omega_3)) - \pi(\nabla_{\Omega_1}\Omega_2,\Omega_3) - \pi(\Omega_2,\nabla_{\Omega_1}\Omega_3) \\ &= 2\alpha(\Omega_1)\pi(\Omega_2,\Omega_3) + \alpha(\Omega_2)\pi(\Omega_1,\Omega_3) + \alpha(\Omega_3)\pi(\Omega_1,\Omega_2), \end{aligned} \quad (3.2)$$

putting $\Omega_3 = \xi$ and using (2.17) we compute,

$$\pi(\phi\Omega_1,\Omega_2) = \alpha(\xi)\pi(\Omega_1,\Omega_2). \quad (3.3)$$

Replacing Ω_1 by $\phi\Omega_1$ in (3.3) and using (2.1) we get,

$$\alpha(\xi)\pi(\phi\Omega_1,\Omega_2) = \pi(\Omega_1,\Omega_2). \quad (3.4)$$

We arrive at, $[\alpha(\xi)^2 - 1]\pi(\Omega_1,\Omega_2) = 0$, by using the equations (3.3) and (3.4). which implies that $\pi(\Omega_1,\Omega_2) = 0$ for all Ω_1, Ω_2 on M where $\{\alpha(\xi)\}^2 \neq 1$. Therefore M is

$\mathcal{TG}$ submanifold. The converse is trivial.

Corollary 3.1. *If M is an invariant submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$, it is observed that M becomes $\mathcal{TG}$ if and only if M satisfies parallel condition with respect to $\bar{\nabla}$.*
Proof. By the virtue of references [13], [15], [14], M is $\mathcal{TG}$ if and only if M satisfies parallel condition with respect to $\bar{\nabla}$.

With respect to the connection $\tilde{\bar{\nabla}}$, let us consider M as an invariant submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$. Then we have

Theorem 3.2. [1] *Let M be an invariant submanifolds of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\tilde{\bar{\nabla}}$. Then*

1. *M admits induced $\mathcal{SSMC}$ $\tilde{\nabla}$,*

2. *The second fundamental forms with respect to $\mathcal{LCC}$ and $\mathcal{SSMC}$ are equal, i.e. $\pi = \tilde{\pi}$.*

Proof. Let M be an invariant submanifolds of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\tilde{\bar{\nabla}}$. Let us consider M admits induced $\mathcal{SSMC}$ $\tilde{\nabla}$. By virtue of (3.1), we get

$$\tilde{\bar{\nabla}}_{\Omega_1}\Omega_2 = \bar{\nabla}_{\Omega_1}\Omega_2 + \eta(\Omega_2)\Omega_1 - g(\Omega_1, \Omega_2)\xi. \tag{3.5}$$

By using (2.13) in (3.5), we get

$$\tilde{\bar{\nabla}}_{\Omega_1}\Omega_2 = \nabla_{\Omega_1}\Omega_2 + \pi(\Omega_1, \Omega_2) + \eta(\Omega_2)\Omega_1 - g(\Omega_1, \Omega_2)\xi. \tag{3.6}$$

Now Gauss formula (2.13) with respect to semi-symmetric metric connection is given by,

$$\tilde{\bar{\nabla}}_{\Omega_1}\Omega_2 = \tilde{\nabla}_{\Omega_1}\Omega_2 + \tilde{\pi}(\Omega_1, \Omega_2) \tag{3.7}$$

Equating (3.6) and (3.7), we get (3.1) and

$$\pi(\Omega_1, \Omega_2) = \tilde{\pi}(\Omega_1, \Omega_2). \tag{3.8}$$

Hence the proof.

By virtue of above theorem (3.2) we prove following results:

Theorem 3.3. *Let M be an invariant submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to $\tilde{\bar{\nabla}}$ is $\mathcal{TG}$ if and only if it is Chaki-pseudo parallel with respect to $\mathcal{SSMC}$.*

Proof. Suppose that M is Chakki-pseudo parallel with respect to $\tilde{\tilde{\nabla}}$. By referring to Definition (2.13) and utilizing Theorem (3.2), we obtain the following result:

$$(\tilde{\nabla}_{\Omega_1}\pi)(\Omega_2,\Omega_3) = 2\alpha(\Omega_1)\pi(\Omega_2,\Omega_3) + \alpha(\Omega_2)\pi(\Omega_1,\Omega_3) + \alpha(\Omega_3)\pi(\Omega_1,\Omega_2). \quad (3.9)$$

In view of (3.1) and (2.17) we have from (3.9) that

$$\begin{aligned}&(\nabla_{\Omega_1}\pi)(\Omega_2,\Omega_3) + g(\pi(\Omega_2,\Omega_3),\xi) - g(\Omega_1,\pi(\Omega_2,\Omega_3))\xi\\ &- \eta(\Omega_2)\pi(\Omega_1,\Omega_3) - \eta(\Omega_3)\pi(\Omega_1,\Omega_2) - \eta(\Omega_3)\pi(\Omega_1,\Omega_2)\\ =&2\alpha(\Omega_1)\pi(\Omega_2,\Omega_3) + \alpha(\Omega_2)\pi(\Omega_1,\Omega_3) + \alpha(\Omega_3)\pi(\Omega_1,\Omega_2),\end{aligned}$$

which implies that,

$$\begin{aligned}&\nabla^{\perp}_{\Omega_1}\pi(\Omega_2,\Omega_3) - \pi(\nabla_{\Omega_1}\Omega_2,\Omega_3) + \pi(\Omega_2,\nabla_{\Omega_1}\Omega_3) + g(\pi(\Omega_2,\Omega_3),\xi)\\ &- g(\Omega_1,\pi(\Omega_2,\Omega_3))\xi - \eta(\Omega_2)\pi(\Omega_1,\Omega_3) - \eta(\Omega_3)\pi(\Omega_1,\Omega_2)\\ =&2\alpha(\Omega_1)\pi(\Omega_2,\Omega_3) + \alpha(\Omega_2)\pi(\Omega_1,\Omega_3) + \alpha(\Omega_3)\pi(\Omega_1,\Omega_2).\end{aligned} \quad (3.10)$$

Putting $\Omega_3 = \xi$ in above equation and using (2.17) we get,

$$-\pi(\Omega_2,\nabla_{\Omega_1}\xi) - \pi(\Omega_1,\Omega_2) = \alpha(\xi)\pi(\Omega_1,\Omega_2), \quad (3.11)$$

By virtue of (2.6), we have from above equation that

$$\pi(\Omega_2,\phi\Omega_1) + \{\alpha(\xi)+1\}\pi(\Omega_1,\Omega_2) = 0, \quad (3.12)$$

Replacing Ω_1 by $\phi\Omega_1$ in above equation and using (2.1) and (2.17) we get

$$\pi(\Omega_2,\Omega_1) + \{\alpha(\xi)+1\}\pi(\phi\Omega_1,\Omega_2) = 0, \quad (3.13)$$

From (3.13) and (3.12) we get, $[\{\alpha(\xi)+1\}^2 - 1]\pi(\Omega_1,\Omega_2) = 0$. Which implies that $\pi(\Omega_1,\Omega_2) = 0$ as $\{\alpha(\xi)+1\}^2 \neq 1$. The converse part is also trivial.

4. Chaki-pseudo parallel invariant submanifold of Lorentzian Para - Kenmotsu manifold with Schouten-van Kampen connection($\mathcal{SVKC}$)

The Schouten-van Kampen connection($\mathcal{SVKC}$) $\hat{\hat{\nabla}}$ and $\bar{\nabla}$ of $\overset{\star}{M}{}^n$ are related by [10] and [26],

$$\hat{\hat{\nabla}}_{\Omega_1}\Omega_2 = \bar{\nabla}_{\Omega_1}\Omega_2 + \eta(\Omega_2)\Omega_1 - g(\Omega_1,\Omega_2)\xi. \quad (4.1)$$

We consider M as an invariant submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}{}^n$ with respect to the $\mathcal{SVKC}$ $\hat{\hat{\nabla}}$. We prove the following result:

Theorem 4.1. *Let M be a invariant submanifold of the $\mathcal{LPKM}$ $\overset{\star}{M}^n$ using the $\hat{\bar{\nabla}}$ connection. Consequently, M possesses an induced $\mathcal{SVKC}$ denoted by $\hat{\nabla}$. Moreover, the $\mathcal{SFF}$ computed using both the $\mathcal{LCC}$ and the $\mathcal{SVKC}$ are identical, denoted by π and $\hat{\pi}$ respectively, i.e., $\pi = \hat{\pi}$.*

Proof. By virtue of (4.1), (2.13) and (2.15) we have

$$\hat{\nabla}_{\Omega_1}\Omega_2 + \hat{\pi}(\Omega_1, \Omega_2) = \nabla_{\Omega_1}\Omega_2 + \pi(\Omega_1, \Omega_2) + \eta(\Omega_2)\Omega_1 - g(\Omega_1, \Omega_2)\xi. \tag{4.2}$$

for $\Omega_1, \Omega_2 \in \Gamma(TM)$.

Now by equating tangential and normal components of (4.2) we have

$$\hat{\bar{\nabla}}_{\Omega_1}\Omega_2 = \nabla_{\Omega_1}\Omega_2 + \eta(\Omega_2)\Omega_1 - g(\Omega_1, \Omega_2)\xi, \tag{4.3}$$

and

$$\hat{\pi}(\Omega_1, \Omega_2) = \pi(\Omega_1, \Omega_2). \tag{4.4}$$

Theorem 4.2. *If M be a invariant submanifold of $\overset{\star}{M}^n$ with respect to the $\mathcal{SVKC}$ $\hat{\bar{\nabla}}$, then M is said to be $\mathcal{TG}$ if and only if it is Chaki-pseudo parallel with respect to the same connection $\hat{\bar{\nabla}}$.*

Proof. Let M be the Chaki-pseudo parallel with respect to Schouten-van Kampen connection. By using the Theorem (2.17) and Definition (2.13) and we have,

$$(\hat{\nabla}_{\Omega_1}\hat{\pi})(\Omega_2, \Omega_3) = 2\alpha(\Omega_1)\pi(\Omega_2, \Omega_3) + \alpha(\Omega_2)\pi(\Omega_1, \Omega_3) + \alpha(\Omega_3)\pi(\Omega_1, \Omega_2). \tag{4.5}$$

Putting $\Omega_3 = \xi$ in (4.5) and using (2.17), (4.3) and (4.4) we compute.

$$\alpha(\xi)\pi(\Omega_1, \Omega_2) = 0.$$

We can see that $\pi(\Omega_1, \Omega_2) = 0$ as $\alpha(\xi) \neq 0$. Therefore M is $\mathcal{TG}$. The converse part also becomes trivial.

Corollary 4.1. *If M is an invariant submanifold of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$, then M becomes $\mathcal{TG}$ if and only if it satisfies parallel condition with respect to the $\mathcal{SVKC}$ $\hat{\bar{\nabla}}$.*

5. Chaki-pseudo parallel invariant submanifolds of Lorentzian para-Kenmotsu manifold using Tanaka Webster connection($\mathcal{TWC}$)

The Tanaka-Webster connection($\mathcal{TWC}$) $\overset{\star}{\bar{\nabla}}$ and $\bar{\nabla}$ of $\overset{\star}{M}^n$ are related by [7] and [8],

$$\overset{\star}{\bar{\nabla}}_{\Omega_1}\Omega_2 = \bar{\nabla}_{\Omega_1}\Omega_2 + \eta(\Omega_1)\phi\Omega_2 + \eta(\Omega_2)\Omega_1 - g(\Omega_1, \Omega_2)\xi. \tag{5.1}$$

Theorem 5.1. *Consider a invariant submanifold M of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$, using the $\mathcal{TWC}$. In this case, the following properties hold:*

1. *The induced $\mathcal{TWC}$, denoted by $\overset{\star}{\nabla}$, exists on M.*
2. *The $\mathcal{SFF}$ of M with respect to the $\mathcal{LCC}$ and the $\mathcal{TWC}$ are equivalent, that is, $\pi = \overset{\star}{\pi}$.*

Proof. By virtue of (5.1), (2.13) and (2.16) we have

$$\overset{\star}{\bar{\nabla}}_{\Omega_1}\Omega_2 = \nabla_{\Omega_1}\Omega_2 + \eta(\Omega_1)\phi\Omega_2 - \eta(\Omega_2)\phi\Omega_1 - g(\phi\Omega_1, \Omega_2)\xi, \tag{5.2}$$

and

$$\overset{\star}{\pi}(\Omega_1, \Omega_2) = \pi(\Omega_1, \Omega_2). \tag{5.3}$$

Hence the proof.

Now we derive following results:

Theorem 5.2. *Consider an invariant submanifold M of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with respect to the $\mathcal{TWC}$ $\overset{\star}{\bar{\nabla}}$. Then M is totally geodesic if and only if M is Chaki-pseudo parallel with respect to $\overset{\star}{\bar{\nabla}}$.*

Proof. Suppose that M is Chakki-pseudo parallel with respect to $\mathcal{TWC}$. Then by virtue of Definition (2.13) and (2.17) we have

$$(\overset{\star}{\nabla}_{\Omega_1}\overset{\star}{\pi})(\Omega_2, \Omega_3) = 2\alpha(\Omega_1)\pi(\Omega_2, \Omega_3) + \alpha(\Omega_2)\pi(\Omega_1, \Omega_3) + \alpha(\Omega_3)\pi(\Omega_1, \Omega_2). \tag{5.4}$$

In view of (2.17), (5.2) and (5.3), putting $\Omega_3 = \xi$ we have from (5.4)

$$2\pi(\phi\Omega_1, \Omega_2) = \alpha(\xi)\pi(\Omega_1, \Omega_2). \tag{5.5}$$

Replacing Ω_1 by $\phi\Omega_1$ and using (2.1) we compute

$$\alpha(\xi)\pi(\phi\Omega_1, \Omega_2) = 2\pi(\Omega_1, \Omega_2). \tag{5.6}$$

From (5.5) and (5.6) we get $[\alpha(\xi)^2 - 4]\pi(\Omega_1, \Omega_2) = 0$. which imply $\pi(\Omega_1, \Omega_2) = 0$, if $\{\alpha(\xi)\}^2 \neq 4$, i.e. M is $\mathcal{TG}$. The converse is also holds trivial.

Corollary 5.1. *Consider an invariant submanifold M of a $\mathcal{LPKM}$ $\overset{\star}{M}^n$ with* *respect to the $\mathcal{TWC}$ $\overset{\star}{\bar{\bar{\nabla}}}$. Then M is totally geodesic if and only if M is parallel with respect to $\overset{\star}{\bar{\bar{\nabla}}}$.*

Proof. By the virtue of references [13], [15], [14], M is totally geodesic if and only if M is parallel with respect to $\overset{\star}{\bar{\bar{\nabla}}}$.

6. Conclusion

The study of Chaki-pseudo parallel submanifolds is a relatively good concept in differential geometry. Our findings in this paper further contribute to the understanding of this concept in $\mathcal{LPKM}$. We have investigated Chaki-pseudo parallel invariant submanifolds of $\mathcal{LPKM}$ using several different connections, including the Levi-Civita, Semisymmetric metric, Schouten-van Kampen, and Tanaka Webstar connections. Our main finding is that an invariant submanifold of a $\mathcal{LPKM}$ is Chaki-pseudo parallel if and only if it is $\mathcal{TG}$ with respect to any of these connections.

Additionally, we observed that an invariant submanifold is $\mathcal{TG}$ if and only if it is parallel. Therefore, we can conclude that the concepts of Chaki-pseudo parallel and parallel submanifolds are equivalent in terms of the $\mathcal{TG}$ property across different connections. $\mathcal{LPKM}$ pave the way for future research in this area. In particular, it would be interesting to investigate the properties of Chaki-pseudo parallel submanifolds in other types of manifolds and with respect to other connections.

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