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ANALYSIS OF NEWTONIAN AND NON-NEWTONIAN BLOOD FLOW THROUGH MULTIPLE STENOSES IN NARROW ARTERY

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Abstract: This study presents a mathematical model aimed at investigating the effect of blood flow parameters in a narrow artery with multiple stenoses. Blood is modelled as a non-Newtonian Kuang-Luo (K-L) fluid. Numerical expressions for blood flow characteristics including blood flow rate, skin friction and resistance to blood flow have been derived. These expressions have been solved using MATLAB R2022a software and analysed graphically. The flow properties exhibit distinct behaviours in response to changes in plasma viscosity and yield stress. Furthermore, it was noted that variation in skin friction is more pronounced in Newtonian fluid compared to non-Newtonian K-L fluid. Additionally, the effect of yield stress on all flow quantities is comparatively lesser than that of plasma viscosity. The findings of this study have been validated against existing models.

Keywords and Phrases: K-L fluid model, narrow artery, blood flow, multiple stenoses.

2020 Mathematics Subject Classification: 92B05, 76Z05.

1. Introduction

Cardiovascular diseases such as coronary arterial disease, cerebrovascular disease and peripheral artery disease are among the deadliest diseases worldwide. These diseases occurs as a result of blockage in one or multiple arteries, leading to restricted blood flow to the brain or heart. Stenosis is a condition characterised by narrowing and hardening of the arteries due to the accumulation of cholesterol, fat, and other substances in their walls. When stenosis occurs, the affected artery cannot supply enough blood to the heart and surrounding tissues. Among various cardiovascular diseases, stenosis play a significant role in impeding blood flow within the arteries [13].

In the past, extensive research has been conducted to gain a comprehensive understanding of blood flow within cardiovascular system, with particular attention given to the challenges posed by stenosed arteries. These arteries experience blood flow that can be described as both Newtonian and non-Newtonian fluid, their behaviour computed using the fundamental Navier-Stokes equation. Young [16] investigated the impact of time-dependent stenosis on flow through a tube, providing an approximate solution for mild stenosis. Shukla et al. [5] explored the effects of stenosis height on resistance to flow and wall shear stress, observing that both characteristics increase with higher stenosis heights for power-law and Casson-model of non-Newtonian fluid. Further, Misra et al. [5, 6] conducted a study on blood flow through multiple stenoses and analysed pathological data.

In recent years, numerous researchers have focused on exploring various non-Newtonian models, including Kuang-Luo, power-law, Herschel-Bulkley and Casson fluid models, resulting in significant advancements in the field of cardiovascular diseases [9,15]. Building upon the work of Srivab [13], Bali and Gupta [1] extended the study of blood flow in non-symmetrical stenosed narrow artery in the presence of K-L fluid model. The K-L fluid model is an improved model compared to the Casson and Herschel-Bulkley models because these models have only one parameter yield stress and power law index respectively, whereas the K-L fluid model has one more parameter plasma viscosity including yield stress. Kumar et al. [2,3] investigated porous and two layered elastic stenosis arteries using non-Newtonian viscoelastic fluid and analysing the impact of stenosis on wave propagation and filtration velocity of tissue fluid. Shah [10] developed a three layered stenosed model, considering arterial blood as Casson's and Bingham's plastic fluid. Owasit and Sriyab [8] formulated a mathematical model for the non-Newtonian power-law fluid through two different types of stenoses, namely bell and cosine shapes, observing slightly varied behaviour in the combined geometry. Nadeem et al. [7] examined four distinct stenosis formulations, namely triangular, trapezoidal, overlapping (wshape) and composite formations, in elliptical cross-sectional artery. Meanwhile, Ponalagusamy and Manchi [9] analysed six types of mild stenosis, including symmetric, triangular, trapezoidal, elliptical, bell-shaped and composite formations. They also considered plasma layer as a Newtonian fluid and core region as a K-L non-Newtonian fluid, obtaining analytical expression for the blood velocity in the various stenotic regions. Singh and Kumar [12] considered Newtonian fluid in an inclined artery and predicted blood flow parameters in the presence of multiple stenoses.

In previous research studies, researchers discussed only the effect of yield stress and plasma viscosity on non-Newtonian fluids, but the effect of plasma viscosity on Newtonian fluids was not previously addressed. In our work, we have developed a mathematical model to investigate the effect of plasma viscosity in both non-Newtonian and Newtonian cases. Additionally, we analyse the effect of yield stress and plasma viscosity on blood flow characteristic in a narrow artery with newly constructed stenoses. Blood is modelled as K-L fluid because Kuang and Luo [4] suggested that one of the best model to demonstrate blood flow with the main properties plasma viscosity and yield stress which play an important role in blood flow. These parameters make this fluid remarkably similar to blood. Numerical expressions are derived to evaluate important aspects of blood flow, including the blood flow rate, resistance to blood flow and skin friction.

2. Mathematical Formulation

Consider an axially symmetric artery having multiple stenoses, with cylindrical polar coordinates (r, θ, z) . The blood is considered as a non-Newtonian K-L fluid with a constant density and viscosity and blood flow is taken as laminar and steady. Further, considering the radius of the artery in stenosed and normal region is R(z)and R_0 respectively. δ_1 and δ_2 are the height of the first stenosis and second stenosis respectively. d and l_0 are the lengths of the first non-stenosis segment and mid of the second stenosis in rigid tube respectively. L_1 and L_2 are the lengths of the first and second stenosis segments. L is considered the total length of the arterial segment.





Figure 2: Stenosed geometry for different values of stenosis height (δ)

Schematic diagram of the problem represented by Fig. 1 and geometry of the artery for various height is depicted through the Fig. 2. Mathematical expression of that stenosed artery described by Eq. (1). To the best of our knowledge the geometry of the stenosis has not been considered before.

$$\frac{R(z)}{R_0} = \begin{cases}
1 - \frac{2\delta_1}{R_0 L_1} (z - d), & d \le z \le d + \frac{L_1}{2} \\
1 - \frac{\delta_1}{2R_0} \left[1 + \cos \frac{2\pi}{5R_0} \left(z - d - \frac{L_1}{2} \right) \right], & d + \frac{L_1}{2} \le z \le d + L_1 \\
1 - \frac{2\delta_2}{R_0 L_2} \left(z - l_0 + \frac{L_2}{2} \right), & l_0 - \frac{L_2}{2} \le z \le l_0 \\
1 - \frac{\delta_2}{2R_0} \left[1 + \cos \frac{2\pi}{5R_0} (z - l_0) \right], & l_0 \le z \le l_0 + \frac{L_2}{2} \\
1, & \text{otherwise}
\end{cases}$$
(1)

In general, \bar{v}_z , \bar{v}_r and \bar{v}_θ are the three components of the velocity along the z, r and θ respectively. In the case of axisymmetric flow, \bar{v}_θ will be zero, therefore the governing equations for the axial and radial components with the equation of continuity in the cylindrical coordinate system are:

$$\frac{\partial \bar{v}_z}{\partial \bar{z}} + \frac{\bar{v}_r}{\bar{r}} + \frac{\partial \bar{v}_r}{\partial \bar{r}} = 0 \tag{2}$$

$$\frac{\partial \bar{v}_z}{\partial \bar{t}} + \bar{v}_z \frac{\partial \bar{v}_z}{\partial \bar{z}} + \bar{v}_r \frac{\partial \bar{v}_z}{\partial \bar{r}} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial \bar{z}} + \frac{\mu}{\rho} \left(\frac{\partial^2 \bar{v}_z}{\partial \bar{r}^2} + \frac{1}{r} \frac{\partial \bar{v}_z}{\partial \bar{r}} + \frac{\partial^2 \bar{v}_z}{\partial \bar{z}^2} \right)$$
(3)

$$\frac{\partial \bar{v}_r}{\partial \bar{t}} + \bar{v}_z \frac{\partial \bar{v}_r}{\partial \bar{z}} + \bar{v}_r \frac{\partial \bar{v}_r}{\partial \bar{r}} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial \bar{r}} + \frac{\mu}{\rho} \left(\frac{\partial^2 \bar{v}_r}{\partial \bar{r}^2} + \frac{1}{r} \frac{\partial \bar{v}_r}{\partial \bar{r}} + \frac{\partial^2 \bar{v}_r}{\partial \bar{z}^2} - \frac{\bar{v}_r}{\bar{r}^2} \right) \tag{4}$$

After using above assumptions Eqs. (2) - (4) reduce to:

$$\bar{v}_z = \bar{v} \tag{5}$$

$$-\frac{\partial \bar{p}}{\partial \bar{z}} + \mu \left(\frac{\partial^2 \bar{v}}{\partial \bar{r}^2} + \frac{1}{r} \frac{\partial \bar{v}}{\partial \bar{r}} \right) = 0 \tag{6}$$

$$-\frac{\partial \bar{p}}{\partial \bar{r}} = 0 \tag{7}$$

The boundary conditions are:

$$\bar{\tau}$$
 is finite at $\bar{r} = 0$ (8)

$$\bar{v} = 0$$
 at $\bar{r} = R_0$ (9)

$$\bar{v} = 0 \quad \text{at} \quad \bar{r} = R(\bar{z}) \tag{10}$$

Now, we introduce following non-dimensional quantities:

$$Z = \frac{\bar{z}}{L}, v = \frac{\bar{v}_z}{V}, L_{i(1,2)} = \frac{\bar{L}_i}{L}, l_0 = \frac{\bar{l}_0}{L}, \delta_{i(1,2)} = \frac{\bar{\delta}_{i(1,2)}}{R_0}, r = \frac{\bar{r}}{R_0}$$
(11)

The non-Newtonian behaviour of the blood is expressed by K - L fluid model [Luo and Kuang (4)] i.e. a relationship between shear rate and shear stress, which is defined as follows:

$$\begin{aligned} \tau &= \tau_y + \eta_2 \dot{\gamma}^{\frac{1}{2}} + \eta_1 \dot{\gamma} \quad \tau \ge \tau_y \\ \dot{\gamma} &= 0 \qquad \tau < \tau_y \end{aligned}$$
 (12)

where τ_y, η_1, η_2 and $\dot{\gamma}$ are yield stress, plasma viscosity, other chemical substance and shear rate respectively.

3. Solution to the Problem

After simplification of Eq. (6) we can write it as

$$-\frac{\partial p}{\partial z} = \frac{1}{r}\frac{d}{dr}(r\tau) \tag{13}$$

where τ and p is the wall shear stress or skin friction and pressure term respectively. Solve Eq. (13) for skin friction (τ) and using boundary conditions from Eqs. (8) - (10), skin friction in stenosed artery obtained as:

$$\tau_s = -\frac{R}{2} \frac{dp}{dz} \tag{14}$$

Solving Eq. (12) of K - L fluid model for shear rate, we obtained:

$$\dot{\gamma}(\tau) = \begin{cases} \frac{1}{2\eta_1^2} \left(\eta_2^2 - 2\eta_1(\tau_y - \tau) - \eta_2 \sqrt{\eta_2^2 - 4(\tau_y - \tau)\eta_1} \right); & \tau_y \le \tau \\ 0; & \tau_y > \tau \end{cases}$$
(15)

Blood flow rate (Q) is defined by Rabinowitsch equation:

$$Q = \frac{\pi R^3}{\tau_s^3} \int_{\tau=0}^{\tau=\tau_s} \dot{\gamma}(\tau) \tau^2 d\tau \tag{16}$$

Substituting Eq. (15) into Eq. (16), blood flow rate of non-Newtonian blood in stenosed artery defined as follows:

$$Q_s = \frac{\pi R^3}{\tau_s^3} \int_{\tau=0}^{\tau=\tau_s} \frac{1}{2\eta_1^2} \left(\eta_2^2 - 2\eta_1(\tau_y - \tau) - \eta_2 \sqrt{\eta_2^2 - 4(\tau_y - \tau)\eta_1} \right) \tau^2 d\tau \qquad (17)$$

Integrating Eq. (17) with respect to τ and simplifying it became as:

$$Q_{s} = \frac{\pi R^{3}}{4\eta_{1}^{2}} \left[\frac{2\eta_{2}^{2}}{3} - \frac{\eta_{1}}{3} (4\tau_{y} - 3\tau_{s}) - \frac{2\eta_{2}}{\sqrt{\eta_{1}}} \left\{ \frac{4\eta_{1}}{7} \sqrt{\tau_{s}} + \frac{2}{5} \left(\frac{4\eta_{1}\tau_{y} - \eta_{2}^{2}}{\sqrt{\tau_{s}}} \right) + \frac{1}{12} \left(\frac{4\eta_{1}\tau_{y} - \eta_{2}^{2}}{\eta_{1}\sqrt{\tau_{s}^{3}}} \right) \right\} \right]$$
(18)

Since $\frac{1}{\sqrt{\tau_s}} \ll 1$ and $\frac{1}{\sqrt{\tau_s^3}} \ll 1$ as Sriyab [13], Eq. (18) can be written as:

$$Q_s = \frac{\pi R_0^3}{4\eta_1^2} \left[\frac{2\eta_2^2}{3} - \frac{\eta_1}{3} (4\tau_y - 3\tau_s) - \frac{8\sqrt{\eta_1} \times \eta_2}{7} \sqrt{\tau_s} \right] \left(\frac{R}{R_0} \right)^3 \tag{19}$$

Now, Eq. (19) solved for skin friction (τ_s) of non-Newtonian blood in stenosed artery is obtained as:

$$\tau_s = \frac{1}{2}g^2 - f(R) + g\sqrt{\frac{1}{4}g^2 - f(R)}$$
(20)

where $g = \frac{8}{7} \frac{\eta_2}{\sqrt{\eta_1}}$ and $f(R) = \frac{2}{3\eta_1} (\eta_2^2 - 2\eta_1 \tau_y) - \frac{4\eta_1 Q_s}{\pi R^3}$. Skin friction of Newtonian blood in stenosed artery is defined as:

$$\tau_s^{N_e} = \frac{1}{2}g^2 - f_1(R) + g\sqrt{\frac{1}{4}g^2 - f_1(R)}$$
(21)

where $f_1(R) = \frac{2\eta_2^2}{3\eta_1} - \frac{4\eta_1 Q_s}{\pi R^3}$.

Resistance of flow (λ) is defines as the ratio of pressure difference to the blood flow rate:

$$\lambda = \frac{\Delta P}{Q} \tag{22}$$

Using Eq. (20) into Eq. (14) we obtained:

$$-\frac{dp}{dz} = \frac{1}{R} \left(g^2 - 2f(R) + 2g\sqrt{\frac{1}{4}g^2 - f(R)} \right)$$
(23)

Integrating Eq. (23) with respect to z, we get pressure difference ΔP along the total arterial length L:

$$\Delta P = \frac{g^2}{R_0} \int_0^L \frac{1}{(R/R_0)} dz - \frac{2}{R_0} \int_0^L \frac{f(R)}{(R/R_0)} dz + \frac{g}{R_0} \int_0^L \frac{\sqrt{g^2 - 4f(R)}}{(R/R_0)} dz \quad (24)$$

The value of ΔP from Eq. (24) substituting into Eq. (22), we obtained resistance of blood flow for non-Newtonian fluid model in stenosed artery:

$$\lambda_s = \frac{1}{R_0 Q_s} [g^2 I_1 - 2I_2 + gI_3]$$
(25)

where $I_1 = \int_0^L \frac{1}{(R/R_0)} dz$, $I_2 = \int_0^L \frac{f(R)}{(R/R_0)} dz$ & $I_3 = \int_0^L \frac{\sqrt{g^2 - 4f(R)}}{(R/R_0)} dz$. Resistance of blood flow for Newtonian fluid model in stenosed artery, we put $\tau_y = 0$ into the Eq. (25) and written as:

$$\lambda_s^{N_e} = \frac{1}{R_0 Q_s} [g^2 I_1 - 2I_4 + gI_5]$$
(26)

where $I_4 = \int_0^L \frac{f_1(R)}{(R/R_0)} dz \& I_5 = \int_0^L \frac{\sqrt{g^2 - 4f_1(R)}}{(R/R_0)} dz$. Substitute the values of R/R_0 , f(R) and $f_1(R)$ into the I_1 , I_2 , I_3 , $I_4 \& I_5$ and solved all integrations numerically through the Gauss-Kronrod quadrature formula by using MATLAB R2022a.

4. Numerical Results and Discussions

Our study aims to explore the effect of plasma viscosity and yield stress on blood flow rate, skin friction and resistance to blood flow. These flow quantities,

Parameters	Numerical values
Blood flow (Q)	$0-5 \ cm^3/s$
Artery length (L)	$5 \ cm$
Stenosis height (δ)	$0 - 0.01 \ cm$
Radius of the artery (R_0)	$0.01 \ cm$

Table 1: Range of the parameters

expressed by Eqs. (19), (20), (21), (25) and (26) evaluated numerically by using MATLAB R2022a software for the various parametric values (Sriyab [13]) provided in Tab. 1.

The geometry of the problem for different stenoses height has been shown in Fig. 2. It is observed, radius of stenosis region (R/R_0) decrease as the height of stenoses increases. The effects of various parameters are visually presented in Figs. 3-10.

The effects of plasma viscosity (η_1) and yield stress (τ_y) on blood flow rate in stenosed artery (Q_s) with axial distance (z) are shown in Figs. 3 and 4 respectively. Fig. 3 exhibits that blood flow rate decreases significantly with the increase of plasma viscosity and Fig. 4 depicts that blood flow rate decreases slightly with the increase of yield stress. We have observed a significant impact of plasma viscosity on blood flow rate. We conclude that influence of yield stress on blood flow rate aligns with predictions from the Herschel-Bulkley, Casson and K - L models.



The variation in skin friction in stenosed artery (τ_s) with axial distance (z) for the different value of plasma viscosity (η_1) and yield stress (τ_y) have been shown in the Figs. 5 and 6. In Fig. 5 noticed that skin friction decrease vaguely as increases plasma viscosity. Fig. 6 exhibits the impact of yield stress on skin friction. We

have observed that it increases as yield stress increases. Fig. 7 shows the effects of plasma viscosity on skin friction in a stenosed artery for Newtonian blood $(\tau_s^{N_e})$. It is observed that $\tau_s^{N_e}$ decrease significantly as plasma viscosity increases, but the variation is more pronounced in Newtonian fluid compared to non-Newtonian K - L fluid, as depicted in the Figs. 5 and 7.



Fig. 8 and 9 shows the variation of resistance to blood flow along with stenosis height $(\delta/(R_0))$ for different values of plasma viscosity and yield stress. All these figures exhibit that the resistance to blood flow increase significantly with the increase of stenosis height $(\delta/(R_0))$ and It is observed that the height of stenoses plays a significant role in resistance to blood flow with both parameters. It is also noticed that resistance to blood flow increases slowly with the increase of plasma viscosity but no significant change reported for yield stress. The effect of stenosis

height $(\delta/(R_0))$ on resistance to blood flow in stenosed artery for Newtonian blood $(\lambda_s^{N_e})$ is shown in the Fig. 10 for different values of plasma viscosity. It is observed that $\lambda_s^{N_e}$ decreases slowly as increases η_1 with the stenosis height. In Figs. 8 and 10 our analysis reveals that under same parametric values, the variation in resistance to blood flow is greater in non-Newtonian K - L fluid model than the Newtonian fluid model. The results obtained from the present study provide valuable insights



into the blood flow rate, skin friction and resistance to blood flow. These results are shown above and compared with a previous study conducted by Sriyab [13, 14], Bali and Gupta [1], which analysed the steady flow of non-Newtonian blood in narrow artery. The comparison of these results indicates that the trend of the graph obtained for Herschel-Bulkley and Casson fluid models.

5. Conclusion

Blood flow through multiple stenoses in a narrow artery has been studied in this research. Blood is considered a non-Newtonian K-L fluid with no-slip conditions on the arterial wall. Important blood flow characteristics such as blood flow rate, skin friction and resistance to blood flow have been investigated for the various blood flow parameters, namely plasma viscosity, yield stress and stenosis height. The numerical results are compared with the results of Sriyab [13] and also with the results of Bali and Gupta [1]. The main findings of our work are as follow:

- The height of the stenosis is an important parameter in blood flow problems.
- Blood flow rate decreases as plasma viscosity and yield stress increase, and it follows arterial shape.

- Variation in skin friction is more pronounced if blood behaves as a Newtonian fluid compared to a non-Newtonian K-L fluid.
- The effect of yield stress on blood flow rate, skin friction and resistance to blood flow are less as compared to plasma viscosity.

Our study provides insight into the influence of plasma viscosity, yield stress and stenosis height in both non-Newtonian K-L fluid and Newtonian fluid. These findings suggest that the numerical simulations used in the present study are consistent and underscore the potential for numerical simulations to advance our understanding of blood flow in complex geometries.

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