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A PROPOSED NUMERICAL METHOD FOR SOLVING REAL-LIFE MODELS: DEVELOPMENT AND ANALYSIS

S. E. Fadugba, D. K. Okyere* and M. C. Kekana**

Department of Physical Sciences,
Landmark University, Omu-Aran, NIGERIA

E-mail : fadugba.sunday@lmu.edu.ng

*African Institute for Mathematical Sciences, GHANA

E-mail : dokyere@aims.edu.gh

**Tshwane University of Technology,
Arcadia Campus, Pretoria, SOUTH AFRICA

E-mail : kekanamc@tut.ac.za

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Abstract: In this paper, the development and analysis of a proposed method for solving physical models arising from real-life scenarios are presented. The proposed method is derived via the transcendental function of exponential type, examined and studied for its properties. In addition, the effectiveness of the method is evaluated by applying it to three numerical examples that originated from real-world scenarios. Moreover, this study presents a comparison of the outcomes generated by the proposed method and the existing method, in the context of the exact solution. The study concludes that the proposed method solves real life problems with the expected level of accuracy and, therefore, can be considered among the numerous methods that are appropriate and suitable for solving first-order initial value problems (IVPs).

Keywords and Phrases: Accuracy, consistency, convergence, initial value problem, local truncation error, stability.

2020 Mathematics Subject Classification: 34A12, 65L05, 65L20, 65L70.

1. Introduction

Differential equations are mathematical representations of the relationship between a quantity's current value and its rate of change [4]. Differential equations can be classified as ordinary or partial. Partial differential equations explain the behavior of functions of many variables, whereas ordinary differential equations deal with functions of a single variable [6]. To correctly solve the equations, a variety of approaches, including analytical, numerical, and qualitative ones, are used in the study of differential equations. It is widely recognized that a large proportion of differential equations that represent real-world problems cannot be solved using analytical methods, and an alternative is to use numerical method to get approximate solutions of the differential equations. The development and analysis of new numerical methods for solving ordinary differential equations (ODEs) is an active area of research, and there is ongoing work towards improving the accuracy, stability, and efficiency of these methods. In this regard, numerous algorithms have been suggested in scholarly works, taking into consideration the characteristics and specific form of the differential equations such as

$$y'(x) = f(x, y), y(x_0) = y_0, x \in [a, b] \ , y \in (-\infty, \infty) \ . \tag{1.1}$$

that need to be solved. Examples of these algorithms can be found in literature [1, 2, 3, 5] - [7 - 21] just to mention few. In this paper, we assessed the effectiveness of a new numerical method involving exponential transcendental functions for solving first-order initial value problems in ordinary differential equations. What follows is a summary of the remaining sections of this study. The second section outlines how the proposed method was derived, while the third section examines the properties of the method. The fourth section includes numerical examples and a discussion of the results, and the fifth section provides a conclusion for the paper.

2. Development of the Proposed Numerical Method

Let us examine an interpolating function with a particular form

$$F(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 e^{-4x} \ , \tag{2.1}$$

for the solution of the initial value problem (1.1), where α_0, α_1, α_2, α_3 represent constants that are not specified. The range of values $[a, b]$ over which the integration is performed is defined as $a = x_0 \leq x \leq x_n = b$. We define the step size as

$$h = \frac{b-a}{N} \ . \tag{2.2}$$

We define the mesh point as

$$x_n = x_0 + nh, n = 1, 2, ..., N \ , x_{n+1} = x_0 + (n+1)h, n = 0, 1, .., N-1 \ . \tag{2.3}$$

From (2.3), setting $x_0 = 0$, we have that:

$$x_n = nh, n = 1, 2, ..., N \ , x_{n+1} = (n+1)h, n = 0, 2, ..., N-1 \ . \tag{2.4}$$

Substituting $x = x_n$ and $x = x_{n+1}$ into (2.1) yields, respectively

$$F(x_n) = \alpha_0 + \alpha_1 x_n + \alpha_2 x_n^2 + \alpha_3 e^{-4x_n} \ , \tag{2.5}$$

and

$$F(x_{n+1}) = \alpha_0 + \alpha_1 x_{n+1} + \alpha_2 x_{n+1}^2 + \alpha_3 e^{-4x_{n+1}} \ . \tag{2.6}$$

Differentiating (2.5) three times with respect to x_n, gives

$$\begin{aligned} F'(x_n) &= f_n = \alpha_1 + 2\alpha_2 x_n - 4\alpha_3 e^{-4x_n} \\ F''(x_n) &= f_n^{(1)} = 2\alpha_2 + 16\alpha_3 e^{-4x_n} \\ F'''(x_n) &= f_n^{(2)} = -64\alpha_3 e^{-4x_n} \ . \end{aligned} \tag{2.7}$$

Solving (2.7), yields

$$\alpha_1 = f_n - x_n f_n^{(1)} - \frac{1}{16}(1+4x_n) f_n^{(2)} \ , \ \ \alpha_2 = \frac{1}{2} f_n^{(1)} + \frac{1}{8} f_n^{(2)} \quad \alpha_3 = -\frac{1}{64} e^{4x_n} f_n^{(2)} \ . \tag{2.8}$$

Subtracting (2.5) from (2.6), we obtain

$$F(x_{n+1}) - F(x_n) = \alpha_1(x_{n+1} - x_n) + \alpha_2(x_{n+1}^2 - x_n^2) + \alpha_3(e^{-4x_{n+1}} - e^{-4x_n}) \ . \tag{2.9}$$

Substituting (2.4) and (2.8) into (2.9), we get

$$\begin{aligned} F(x_{n+1}) - F(x_n) &= \frac{1}{64}(1 - e^{-4h}) f_n^{(2)} + \frac{1}{16} h (16 f_n - f_n^{(2)}) \\ &+ \frac{1}{8} h^2 (4 f_n^{(1)} + f_n^{(2)}) \\ &\equiv y_{n+1} - y_n. \end{aligned} \tag{2.10}$$

Setting

$$D_1 = \frac{1}{64}(1 - e^{-4h}) f_n^{(2)} \ , \ \ D_2 = \frac{1}{2}(16 f_n - f_n^{(2)}) \ , \ \ D_3 = h(4 f_n^{(1)} + f_n^{(2)}) \ , \tag{2.11}$$

Therefore, (2.10) becomes

$$y_{n+1} = y_n + D_1 + \frac{1}{8} h (D_2 + D_3) \ . \tag{2.12}$$

Equation (2.12) is the newly proposed numerical method.

3. Analysis of the Properties of the proposed method

3.1. Local truncation error analysis of the proposed method

Let us consider the Taylor's series expansion of the form,

$$y(x_n+h) = y(x_n) + hf(x_n, y(x_n)) + \frac{1}{2!}h^2 f^{(1)}(x_n, y(x_n)) + \frac{1}{3!}h^3 f^{(2)}(x_n, y(x_n)) + \frac{1}{4!}h^4 f^{(3)}(x_n, y(x_n)) + O(h^5)\ . \tag{3.1}$$

The expression for the local truncation error of the proposed scheme is as follows:

$$\tau_{n+1} = y(x_n+h) - y_{n+1}\ . \tag{3.2}$$

Substituting (2.11), (2.12) and (3.1) into (3.2), we get

$$\begin{aligned}\tau_{n+1} &= y(x_n) + hf(x_n, y(x_n)) + \frac{1}{2!}h^2 f^{(1)}(x_n, y(x_n)) + \frac{1}{3!}h^3 f^{(2)}(x_n, y(x_n)) \\ &\quad + \frac{1}{4!}h^4 f^{(3)}(x_n, y(x_n)) + O(h^5) - \left(y_n + D_1 + \frac{1}{8}h(D_2+D_3)\right)\ .\end{aligned} \tag{3.3}$$

Using the Maclaurin's series of e^{-4h} and the application of localizing assumptions that led to the removal of terms up to h^3, (3.3) becomes

$$\tau_{n+1} = \frac{1}{4!}h^4\left(f^{(3)}(x_n, y(x_n)) - 4f_n^{(2)}\right) + O(h^5)\ . \tag{3.4}$$

3.2. Order of accuracy and consistency analysis of the proposed method

Based on (3.4), it can be determined that the proposed scheme possesses a third-order accuracy. In other words, the order of accuracy of the proposed method is 3. In accordance with [18], for a numerical method to be considered consistent, it must have a minimum accuracy of $p = 1$. From equation (3.4), it is evident that the method being proposed demonstrates consistency because its accuracy is of order $p = 3$, the increment function $\phi(x_n, y_n; 0) = f(x_n, y_n) = f_n$. and $\lim_{h\to 0}\frac{\tau_{n+1}}{h} = 0$.

3.3. Stability analysis of the proposed method

Numerical stability refers to the ability of a method to reduce or eliminate small variations in the input data [20]. To discuss the stability of the proposed method, consider the IVP with its exact solution given by

$$y' = \mu y\ , \quad y(0) = 1\ , y(x) = e^{\mu x}\ , \quad \mu < 0\ , \tag{3.5}$$

where μ is a complex constant. Using the proposed method (2.12), we arrive at the numerical approximation

$$y_{n+1} = \lambda y_n \ . \tag{3.6}$$

where the proposed method's stability region λ is given by

$$\begin{aligned} \lambda &= 1 + \frac{1}{192} h\left(192\mu + 96h\mu^2 + 32h^2\mu^3 \right) \\ &= 1 + \frac{1}{192}\left(192z + 96z^2 + 32z^3 \right), \ z = \mu h. \end{aligned} \tag{3.7}$$

By comparing the exact solution at $x = x_{n+1}$ in (3.5) and (3.6), it is evident that (3.7) constitutes the fourth component in the series expansion of $e^{\mu h}$. Hence, for the proposed method to maintain stability, it is required that

$$||\lambda|| < 1 \ . \tag{3.8}$$

The proposed third-order method is considered stable based on the information conveyed in equations (3.7) and (3.8). The stability region of the proposed method is plotted in the Figure 1.

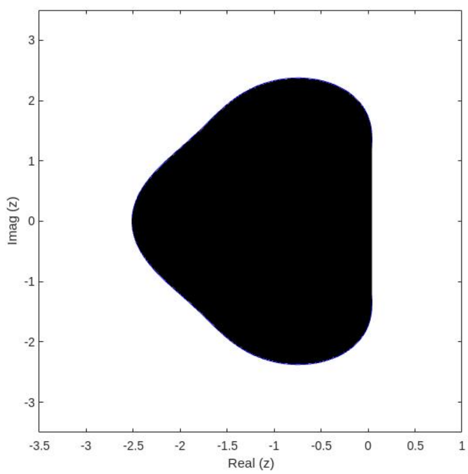

Figure 1: The stability region for the proposed numerical method (shaded).

3.4. Zero stability analysis of the proposed method

From the LHS of (2.12), it is readily established that

$$\rho_0 = -1 \quad \text{and} \quad \rho_1 = 1 \ . \tag{3.9}$$

So, the first characteristic polynomial is obtained as

$$\begin{aligned}\beta(r) &= \rho_1 r + \rho_0 \\ &= r - 1 \ .\end{aligned} \tag{3.10}$$

This implies that

$$\begin{aligned}r - 1 &= 0 \\ r &= 1 \ .\end{aligned} \tag{3.11}$$

Since (2.12) satisfies the Dahlquist root condition, hence this confirms the zero stability property of the method.

3.5. Convergence analysis of the proposed method

Simplifying (2.12) further yields,

$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_n) + \frac{h}{2} f^{(1)}(x_n, y_n) + \frac{h^2}{3!} f^{(2)}(x_n, y_n). \tag{3.12}$$

The increment function of (2.12) is given by

$$\phi(x_n, y_n; h) = f(x_n, y_n) + C_1 f_n^{(1)}(x_n, y_n) + C_2 f_n^{(2)}(x_n, y_n) \ , C_1 = \frac{h}{2}, C_2 = \frac{h^2}{3!} \ . \tag{3.13}$$

Suppose

$$\phi(x_n, \bar{y}_n; h) = f(x_n, \bar{y}_n) + C_1 f_n^{(1)}(x_n, \bar{y}_n) + C_2 f_n^{(2)}(x_n, \bar{y}_n) \ . \tag{3.14}$$

Subtracting (3.14) from (3.13), we get

$$\begin{aligned}\phi(x_n, y_n; h) - \phi(x_n, \bar{y}_n; h) = f(x_n, y_n) - f(x_n, \bar{y}_n) + C_1[f_n^{(1)}(x_n, y_n) - f_n^{(1)}(x_n, \bar{y}_n)] \\ + C_2[f_n^{(2)}(x_n, y_n) - f_n^{(2)}(x_n, \bar{y}_n)] \ .\end{aligned} \tag{3.15}$$

Let $\hat{y}_n$ be defined as a point in the interior of the interval whose end points are y_n and $\bar{y}_n$. Applying the mean value theorem, (3.14) becomes

$$\phi(x_n, y_n; h) - \phi(x_n, \bar{y}_n; h) = [P + Q + R](y_n - \bar{y}_n) \ . \tag{3.16}$$

Taking the norm of (3.16), one gets

$$\begin{aligned}||\phi(x_n, y_n; h) - \phi(x_n, \bar{y}_n; h)|| &= ||(P + Q + R)(y_n - \bar{y}_n)|| \\ &\le L||y_n - \bar{y}_n|| \ ,\end{aligned} \tag{3.17}$$

with

$$P = \sup_{(x_n,y_n)\in D} \frac{\partial f}{\partial y}(x_n, \hat{y}_n), Q = C_1 \sup_{(x_n,y_n)\in D} \frac{\partial f^{(1)}}{\partial y}(x_n, \hat{y}_n), R = C_2 \sup_{(x_n,y_n)\in D} \frac{\partial f^{(2)}}{\partial y}(x_n, \hat{y}_n),$$

where the Lipschitz constant is given by

$$L = ||P + Q + R|| \ .$$

Equation (3.17) shows that (2.12) is convergent and the increment function (3.13) is Lipschitzian.

4. Numerical Examples and Discussion of Results

4.1. Numerical Examples

Consider the following physical models with their exact solutions:

Example 4.1.

$$V'(t) = rV(t), V(0) = 100, V(t) = 100\ e^{0.05t}, 0 \le t \le 7 \qquad (4.1)$$

The comparative results and errors analyses of the proposed method (PNM) and FIM [14] are displayed in Figures 2 and 3, respectively.

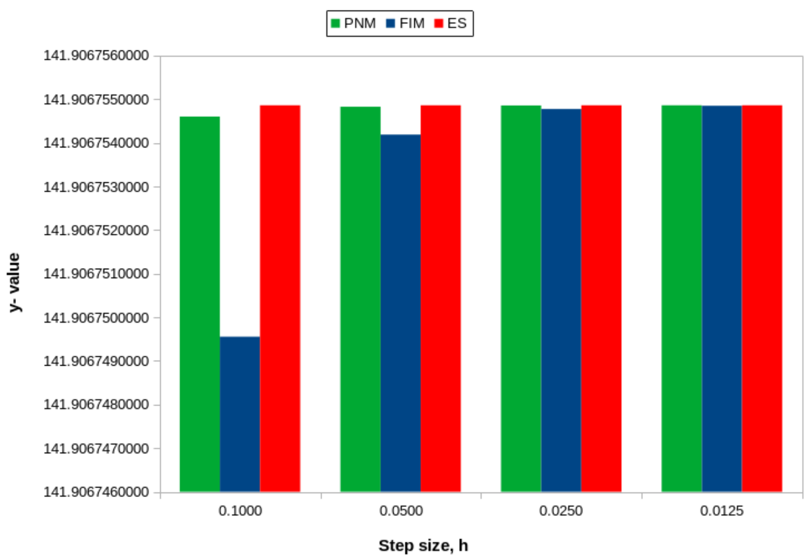

Figure 2: The comparative results analyses of *PNM*, *FIM* [14] and exact solution *ES*.

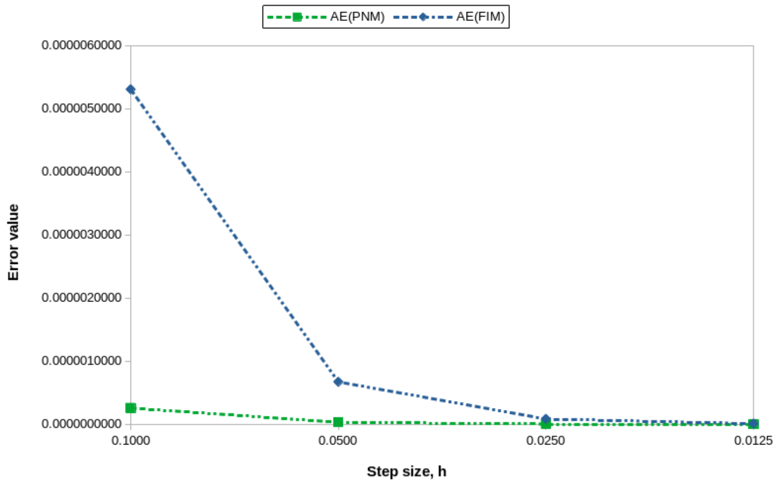

Figure 3: The error incurred in *PNM* and *FIM*.

Example 4.2.

$$N'(t) = -\frac{\ln(2)}{10}N(t)\ , N(0) = 500\ , 0 \leq t \leq 30, N(t) = 500\ e^{-0.0693t}\ . \qquad (4.2)$$

The comparative results and errors analyses of the proposed method (PNM) and FIM [14] are displayed in Figures 4 and 5, respectively.

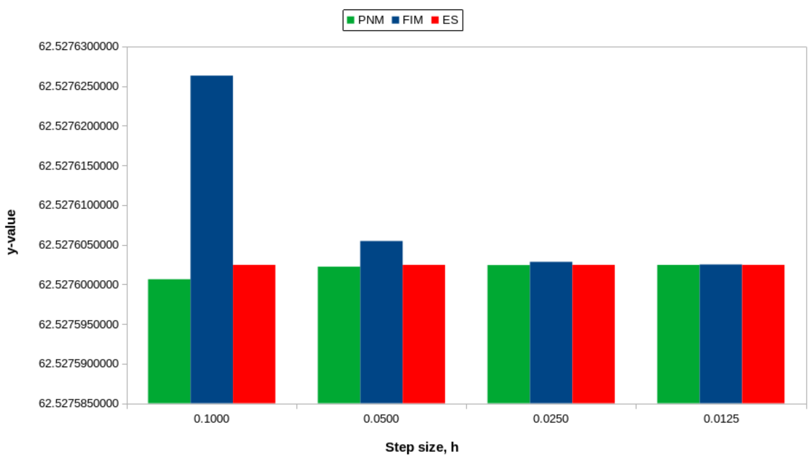

Figure 4: The comparative results analyses of *PNM*, *FIM* [14] and exact solution *ES*.

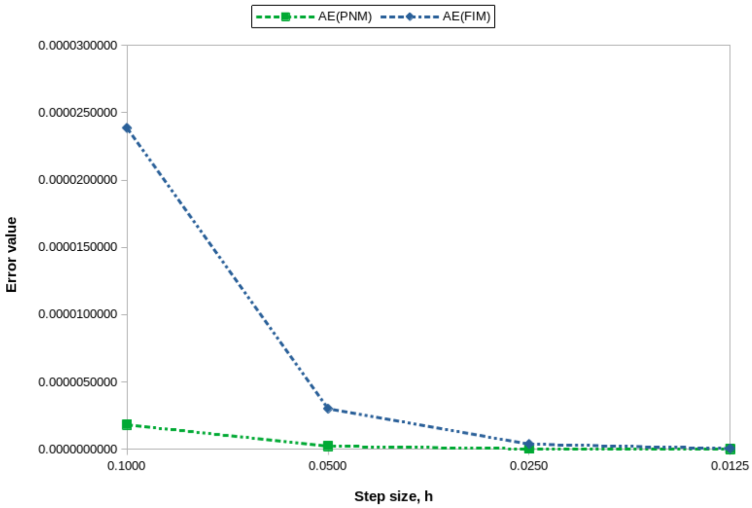

Figure 5: The error incurred in *PNM* and *FIM*.

Example 4.3.

$$C'(t) = -0.001C^2, C(0) = 10, 0 \leq t \leq 5, C(t) = \frac{10}{1+10kt}\ . \tag{4.3}$$

The comparative results and errors analyses of the proposed method (PNM) and FIM [14] are displayed in Figures 6 and 7, respectively.

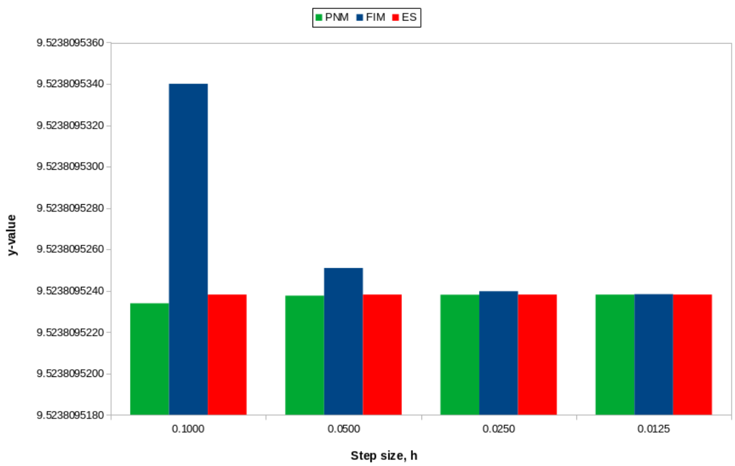

Figure 6: The comparative results analyses of *PNM*, *FIM* [14] and exact solution *ES*.

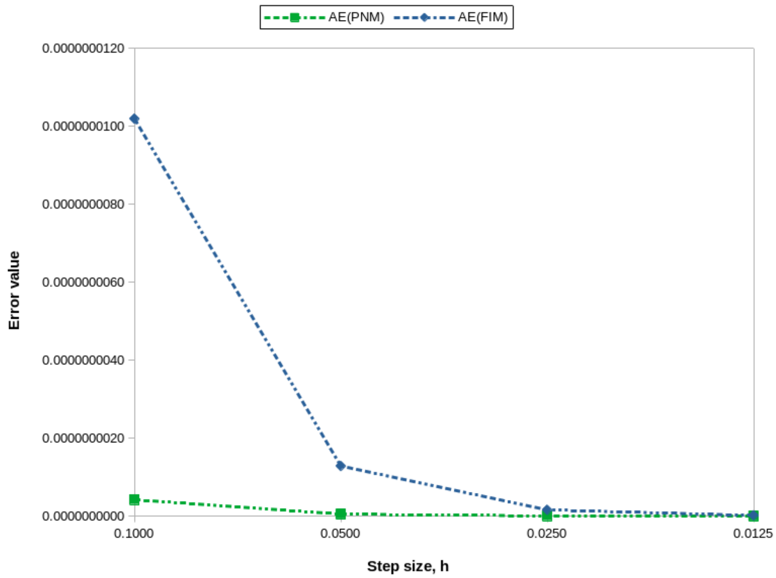

Figure 7: The error incurred in *PNM* and *FIM* [14].

4.2. Discussion of Results

As demonstrated by Figures 2, 4 and 6, the proposed numerical method aligns with the exact solution as the step size becomes smaller. The progression of the computation with various step sizes, as shown in Figures 3, 5 and 7, clearly indicates that the error incurred by the proposed numerical method is smaller as compared to FIM [14] but both errors approach to 0 as the step size becomes smaller.

5. Conclusion

In this paper, we developed a new method for solving physical models arising from real-life scenarios by using the transcendental function of exponential type. The properties of developed numerical method were analyzed in terms of its consistency, order of accuracy, convergence, zero stability, and linear stability. It was discovered that the method was linearly stable, consistent, convergent and had a third-order accuracy. The method used a single step approach that employed MATLAB software to obtain numerical solutions. To evaluate the performance of the method, three real-life models were solved and the absolute relative errors at the final nodal point were calculated. The findings showed that the new method outperformed the existing method and gave good results for the test problems considered as shown in Figures 2 - 7. The results further revealed that the derived method performed well as the step size, h, decreases. In fact the proposed numerical method is an improvement of [14]. This indicates that the proposed numerical method can be considered suitable for solving initial value problem of first order, with the expected level of accuracy.

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