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SOME ASPECTS OF ROUGHLY PRIME SUBMODULES

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Abstract: Let R be a commutative ring with unity and M be an R-module. The aim of this article is to introduce and investigate certain properties of a new notion of prime submodules of a module, namely roughly prime submodules. A submodule N of M is said to be a roughly prime submodule of M if for rm $\in$ N, r $\in$ R and m $\in$ M, implies that either m $\in$ N + Z(M) or r $\in$ (N + Z(M): M) where Z(M) is the singular submodule of M. The interaction of this submodule with other classes of modules as well as its characteristics in terms of direct sum, intersection and homomorphic image are studied along with the exploration of its behaviour in quotient modules.

Keywords and Phrases: Prime submodules, Singular submodules, Quotient Submodules.

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1. Introduction

Let M be an R-module. Prime submodules, which came as a generalization of prime ideals of rings was first introduced by Dauns in 1978 [7]. A proper submodule N of M is said to be prime if for any r $\in$ R and m $\in$ M with rm $\in$ N, we have m $\in$ N or r $\in$ (N : $_R$M) where (N : $_R$M) = {x $\in$ R: xM $\subseteq$ N}. This has garnered immense attention from researchers which has led to the introduction of several related concepts like fully prime submodules, approximately prime submodules, almost prime submodules etc. A proper submodule N of M is said to be semiprime

if for $r^2m \in N$, where $r \in R$ and $m \in M$, then $rm \in N$ [13]. A proper submodule N of M is said to be weakly semiprime if for $r^2M \subseteq N$, where $r \in R$ and $m \in M$, then $rM \subseteq N$ [13]. The singular submodule Z(M) of R-module M is defined as: $Z(M) = \{m \in M : ann(m) \leq_e R\}$ [9]. M is called a singular module if Z(M) = M and it is called non-singular if Z(M) = 0 [9]. Also, the map $M \to \frac{M}{L}$ carries $Z(M) \to Z(\frac{M}{L})$ for any submodule L of M [9]. A submodule N of M is d-closed if $Soc(\frac{M}{N}) = 0$ where Soc denotes the socle of M [8]. In [10] Goodearl defined an s-closed submodule of a module M as a submodule N for which $\frac{M}{N}$ is non-singular. A non-zero submodule N of M is called essential if $N \cap L \neq \{0\}$ for every non-zero submodule L of M [9]. If all the submodules of R-module M is essential, then M is called a uniform module [12]. An R-module M is called chained if for every submodule D, H of M, either $H \leq D$ or $D \leq H$. An R-module M is called a multiplication module if every submodule of M is of the form IM, for some ideal I of R. Also, since $(N:M) = \{r \in R: rM \subset N\}$, so M is a multiplication module if and only if N = (N:M)M [1].
The first section of this paper deals with the various properties of roughly prime submodules and its relation with other types of modules. The direct sum and intersection of this type of module is studied along with the introduction of roughly prime ideals. The second section further investigates the behaviour of roughly prime submodules in quotient modules.
Throughout this article, R will denote a commutative ring with unity and $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ denotes the set of integers, rationals and reals respectively.

2. Main Results

In this section, we define roughly prime submodules and explore the various properties exhibited by this class of module.

2.1. Definition

Let M be an R-module. A submodule P of M is said to be roughly prime if for $rm \in P$, $r \in R$ and $m \in M$, implies either $m \in P + Z(M)$ or $r \in (P+Z(M):M)$ where Z(M) is the singular submodule of M.

Proposition 2.1. *Let M be an R-module. Then every prime submodule of M is also roughly prime.*

Proof. Let P be a prime submodule of M and $rm \in P$, $r \in R$ and $m \in M$. This implies either $m \in P$ or $r \in (P:M)$. If $m \in P$, then $m \in P + Z(M)$. If $r \in (P:M)$, then $r \in (P + Z(M):M)$. Thus, P is a roughly prime submodule of M.

The converse need not be true in general.

Example. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{12}$. Here $R = \mathbb{Z}$ and $M = \mathbb{Z}_{12}$ and the singular submodule $Z(\mathbb{Z}_{12}) = \mathbb{Z}_{12}$. The submodule $P = \{0, 4, 8\}$ of M is not a prime submodule of M as $2.2 \in P$ but $2 \notin P$ and $2M \not\subseteq P$. But P is a roughly prime submodule of M.

Proposition 2.2. *Let M be an R-module. If M is a non-singular module, then every roughly prime submodule of M is also prime.*
Proof. Using the fact that for a non-singular module Z(M) = 0.

Remark 2.3. *Direct sum of roughly prime submodules need not be roughly prime.*

Example. Consider the $\mathbb{Z}$-module M = $\mathbb{Z} \oplus \mathbb{Q}$. Here Z(M) = 0. The submodules $P_1 = 3\mathbb{Z}$ and $P_2 =< 0 >$ are roughly prime submodules of $\mathbb{Z}$ and $\mathbb{Q}$ respectively. Consider the direct sum P = $3\mathbb{Z}\oplus < 0 >$. Then 3.(1,0) ∈ P but (1,0) ∉ P + Z(M) and $3 \notin (P + Z(M) : M)$.

Theorem 2.4. *Let M be an R-module. Then a submodule N of M is roughly prime if and only if for every submodule K of M and for every ideal I of R such that $IK \subseteq N$ implies either $K \subseteq\ N + Z(M)$ or $I \subseteq (N + Z(M) : M)$.*
Proof. Let N be a roughly prime submodule of M. Consider $IK \subseteq N$ where I is an ideal of R and K is a submodule of M. If $K \nsubseteq N + Z(M)$ then ∃ k ∈ K such that $k \notin N + Z(M)$. As $IK \subseteq N$, so for any $i \in I$, $ik \in N$. But N being roughly prime, $i \in (N + Z(M) : M)$ as $k \notin N + Z(M)$.
Conversely, let rm ∈ N, r ∈ R and m ∈ M. Then $< r >< m >\subseteq N$. Then by given assumption, either $< m >\subseteq N + Z(M)$ or $< r >\subseteq (N + Z(M) : M)$.

Corollary 2.5. *Let M be an R-module and N be a submodule of M. Then N is roughly prime if and only if for every submodule K of M, such that rK ⊆ N for any r ∈ R, implies that either K ⊆ N + Z(M) or rM ⊆ N + Z(M).*

Remark 2.6. *Submodule of a roughly prime module need not be roughly prime in general.*

Example. $\mathbb{Z}$ as a $\mathbb{Z}$-module is roughly prime. Here R = $\mathbb{Z}$ and M = $\mathbb{Z}$. The submodule P = $2\mathbb{Z}$ of M is also roughly prime. But the submodule $P' = 4\mathbb{Z}$ of M is not roughly prime as 2.2 ∈ P', 2 ∈ R and 2 ∈ M but 2 ∉ P' + Z(M) and 2 ∉ (P'+Z(M):M).

Theorem 2.7. *Let M be a singular R-module. Then every submodule of M is roughly prime.*
Proof. Since M is a singular module, so Z(M) = M. Let N be any arbitrary submodule of M. Consider rm ∈ N, r ∈ R and m ∈ M. Then m ∈ Z(M), which implies m ∈ N + Z(M). Therefore, for every rm ∈ N, r ∈ R and m ∈ M, we get m ∈ N + Z(M). Thus, N is roughly prime.

Remark 2.8. *Intersection of roughly prime submodules of a module need not be* *roughly prime in general.*

Example. Consider the roughly prime submodules $N_1 = 2\mathbb{Z}$ and $N_2 = 3\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$. Here $Z(\mathbb{Z}) = 0$. Then $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ is not roughly prime as $2.3 \in 6\mathbb{Z}$ but $3 \notin 6\mathbb{Z} + Z(\mathbb{Z})$ and $2\mathbb{Z} \nsubseteq 6\mathbb{Z} + Z(\mathbb{Z})$.

Proposition 2.9. *Let M be a chained R-module and N_1 and N_2 be two roughly prime submodules of M. Then $N_1 \cap N_2$ is also roughly prime.*
Proof. Since M is a chained R-module, either $N_1 \leq N_2$ or $N_2 \leq N_1$. Therefore either $N_1 \cap N_2 = N_1$ or $N_1 \cap N_2 = N_2$. The result thus follows.

Corollary 2.10. *Let M be a chained R-module and N, Z(M) be roughly prime submodules of M. Then the singular submodule of N is also roughly prime.*
Proof. Follows from the previous proposition and the fact that Z(N) = N ∩ Z(M).

Remark 2.11. *Let A and B be R-modules. Then for any homomorphism ϕ : A → B, ϕ (Z(A)) is a submodule of Z(B)* [9].

Theorem 2.12. *Let M be an R-module and N be a submodule of M. Let f : M → M be a surjective homomorphism. If N is a roughly prime submodule of M, then so is f(N).*
Proof. Let rm ∈ N, where r ∈ R and m ∈ M. Then f(rm) ∈ f(N). This implies rf(m) ∈ f(N). Since N is roughly prime, rm ∈ N implies m ∈ N + Z(M) or rM ⊆ N + Z(M). If m ∈ N + Z(M), then f(m) ∈ f(N + Z(M)). This implies f(m) ∈ f(N) + f(Z(M)) as f is a homomorphism. Thus, f(m) ∈ f(N) + Z(M) (using remark 2.11). Again, if rM ⊆ N + Z(M) then f(rM) ⊆ f(N + Z(M)) which implies rf(M) ⊆ f(N) + f(Z(M)). f being surjective, f(M) = M and thus, rM ⊆ f(N) + Z(M).

Theorem 2.13. *Let M, M' be R-modules and f : M → M' be an isomorphism. If N' is a roughly prime submodule of M', then $f^{-1}(N')$ is a roughly prime submodule of M.*
Proof. Let rm ∈ f^{-1}(N$'$), where r ∈ R and m ∈ M. Then f(rm) ∈ N$'$, i.e, rf(m) ∈ N$'$. Since N$'$ is roughly prime in M$'$, so either f(m) ∈ N$'$ + Z(M$'$) or rM$'$ ⊆ N$'$ + Z(M$'$). If f(m) ∈ N$'$ + Z(M$'$), then m ∈ f^{-1} (N$'$ + Z(M$'$)). As f is an isomorphism, so f^{-1} is also an isomorphism. Therefore, m ∈ f^{-1} (N$'$) + f^{-1} (Z(M$'$)) which implies m ∈ f^{-1} (N$'$) + Z(M). If rM$'$ ⊆ N$'$ + Z(M$'$), then f^{-1}(rM$'$) ⊆ f^{-1} (N$'$ + Z(M$'$)). This implies rf^{-1} (M$'$) ⊆ f^{-1} (N$'$) + f^{-1} (Z(M$'$)) which ultimately gives rM ⊆ f^{-1} (N$'$) + Z(M) (using remark 2.11).

Let M be an R-module. Recall that an element m ∈ M is said to be a torsion element if there exists r ∈ R such that r ≠ 0 and rm = 0. The set τ of torsion elements of M is a submodule of M. Clearly Z(M) ≤ τ .

Theorem 2.14. *Let M be an R-module. If R is an integral domain, then the submodule τ of the torsion elements of M is roughly prime.*

Proof. Let rm $\in \tau$, where r ($\neq 0$) $\in$ R and m $\in$ M. Then there exists $r_1 \in$ R such that $r_1 \neq 0$ and r_1 (rm) = 0. This implies (r_1 r)m = 0. As R is an integral domain, r_1 r $\neq 0$. Thus, m $\in \tau$ which implies m $\in \tau$ + Z(M).

Theorem 2.15. *Let M be an R-module and N be a submodule of M. If the submodule N + Z(M) is roughly prime, then N is also a roughly prime submodule of M.*

Proof. Let rm $\in$ N, r $\in$ R and m $\in$ M. Then rm $\in$ N + Z(M). Since N + Z(M) is roughly prime, this implies either m $\in$ N + Z(M) + Z(M) or rM $\subseteq$ N + Z(M) + Z(M), i.e, m $\in$ N + Z(M) or rM $\subseteq$ N + Z(M). Therefore, for rm $\in$ N, we get either m $\in$ N + Z(M) or rM $\subseteq$ N + Z(M). Thus, N is roughly prime.

Proposition 2.16. *Let M be an R-module. If all the ideals of R are essential in R, then the singular submodule of M is itself roughly prime.*

Proof. Let rm $\in$ Z(M), r $\in$ R and m $\in$ M.

If m $\in$ Z(M)+ Z(M) = Z(M), then proved.

Suppose m $\notin$ Z(M). Claim rM $\subseteq$ Z(M)+ Z(M) = Z(M). Suppose rM $\nsubseteq$ Z(M). So there exists r$m_1 \in$ rM such that r$m_1 \notin$ Z(M), i.e, Ann(rm_1) $\nleq_e$ R, which is a contradiction to the given hypothesis that all the ideals of R are essential in R. Thus rM $\subseteq$ Z(M) and Z(M) is roughly prime.

Corollary 2.17. *Let M be an R-module such that M is the direct sum of its submodules M_1 , M_2,... .If all the ideals of R are essential in R, then the direct sum of the singular submodules of M_1 , M_2,... is also roughly prime.*

Proof. By previous proposition, we get Z(M) is roughly prime. This implies Z($\oplus$ M_i) is roughly prime $\Rightarrow \oplus\ Z(M_i)$ is roughly prime for i = 1,2,3.....

Proposition 2.18. *Let M be an R-module with Soc(M) = 0. Then a submodule N of M is roughly prime if and only if N is prime.*

Proof. Since every module M with Soc(M) = 0 is non-singular [8], the result thus follows from proposition 2.2.

Corollary 2.19. *Let M be a torsion-free R-module. Then a submodule N of M is prime if and only if N is roughly prime.*

Proof. Since a torsion-free module has zero socle [3], the result follows from previous proposition.

Recall that a submodule K of an R-module M is said to be weakly prime if K $\neq$ M and for every 0 $\neq$ rm $\in$ K, r $\in$ R and m $\in$ M, we get r $\in$ (K:M) or m $\in$ K [6].

Corollary 2.20. *Let M be a torsion-free R module. Then every weakly prime* *submodule of M is roughly prime.*

Proof. Consider a weakly prime submodule N of M. Let rm ∈ N where r ∈ R and m ∈ M. If rm ≠ 0, then either m ∈ N or rM ⊆ N. This implies either m ∈ N + Z(M) or rM ⊆ N + Z(M).
If rm = 0, then r = 0 as M is torsion-free. Thus, rM ⊆ N + Z(M).

Let M be an R-module. Recall that the module M is defined as a local module if M has a unique maximal submodule [11].

Theorem 2.21. *Let M be a local R-module. If the maximal submodule of M is roughly prime, then it is also prime.*

Proof. Let M be a local R-module and K be its maximal submodule such that K is roughly prime. Since Z(M) is a submodule of M and K is the only maximal submodule of M, so Z(M) ≤ K. Let rm ∈ K, where r ∈ R and m ∈ M. As K is roughly prime, either m ∈ K + Z(M) or rM ⊆ K + Z(M). But as Z(M) ≤ K, so m ∈ K + Z(M) implies m ∈ K and rM ⊆ K + Z(M) implies rM ⊆ K. Thus, K is a prime submodule of M.

Theorem 2.22. *Let P be a roughly prime submodule of an R-module M. If the singular submodule is contained in P, then P is also a prime submodule of M.*

Proof. Let P be a roughly prime submodule of M. Consider rm ∈ P where r ∈ R and m ∈ M. Then either m ∈ P + Z(M) or rM ⊆ P + Z(M). This implies either m ∈ P or rM ⊆ P as Z(M) is contained in P. Thus, P is prime.

Recall that a module M over a ring R is said to be fully prime if every proper submodule of M is a prime submodule of M [17].

Theorem 2.23. *Let M be a fully prime R-module. Then every submodule of M is roughly prime.*

Proof. Follows from proposition 2.1.

Theorem 2.24. *Let M be an R-module. If the singular submodule of M is roughly prime, then it is also prime.*

Proof. Let rm ∈ Z(M) where r ∈ R and m ∈ M. This implies either m ∈ Z(M) + Z(M) or rM ⊆ Z(M) + Z(M) as Z(M) is roughly prime. Thus we get either m ∈ Z(M) or rM ⊆ Z(M). Therefore, Z(M) is prime.

A proper submodule K of an R-module M is said to be n-primary submodule, for a fixed positive integer n, if whenever rx ∈ K, r ∈ R and x ∈ M, implies that either x ∈ K or r^n ∈ [K:M] [14].

Proposition 2.25. *Let M be an R-module and N be a 2-primary submodule of N. If N is weakly semi-prime, then N is also roughly prime.*

Proof. Let rm ∈ N where r ∈ R and m ∈ M. Since N is 2-primary, either m ∈

N or r^2M $\subseteq$ N. If m $\in$ N then m $\in$ N + Z(M). If r^2M $\subseteq$ N then rM $\subseteq$ N which implies rM $\subseteq$ N + Z(M).

Proposition 2.26. *Let M be an R-module and N be a submodule of M such that (N+Z(M):M) is a maximal ideal of R. Then N is a roughly prime submodule of M.*
Proof. Let rm $\in$ N, r $\in$ R and m $\in$ M such that r $\notin$ (N+Z(M):M). Since (N+Z(M):M) is a maximal ideal of R, we can write R = $<r>$ + (N+Z(M):M) $\Rightarrow$ there exists t $\in$ R and n $\in$ (N+Z(M):M) such that 1 = tr + n $\Rightarrow$ m = trm + nm $\Rightarrow$ m $\in$ N + (N+Z(M)) $\Rightarrow$ m $\in$ N + Z(M). Therefore, N is roughly prime.

Definition 2.1. *An ideal I of a ring R is called roughly prime ideal of R if I is a roughly prime submodule of the R-module R. Equivalently, we can define it as: for every ab $\in$ I where a, b $\in$ R, either a $\in$ I + Z(R) or b $\in$ I + Z(R).*

Proposition 2.27. *Let P be a roughly prime submodule of an R-module M. If Z(M) $\subseteq$ P, then (P:M) is a roughly prime ideal of R.*
Proof. Let ab $\in$ (P:M) where a, b $\in$ R. This implies abM $\subseteq$ P $\Rightarrow$ abm $\in$ P for every m $\in$ M $\Rightarrow$ either aM $\subseteq$ P + Z(M) or bm $\in$ P + Z(M) for every m $\in$ M $\Rightarrow$ either aM $\subseteq$ P or bm $\in$ P for every m $\in$ M $\Rightarrow$ either aM $\subseteq$ P or bM $\subseteq$ P $\Rightarrow$ either a $\in$ (P:M) or b $\in$ (P:M) $\Rightarrow$ either a $\in$ (P:M) + Z(M) or b $\in$ (P:M) + Z(M) $\Rightarrow$ (P:M) is a roughly prime ideal.

Proposition 2.28. *Let M be a multiplication module and N be a proper submodule of M. Then N is roughly prime if the ideal (N:M) is roughly prime.*
Proof. Let (N:M) be a roughly prime ideal of R. Consider rm $\in$ N, where r $\in$ R and m $\in$ M. This implies r$<m>$ $\subseteq$ N $\Rightarrow$ rIM $\subseteq$ N (for some ideal I of R such that $<m>$ = IM) $\Rightarrow$ rI $\subseteq$ (N:M) $\Rightarrow$ r $\in$ (N:M) + Z(R) or I $\subseteq$ (N:M) + Z(R) $\Rightarrow$ rM $\subseteq$ (N:M)M + Z(R)M or IM $\subseteq$ (N:M)M + Z(R)M $\Rightarrow$ rM $\subseteq$ N + Z(R)M or IM $\subseteq$ N + Z(R)M $\Rightarrow$ rM $\subseteq$ N + Z(M) or IM $\subseteq$ N + Z(M) (since Z(R)M $\subseteq$ Z(M)) $\Rightarrow$ rM $\subseteq$ N + Z(M) or $<m>$ $\subseteq$ N + Z(M).

3. Roughly prime submodules in quotient modules

In this section, we study certain conditions under which submodules of quotient modules become roughly prime.

Proposition 3.1. *Let A be a uniform R-module. Then any quotient module of A is a roughly prime submodule of itself.*
Proof. Let B be any submodule of A and consider the quotient module M = $\frac{A}{B}$. Since A is uniform, B $\leq_e$ A. This implies M is singular [9]. Thus, by theorem 2.7, M is a roughly prime submodule of itself.

Corollary 3.2. *Let M be an R-module and N be an essential submodule of M.*

Then for any submodule K of M containing N, the quotient module $\frac{K}{N}$ is a roughly prime submodule of itself.
Proof. Since N is essential in M so N is essential in K. This implies the quotient module $\frac{K}{N}$ is singular. By theorem 2.7 every submodule of $\frac{K}{N}$ is roughly prime.

Remark 3.3. *Let N be a prime submodule of M. This implies $\frac{M}{N}$ is a torsion-free $\frac{R}{P}$ module where P = (N:M)[17]. By corollary 2.20, every weakly prime submodule of $\frac{M}{N}$ is roughly prime.*

Proposition 3.4. *Let N be an s-closed submodule of an R-module M. Then every submodule of the quotient module $\frac{M}{N}$ is prime if and only if it is roughly prime.*
Proof. Since N is s-closed, the module $\frac{M}{N}$ is non-singular. The result thus follows from proposition 2.1 and proposition 2.2.

Proposition 3.5. *Let N be a d-closed submodule of M. Then every submodule of the quotient module $\frac{M}{N}$ is prime if and only if it is roughly prime.*
Proof. Since N is d-closed, Soc($\frac{M}{N}$) = 0 ⇒ $\frac{M}{N}$ is non-singular. The result follows from proposition 2.1 and proposition 2.2.

Proposition 3.6. *Let M be an R-module and N, K be submodules of M such that $N \subseteq K$. Then the submodule K is roughly prime submodule of M if and only if $\frac{K}{N}$ is roughly prime submodule of $\frac{M}{N}$.*
Proof. Let K be a roughly prime submodule of M containing N. Consider r(m + N) ∈ $\frac{K}{N}$ ⇒ rm + N ∈ $\frac{K}{N}$ ⇒ rm ∈ K ⇒ m ∈ K + Z(M) or rM ⊆ K + Z(M).
Case I. If m ∈ K + Z(M) then m is of the form m = k + x, for some k ∈ K and x ∈ Z(M) ⇒ m-k ∈ Z(M) ⇒ I = ann(m-k) $\leq_e$ R. As I <(m-k)> = 0, therefore I <(m-k) + N> = N ⇒ (m-k) + N ∈ Z($\frac{M}{N}$) ⇒ m + N ∈ $\frac{K}{N}$ + Z($\frac{M}{N}$).
Case II. If rM ⊆ K + Z(M), then rm ∈ K + Z(M) for all m ∈ M ⇒ rm is of the form rm = k' + y for some k' ∈ K and y ∈ Z(M) ⇒ rm - k' ∈ Z(M) ⇒ L = ann(rm-k') $\leq_e$ R. As L <(rm-k')> = 0 therefore L <(rm-k') + N> = N ⇒ (rm-k') + N ∈ Z($\frac{M}{N}$) ⇒ rm + N ∈ $\frac{K}{N}$ + Z($\frac{M}{N}$) ⇒ r($\frac{M}{N}$) ⊆ $\frac{K}{N}$ + Z($\frac{M}{N}$).
Thus, $\frac{K}{N}$ is a roughly prime submodule of $\frac{M}{N}$.
The converse part follows similarly.

Proposition 3.7. *Let M be an R-module and N be a submodule of M. Let $f : \frac{M}{N} \to \frac{M}{N}$ be a surjective homomorphism. If $\frac{M}{N}$ is a roughly prime submodule of M, then so is f($\frac{M}{N}$).*
Proof. Similar to theorem 2.12.

Theorem 3.8. *Let N, N' be submodules of R-modules M and M' respectively and $f : \frac{M}{N} \to \frac{M'}{N'}$ be an isomorphism. If $\frac{M'}{K'}$ is a roughly prime submodule of $\frac{M'}{N'}$, where*

K' is a submodule of N', then f^{-1} $(\frac{M'}{K'})$ is a roughly prime submodule of $\frac{M}{N}$.
Proof. Similar to theorem 2.13.

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