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A NOTE ON CONNECTIVITY PRESERVING SPLITTING OPERATION FOR MATROIDS REPRESENTABLE OVER $GF(p)$

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Abstract: The splitting operation on a p-matroid does not necessarily preserve connectivity. It is observed that there exists a single element extension of the splitting matroid which is connected. In this paper, we define the element splitting operation on p-matroids which consist of a splitting operation followed by a single element extension. It is proved that the element splitting operation on a connected p-matroid yields a connected p-matroid. We give a sufficient condition to yield Eulerian p-matroid from Eulerian p-matroid under the element splitting operation. A sufficient condition to obtain Hamiltonian p-matroid by applying the element splitting operation on p-matroid is also provided. The characterization of the paving p-matroid which are closed under the element splitting operation, is also obtained.

Keywords and Phrases: p-matroid, element splitting operation, Eulerian matroid, connected matroid, hamiltonian matroid, elementary lift, paving matroid.

2020 Mathematics Subject Classification: 05B35, 05C50, 05C83.

1. Introduction

We discuss loopless and coloopless p-matroids, by a p-matroid we mean a vector matroid $M \cong M[A]$ for some matrix A of size $m \times n$ over the field $F = GF(p)$, for prime p. We denote the set of column labels of M (viz. the ground set of M) by E, the set of circuits of M by $\mathcal{C}(M)$, and the set of independent sets of M by $\mathcal{I}(M)$. For undefined, standard terminology in graphs and matroids, see Oxley [13].

Malavadkar et al. [9] defined the splitting operation for p-matroids in the following way:

Definition 1.1. *Let $M \cong M[A]$ be a p-matroid on the ground set E, let $\{a, b\} \subset E$, and $\alpha \neq 0$ in $GF(p)$. The matrix $A_{a,b}$ is constructed from A by appending an extra row to A, which has coordinates equal to α in the columns corresponding to the elements a,b, and zero elsewhere. Define the splitting matroid $M_{a,b}$ to be the vector matroid $M[A_{a,b}]$. The transition of M to $M_{a,b}$ is called the splitting operation.*

A circuit $C \in \mathcal{C}(M)$ containing $\{a, b\}$ is said to be a p-circuit of M, if $C \in \mathcal{C}(M_{a,b})$. And if C is a circuit of M containing either a or b, but it is not a circuit of $M_{a,b}$, then we say that C is an np-circuit of M. For $a, b \in E$, if the matroid M contains no np-circuit, then splitting operation on M with respect to a, b is called trivial splitting.
Note that the class of connected p-matroids is not closed under the splitting operation. This fact is illustrated with the following example.

Example 1.2. The vector matroid $M \cong M[A]$ represented by the matrix A over the field $GF(3)$ is connected, whereas the splitting matroid $M_{1,4} \cong M[A_{1,4}]$ is not connected.

$$A = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad A_{1,4} = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

It is interesting to see that the vector matroid $M'_{1,4} \cong M[A'_{1,4}]$, which is a single element extension of $M_{1,4}$, is connected.

$$A'_{1,4} = \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

This example motivates us to investigate the question: If M is a connected p-matroid and $M_{a,b}$ is the splitting matroid of M, then does there exist a single element extension of the splitting matroid that is connected? In the next section, we answer this question by defining the element splitting operation on a p-matroid M which is a splitting operation on M followed by a single element extension.

2. Element Splitting Operation

In this section, we define the element splitting operation on a p-matroid M and characterize its circuits.

Definition 2.1. *Let $M \cong M[A]$ be a p-matroid on the ground set E,$\{a, b\} \subset E$, and $M_{a,b}$ be the corresponding splitting matroid. Let the matrix $A_{a,b}$ represent $M_{a,b}$ on $GF(p)$. Construct the matrix $A'_{a,b}$ from $A_{a,b}$ by adding an extra column to $A_{a,b}$, labeled as z, which has the last coordinate equal to $\alpha \neq 0$ and the rest are equal to zero. Define the element splitting matroid $M'_{a,b}$ to be the vector matroid $M[A'_{a,b}]$. The transformation of M to $M'_{a,b}$ is called the element splitting operation.*

Various splitting operations on binary matroids are closely studied in [8, 10, 11, 15, 16, 17, 18, 19]. A matroid L is a lift of the matroid M, if there exists a matroid N, and $X \subset E(N)$ such that $N/X = M$, and $N \setminus X = L$. If X is a singleton set, then L is called an elementary lift of M. In the following result, Mundhe et al. [12] showed the equivalence of splitting matroids with elementary lift for binary matroids:

Lemma 2.2. *Let M and L be binary matroids. Then L is an elementary lift of M if and only if L is isomorphic to M_T for some $T \subset E(M)$.*

Lemma 2.2 can be extended to p-matroids by using the similar arguments used to prove it in [12]. Thus a splitting matroid $M_{a,b}$ of p-matroid M is an elementary lift of M. In-depth study on lifted graphic matroid is done in [3, 4, 6].

Remark 2.3. *$rank(A) < rank(A'_{a,b}) = rank(A) + 1$. If the rank functions of M and $M'_{a,b}$ are denoted by r and r', respectively, then $r(M) < r'(M'_{a,b}) = r(M) + 1$.*

Let $C = \{v_1, v_2, \ldots, v_k\}$, where $v_i, i = 1, 2, \ldots, k$ are column vectors of the matrix A, be an np-circuit of M containing only a. Assume $v_1 = a$, without loss of generality. Then there exist non-zero scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in GF(p)$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k \equiv 0(\mod p)$. Let $\alpha_z \in GF(p)$ be such that $\alpha_z + \alpha_1 \equiv 0(\mod p)$. Note that $\alpha_z \neq 0$. Then in the matrix $A'_{a,b}$, we have $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k + \alpha_z z \equiv 0(\mod p)$. Therefore the set $C \cup z = \{v_1, v_2, \ldots, v_k, z\}$ is a dependent set of $M'_{a,b}$. If both $a, b \in C$, then by the similar arguments, we can show that $C \cup z$ is a dependent set of $M'_{a,b}$.

In the next Lemma, we characterize the circuits of $M'_{a,b}$ containing the element z.

Lemma 2.4. *Let C be a circuit of p-matroid M. Then $C \cup z$ is a circuit of $M'_{a,b}$ if and only if C is an np-circuit of M.*

Proof. First assume that $C \cup z$ is a circuit of $M'_{a,b}$. If C is not an np-circuit of M, then it is a p-circuit of M, and hence it is also a circuit of $M_{a,b}$ and $M'_{a,b}$, as well. Thus we get a circuit C contained in $C \cup z$, a contradiction.

Conversely, suppose C is an np-circuit of M. Then C is an independent set of $M'_{a,b}$. As noted earlier, $C \cup z$ is a dependent set of $M'_{a,b}$. On the contrary, assume that $C \cup z$ is not a circuit of $M'_{a,b}$, and $C_1 \subset C \cup z$ be a circuit of $M'_{a,b}$. One of the following two cases occurs.

Case 1: $z \notin C_1$. Then C_1 is a circuit contained in C, which is contradictory to the fact that C is independent in $M'_{a,b}$.

Case 2: $z \in C_1$. Then $C_1 \setminus z$ is a dependent set of M contained in the circuit C which is not possible. Thus $C \cup z$ is a circuit of $M'_{a,b}$.

We denote the collection of circuits described in Lemma 2.4 by $\mathcal{C}_z$.

Theorem 2.5. *Let M be a p-matroid on the ground set E and $\{a, b\} \subset E$. Then $\mathcal{C}(M'_{a,b}) = \mathcal{C}(M_{a,b}) \cup \mathcal{C}_z$.*

Proof. The inclusion $\mathcal{C}(M_{a,b}) \cup \mathcal{C}_z \subset \mathcal{C}(M'_{a,b})$ follows from the Definition 2.1 and Lemma 2.4. For the other inclusion, let $C \in \mathcal{C}(M'_{a,b})$. If $z \notin C$, then $C \in \mathcal{C}(M_{a,b})$. Otherwise, $C \in \mathcal{C}_z$.

Example 2.6. Consider the matroid R_8, the vector matroid of the following matrix A over field $GF(3)$.

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \end{pmatrix} \end{array} \qquad A'_{3,5} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

For $a = 3$, $b = 5$ and $\alpha = 1$ the representation of element splitting matroid $M'_{3,5}$ over $GF(3)$ is given by the matrix $A'_{3,5}$. The collections of circuits of M, $M_{3,5}$ and $M'_{3,5}$ are given in the following table.

Circuits of M	**Circuits of $M_{3,5}$**	**Circuits of $M'_{3,5}$**
$\{1,2,3,4,5\}$, $\{1,2,7,8\}$, $\{1,4,6,7\}$	$\{1,2,3,4,5\}$, $\{1,2,7,8\}$, $\{1,4,6,7\}$	$\{1,2,3,4,5\}$, $\{1,2,7,8\}$, $\{1,4,6,7\}$
$\{2,4,6,8\}$,$\{3,5,6,7,8\}$	$\{2,4,6,8\}$,$\{3,5,6,7,8\}$	$\{2,4,6,8\}$,$\{3,5,6,7,8\}$
-	$\{1,2,3,5,6,7\}$, $\{1,2,3,5,6,8\}$	$\{1,2,3,5,6,7\}$, $\{1,2,3,5,6,8\}$
-	$\{1,3,4,5,6,8\}$, $\{1,3,4,5,7,8\}$	$\{1,3,4,5,6,8\}$, $\{1,3,4,5,7,8\}$
-	$\{2,3,4,5,6,7\}$, $\{2,3,4,5,7,8\}$	$\{2,3,4,5,6,7\}$, $\{2,3,4,5,7,8\}$
$\{1,2,3,4,6\}$, $\{1,2,3,4,7\}$	-	$\{1,2,3,4,6,9\}$, $\{1,2,3,4,7,9\}$
$\{1,2,3,4,8\}$, $\{1,2,5,6\}$	-	$\{1,2,3,4,8,9\}$, $\{1,2,5,6,9\}$
$\{1,3,5,7\}$, $\{1,3,6,8\}$	-	$\{1,3,5,7,9\}$, $\{1,3,6,8,9\}$
$\{1,4,5,8\}$, $\{1,5,6,7,8\}$	-	$\{1,4,5,8,9\}$, $\{1,5,6,7,8,9\}$
$\{2,3,5,8\}$, $\{2,3,6,7\}$	-	$\{2,3,5,8,9\}$, $\{2,3,6,7,9\}$
$\{2,4,5,7\}$, $\{2,5,6,7,8\}$	-	$\{2,4,5,7,9\}$, $\{2,5,6,7,8,9\}$
$\{3,4,5,6\}$, $\{3,4,7,8\}$	-	$\{3,4,5,6,9\}$, $\{3,4,7,8,9\}$
$\{4,5,6,7,8\}$	-	$\{4,5,6,7,8,9\}$

3. Independent sets, Bases and Rank function of $M'_{a,b}$

In this section, we describe independent sets, bases and rank function of $M'_{a,b}$. Let $\mathcal{I}_z = \{I \cup z : I \in \mathcal{I}(M)\}$.

Lemma 3.1. *Let $M \cong M[A]$ be a p-matroid with the ground set E and $M'_{a,b}$ be its element splitting matroid. Then $\mathcal{I}(M'_{a,b}) = \mathcal{I}(M_{a,b}) \cup \mathcal{I}_z$*

Proof. Notice that $\mathcal{I}(M_{a,b}) \cup \mathcal{I}_z \subseteq \mathcal{I}(M'_{a,b})$. For other inclusion, assume $T \in \mathcal{I}(M'_{a,b})$. If $z \notin T$, then $T \in \mathcal{I}(M_{a,b})$. And if $z \in T$, then $T \setminus \{z\} \in \mathcal{I}(M_{a,b})$. That is $T = I \cup z$ for some $I \in \mathcal{I}(M_{a,b})$.

Case 1 : $I \in \mathcal{I}(M)$. Then $T \in \mathcal{I}_z$.

Case 2 : $I = C \cup I'$ where C is an np-circuit of M and $I' \in \mathcal{I}(M)$. Then by Lemma 2.4, $C \cup z$ is a circuit of $M'_{a,b}$ contained in T, a contradiction.

Lemma 3.2. *Let M be a p-matroid and $\{a,b\} \subset E$. Then $\mathcal{B}(M'_{a,b}) = \mathcal{B}(M_{a,b}) \cup \mathcal{B}_z$, where $\mathcal{B}_z = \{B \cup z : B \in \mathcal{B}(M)\}$.*

Proof. It is easy to observe that $\mathcal{B}(M_{a,b}) \cup \mathcal{B}_z \subseteq \mathcal{B}(M'_{a,b})$. Next assume that $B \in \mathcal{B}(M'_{a,b})$. Then $rank(B) = rank(M) + 1$. If B contains z, then $B \setminus z$ is an independent set of $M_{a,b}$ of size $rank(M)$. Then by similar arguments given in the proof of Lemma 3.1, $B = I \cup z$, for some $I \in \mathcal{I}(M)$. Therefore $B \setminus z$ is a basis of M and $B \in \mathcal{B}_z$. If $z \notin B$, then B is an independent set of size $rank(M) + 1$. Therefore $B \in \mathcal{B}(M_{a,b})$.

In the following lemma, we provide the rank function of $M'_{a,b}$ in terms of the rank function of M.

Lemma 3.3. *Let r and r' be the rank functions of the matroids M and $M'_{a,b}$, respectively. Suppose $S \subseteq E(M)$. Then $r'(S \cup z) = r(S) + 1$, and*

$$\begin{aligned} r'(S) &= r(S), && \text{if S contains no np-circuit of M; and} \\ &= r(S) + 1, && \text{if S contains an np-circuit of M.} \end{aligned} \tag{3.1}$$

Proof. The equality $r'(S \cup z) = r(S) + 1$ follows from the definition. The proof of the Equation (1) is discussed in Corollary 2.13 of [9].

4. Connectivity of element splitting p-matroids

Let M be a matroid having the ground set E, and k be a positive integer. The k-separation of matroid M is a partition $\{S, T\}$ of E such that $|S|, |T| \geq k$ and $r(S) + r(T) - r(M) < k$. For an integer $n \geq 2$, we say M is an n-connected if M has no k- separation, where $1 \leq k \leq n - 1$.

In the following theorem, we provide a necessary and sufficient condition to preserve the connectedness of a p-matroid under element splitting operation. For $a, b \in E$, if the matroid M contains at least one *np*-circuit, then splitting operation on M with respect to a, b is called a non-trivial splitting operation.

Theorem 4.1. *Let M be a connected p-matroid on the ground set E. Then $M'_{a,b}$ is a connected p-matroid on the ground set $E \cup \{z\}$ if and only if $M_{a,b}$ is the splitting matroid obtained by applying non-trivial splitting operation on M.*

Proof. First assume that $M'_{a,b}$ is a connected p-matroid on the ground set $E \cup \{z\}$. On the contrary, suppose $M_{a,b}$ is obtained by applying trivial splitting operation. Then M contains no *np* circuits with respect to the splitting by elements a, b. Now, let $S = \{z\}$ and $T = E$. Then $r'(S) + r'(T) - r'(M'_{a,b}) = 1 + r(E) - (r(M) + 1) = 0 < 1$ gives a 1-separation of $M'_{a,b}$, which is a contradiction.

For converse part, assume that $M_{a,b}$ is the splitting matroid obtained by applying non-trivial splitting operation on M. Suppose that, $M'_{a,b}$ is not connected. It means

$M'_{a,b}$ has 1-separation, say $\{S, T\}$. Then $|S|, |T| \geq 1$ and

$$r'(S) + r'(T) - r'(M'_{a,b}) < 1. \tag{4.1}$$

Case 1 : Assume $S = \{z\}$. Then T contains an np circuit. Then Equation (4.1) gives, $1 + (1 + r(T)) - r(M) - 1 < 1 \implies r(T) < r(M)$, which is not possible.
Case 2 : Assume $|S| \geq 2$, $z \in S$. If T contains no np-circuit then Equation (4.1) yields, $(r(S \setminus z)+1)+r(T)-r(M)-1 < 1$, that is $r(S \setminus z)+r(T)-r(M) < 1$. Therefore $\{S \setminus z, T\}$ gives $1-$separation of M, a contradiction. Further, if T contains an np-circuit, then $r'(S) = r(S \setminus z) + 1$, $r'(T) = r(T) + 1$. By Equation (4.1), we get $(r(S \setminus z) + 1) + (r(T) + 1) - r(M) - 1 < 1$, which gives $r(S \setminus z) + r(T) - r(M) < 0$, which is not possible. So in either case such separation does not exist. Therefore $M'_{a,b}$ is connected.

For $p = 2$, the following sufficient condition for the element splitting operation to preserve the connectedness of the binary matroid by Shikare [17] follows immediately.

Corollary 4.2. *Let M be a connected binary matroid on a set E and $\{a, b\} \subseteq E$ such that there is a circuit of M containing exactly one member of $\{a, b\}$. Then the matroid $M'_{a,b}$ is connected.*

In Example 2.6, the p-matroid $R_8 \cong M[A]$ and its element splitting p-matroid $M'_{3,5} \cong M[A'_{3,5}]$ both are connected. In the next result we give a necessary and sufficient condition to preserve 3-connectedness of a p-matroid under the element splitting operation.

Theorem 4.3. *Let M be a 3-connected p-matroid. Then $M'_{a,b}$ is 3-connected p-matroid if and only if for every $t \in E(M)$ there is an np-circuit of M not containing t.*

Proof. Let $M'_{a,b}$ be 3-connected p-matroid. On contrary, if there is an element $t \in E(M)$ contained in every np-circuit of M. Take $S = \{z, t\}$ and $T = E \setminus S$. Then $r'(S)+r'(T)-r'(M'_{a,b}) = r(\{t\})+1+r(T)-r(M)-1 = r(\{t\})+r(T)-r(M) = 1 < 2$. Because, in this case, $t \in cl(T)$ hence $r(T) = r(M)$. That is $\{S, T\}$ forms a 2-separation of $M'_{a,b}$, a contradiction.
For converse part suppose, for every $t \in E(M)$ there is an np-circuit of M not containing t. On the contrary assume that $M'_{a,b}$ is not a 3-connected matroid. Then there exists a k separation, for $k \leq 2$, of $M'_{a,b}$. By Theorem 4.1, k can not be equal to 1. For $k = 2$, let $\{S, T\}$ be a 2-separation of $M'_{a,b}$. Then $\{S, T\}$ is a partition of $E \cup \{z\}$ such that $|S|, |T| \geq 2$ and

$$r'(S) + r'(T) - r'(M'_{a,b}) < 2. \tag{4.2}$$

Case 1 :Suppose $S = \{z, t\}$, $t \in E(M)$. By hypothesis, T contains an *np*-circuit not containing t. Then Equation (4.2) gives, $(r(\{t\})+1)+(1+r(T))-r(M)-1 < 2 \implies r(t) + r(T) - r(M) < 1$. Thus $\{\{t\}, T\}$ forms a 1-separation of M, which is a contradiction.

Case 2 : Suppose $z \in S$ and $|S| \geq 3$. If T contains no *np*-circuit then Equation (4.2) yields $(r(S \setminus z) + 1) + r(T) - r(M) - 1 < 2 \implies r(S \setminus z) + r(T) - r(M) < 2$. Therefore $\{S \setminus z, T\}$ gives a 2-separation of M, a contradiction.

Further, if T contains an *np*-circuit, then $r'(S) = r(S \setminus z) + 1$, $r'(T) = r(T) + 1$. By Equation (4.2), we get $(r(S \setminus z) + 1) + (r(T) + 1) - r(M) - 1 < 2 \implies r(S \setminus z) + r(T) - r(M) < 1$. Thus, $\{S \setminus z, T\}$ gives a 1-separation of M, a contradiction. So in either case such partition does not exist. Therefore $M'_{a,b}$ is 3-connected.

5. Applications

For Eulerian matroid M on the ground set E there exists disjoint circuits C_1,C_2, $\ldots$,C_k of M such that $E = C_1 \cup C_2 \cup ... \cup C_k$. We call the collection $\{C_1, C_2, \ldots, C_k\}$ a circuit decomposition of M.

Let $\{a, b\} \subset E$. We say a circuit decomposition $\tilde{\mathrm{C}} = \{C_1, C_2, \ldots, C_k\}$ of M an *ep-decomposition* of M if it contains exactly one *np*-circuit with respect to the a, b splitting of M. In the next proposition, we give a sufficient condition to yield Eulerian p-matroids from Eulerian p-matroids after the element splitting operation.

Proposition 5.1. *Let M be Eulerian p-matroid and $a, b \in E$. If M has an ep-decomposition, then $M'_{a,b}$ is Eulerian p-matroid.*

Proof. Let $\tilde{\mathrm{C}} = \{C_1, C_2, \ldots, C_k\}$ be an *ep-decomposition* of M and C_1 be an *np*-circuit in it. Then $C_1 \cup z$ is a circuit of $M'_{a,b}$. Thus $\{C_1 \cup z, C_2, \ldots, C_k\}$ is the desired circuit decomposition of $M'_{a,b}$.

Proposition 5.2. *Let $M'_{a,b}$ is Eulerian p-matroid and $\tilde{C} = \{C_1, C_2, \ldots, C_k\}$ be a circuit decomposition of $M'_{a,b}$. If $\tilde{C}$ contains no member which be a union of an np-circuit and an independent set of M, then M is Eulerian and has an ep-decomposition.*

Proof. Assume, without loss of generality, $z \in C_1$. Then $C_1 \in \mathcal{C}_z$ and $C_1 \setminus z$ is an *np*-circuit of M. We will show $C_1 \setminus z$ contains both a and b. On the contrary assume that $C_1 \setminus z$ contains only a. Then $b \in C_i$ for some $i \in \{2, 3, \ldots, k\}$. Since C_i is also a circuit of $M_{a,b}$ containing only b, by Theorem 2.10 of [9] it must be a union of an *np*-circuit and an independent set of M, which is a contradiction to the hypothesis. Therefore $C_1 \setminus z$ contains both a and b and the collection $\{C_1 \setminus z, C_2, \ldots, C_k\}$ forms an *ep*-decomposition of M.

In Example 2.6, the matroid R_8 is Eulerian with *ep*-decomposition $E = C_1 \cup C_2$, where $C_1 = \{2,4,6,8\}$ is a p-circuit and $C_2 = \{1,3,5,7\}$ is an np-circuit. An element splitting matroid $M'_{3,5}$ is also Eulerian with circuit decomposition $E \cup z = C_1 \cup (C_2 \cup z)$.

M. Borowiecki [2] defined Hamiltonian matroid as a matroid containing a circuit of size $r(M)+1$. This circuit is called the Hamiltonian circuit of the matroid M. In the next corollary, we give a sufficient condition to yield Hamiltonian matroid from Hamiltonian matroid after the element splitting operation.

Corollary 5.3. *If M is Hamiltonian matroid with an np-circuit of size $r(M)+1$, then $M'_{a,b}$ is Hamiltonian.*
Proof. Let C be an np-circuit of M of size $r(M)+1$. Then by Proposition 2.4, $C \cup z$ is a circuit in $M'_{a,b}$ of size $r(M)+2$.

In Example 2.6, the matroid $R_8 \cong M[A]$ is Hamiltonian and its element splitting matroid $M'_{3,5} \cong M[A'_{3,5}]$ is also Hamiltonian.

Let M be a matroid of rank r. M is called a paving matroid, if every circuit of M is of the size r or $r+1$. All binary paving matroids are characterized by Acketa [1]. Oxley [14] gave a characterization of ternary paving matroids. A paving matroid M does not always yield a paving matroid after splitting. In the next proposition, we characterize the element splitting p-matroids $M'_{a,b}$ that are paving.

Proposition 5.4. *Let M be a paving p-matroid of rank r, $\{a,b\} \subset E(M)$. Then the element splitting matroid $M'_{a,b}$ is also paving if and only if every circuit $C \in \mathcal{C}(M)$ of size r is an np-circuit.*
We conclude this paper by proposing following problem:
Rota conjectured that the family of matroids that are representable over finite fields has only finitely many excluded minors [7]. For example, the 4-point line, $U_{2,4}$, is the only excluded minor for the class of binary matroids. In the following example, we demonstrate that there exist a splitting of the ternary matroid $U_{2,4}$, which yields a graphic matroid.

Example 5.5. Let the matrix A represents the ternary matroid $U_{2,4}$ and the vector matroid of $A_{1,3}$ represents the splitting matroid $M[A_{1,3}]$.

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{array} \qquad A_{1,3} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad A'_{1,3} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Observe that

- the splitting matroid $M[A_{1,3}]$ is binary and matrix $B = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$ gives its binary representation.
- $A'_{1,3}/5 = U_{2,4}$.

However, the element splitting operation on $U_{2,4}$ does not give a binary matroid. With this observation, we propose the following question:
For a given ternary matroid M, does there always exist a pair of elements $\{a, b\}$ in $E(M)$ such that the splitting matroid $M_{a,b}$ is binary (graphic)?

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