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A NEW TYPE OF REGULARITY VIA FUZZY α-PREOPEN SET

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Abstract: This paper deals with a new type of set, viz., fuzzy α-preopen set, the class of which is strictly larger than that of fuzzy open set as well as fuzzy α-open set [4]. Using this newly defined fuzzy set, here we introduce and study fuzzy α-precontinuous and fuzzy α-preirresolute functions. It is shown that fuzzy α-preirresolute function is fuzzy α-precontinuous, but the converse may not be true, in general. Next we introduce fuzzy α-preregular space, in which fuzzy open set and fuzzy α-preopen set coincide. Lastly, some applications of the functions defined here are established.

Keywords and Phrases: Fuzzy α-open set, fuzzy α-preopen set, fuzzy α-nbd of a fuzzy point, fuzzy α-precontinuous function, fuzzy α-preregular space, fuzzy α-preirresolute function.

2020 Mathematics Subject Classification: 54A40, 03E72.

1. Introduction

After introducing fuzzy topology by Chang [5], many mathematicians have engaged themselves to introduce different types of fuzzy open-like sets. In [7], fuzzy strongly preopen set and fuzzy strong precontinuous function are introduced and studied by using fuzzy preopen set [8] as a basic tool whereas in [3], fuzzy pre-semiopen set and fuzzy pre-semi-continuous function are introduced and studied by using fuzzy semiopen set [1] introduced by K. K. Azad. In [4], Bin Shahna introduced fuzzy α-open set. Using this set as a basic tool, here we introduce fuzzy α-preopen set, a larger class of sets than that of fuzzy open as well as fuzzy α-open set and fuzzy strongly preopen set and weaker than fuzzy pre-semiopen set. In [5], Chang introduced fuzzy continuous function. Here we introduce fuzzy α-precontinuous function, the class of which is strictly larger than that of fuzzy continuous function as well as fuzzy strong precontinuous function, but weaker than fuzzy pre-semi-continuous function.

2. Preliminaries

Throughout this paper, (X, τ) or simply by X we shall mean a fuzzy topological space. A fuzzy set A is a function from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$ [11]. The support [11] of a fuzzy set A, denoted by $suppA$ or A_0 and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [11] of a fuzzy set A in an fts X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [11] while AqB means A is quasi-coincident (q-coincident, for short) [9] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [5] and fuzzy interior [5] of A respectively. A fuzzy set A in X is called a fuzzy neighbourhood (fuzzy nbd, for short) [9] of a fuzzy point x_t if there exists a fuzzy open set G in X such that $x_t \in G \leq A$. If, in addition, A is fuzzy open, then A is called fuzzy open nbd of x_t. A fuzzy set A is said to be a fuzzy quasi neighbourhood (fuzzy q-nbd, for short) of a fuzzy point x_t in an fts X if there is a fuzzy open set U in X such that $x_t q U \leq A$ [9]. If, in addition, A is fuzzy open, then A is called a fuzzy open q-nbd of x_t [9].

A fuzzy set A in an fts (X, τ) is called fuzzy α-open [4] (resp., fuzzy semiopen [1], fuzzy preopen [8]) if $A \leq int(cl(intA))$ (resp., $A \leq cl(intA), A \leq int(clA)$). The complement of a fuzzy α-open set (resp., fuzzy semiopen, fuzzy preopen) is called fuzzy α-closed [4] (resp., fuzzy semiclosed [1], fuzzy preclosed [8]). The union (intersection) of all fuzzy α-open (resp., fuzzy α-closed) sets contained in (resp., containing) a fuzzy set A is called fuzzy α-interior [4] (resp., fuzzy α-closure [4]) of A, denoted by $\alpha intA$ (resp., αclA). The union of all fuzzy semiopen sets contained in a fuzzy set A is called fuzzy semiinterior [1] of A, denoted by $sintA$ and the intersection of all fuzzy preclosed sets containing a fuzzy set A is called fuzzy preclosure of A [8], denoted by $pclA$. A fuzzy set A in an fts X is called fuzzy α-neighbourhood (fuzzy α-nbd, for short) of a fuzzy point x_t in X if there exists a fuzzy α-open set U in X such that $x_t \in U \leq A$ [4]. The collection of all fuzzy α-open (resp., fuzzy α-closed) sets in an fts X is denoted by $F\alpha O(X)$ (resp., $F\alpha C(X)$).

3. Fuzzy α-Preopen Set : Some Properties

In this section, we introduce a new class of fuzzy open-like set, viz., fuzzy α-preopen set, the class of which is strictly larger than that of fuzzy open set as well as fuzzy α-open set and fuzzy strongly preopen set and weaker than fuzzy pre-semiopen set. Some basic properties of this set is discussed here. Afterwards, we introduce a new type of fuzzy closure-like operator which is an idempotent operator.

Definition 3.1. *A fuzzy set A in an fts (X, τ) is called fuzzy α-preopen (resp., fuzzy strongly preopen* [7]*, fuzzy pre-semiopen* [3]*) if $A \leq \alpha int(clA)$ (resp, $A \leq int(pclA)$, $A \leq sint(clA)$). The complement of fuzzy α-preopen set is called fuzzy α-preclosed set.*
The collection of fuzzy α-preopen (resp., fuzzy α-preclosed) sets in (X, τ) is denoted by $F\alpha PO(X)$ (resp., $F\alpha PC(X)$) and that of fuzzy strongly preopen (resp., fuzzy pre-semiopen) set is denoted by $FSPO(X)$ (resp., $FPSO(X)$).
The union (resp., intersection) of all fuzzy α-preopen (resp., fuzzy α-preclosed) sets contained in (containing) a fuzzy set A is called fuzzy α-preinterior (resp., fuzzy α-preclosure) of A, denoted by $\alpha pintA$ (resp., $\alpha pclA$).

Result 3.2. *Union of two fuzzy α-preopen sets in an fts X is also so.*
Proof. Let $A, B \in F\alpha PO(X)$. Then $A \leq \alpha int(clA), B \leq \alpha int(clB)$. Now $\alpha int(cl(A \bigvee B)) = \alpha int(clA \bigvee clB) \geq \alpha int(clA) \bigvee \alpha int(clB) \geq A \bigvee B$.

Remark 3.3. *Intersection of two fuzzy α-preopen sets need not be so, follows from the next example.*

Example 3.4. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Consider two fuzzy sets U, V defined by $U(a) = 0.4, U(b) = 0.5, V(a) = 0.6, V(b) = 0.4$. Then clearly $U, V \in F\alpha PO(X)$. Let $W = U \bigwedge V$. Then $W(a) = W(b) = 0.4$. Now $\alpha int(clW) \not\geq W \Rightarrow W \notin F\alpha PO(X)$.

Note 3.5. So we can conclude that the set of all fuzzy α-preopen sets in an fts do not form a fuzzy topology.

Remark 3.6. *It is clear from definitions that*
(i) fuzzy open and fuzzy α-open sets are fuzzy α-preopen,
(ii) fuzzy strongly preopen set implies fuzzy α-preopen set which also implies fuzzy pre-semiopen set. But reverse implications are not necessarily true follow from the next examples.

Example 3.7. (i) Consider Example 3.4. Here $U \in F\alpha PO(X)$. But $U \notin \tau$, $U \notin F\alpha O(X)$.
(ii) $F\alpha PO(X) \subsetneq FPSO(X)$
Let $X = \{a, b\}, \tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$. Then $\alpha int(clB) = A \not\geq B \Rightarrow B \notin F\alpha PO(X)$. But $sint(clB) = 1_X \setminus A \geq B \Rightarrow B \in FPSO(X)$.
(iii) $FSPO(X) \subsetneq F\alpha PO(X)$
Let $X = \{a, b\}, \tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.55$. Then (X, τ) is an fts. Consider the fuzzy set C defined by $C(a) = C(b) = 0.5$. Now $int(pclC) = A \not\geq C \Rightarrow C \notin FSPO(X)$. But $\alpha int(clC) = B \geq C \Rightarrow C \in F\alpha PO(X)$.

Now we introduce a new type of fuzzy neighbourhood of a fuzzy point, the class of which is strictly greater than that of fuzzy neighbourhood of the point.

Definition 3.8. *A fuzzy set A in an fts (X, τ) is called fuzzy α-pre neighbourhood (fuzzy α-pre nbd, for short) of a fuzzy point x_α if there exists a fuzzy α-preopen set U in X such that $x_\alpha \in U \leq A$. If, in addition, A is fuzzy α-preopen, then A is called fuzzy α-preopen nbd of x_α.*

Definition 3.9. *A fuzzy set A in an fts (X, τ) is called fuzzy α-pre quasi neighbourhood (fuzzy α-pre q-nbd, for short) of a fuzzy point x_α if there exists a fuzzy α-preopen set U in X such that $x_\alpha q U \leq A$. If, in addition, A is fuzzy α-preopen, then A is called fuzzy α-preopen q-nbd of x_α.*

Remark 3.10. *Since a fuzzy open set is fuzzy α-preopen, we can conclude that*
(i) fuzzy nbd (resp., fuzzy open nbd) of a fuzzy point x_α is a fuzzy α-pre nbd (resp., fuzzy α-preopen nbd) of x_α,
(ii) fuzzy q-nbd (resp., fuzzy open q-nbd) of a fuzzy point x_α is a fuzzy α-pre q-nbd (resp., fuzzy α-preopen q-nbd) of x_α.
But the reverse implications are not necessarily true follow from the following example.

Example 3.11. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Consider two fuzzy sets B, C defined by $B(a) = 0.4, B(b) = 0.5, C(a) = 0.6, C(b) = 0.4$. Then $B, C \in F\alpha O(X)$. Consider two fuzzy points $a_{0.3}$ and $a_{0.45}$. Now$a_{0.3} \in B \leq B \Rightarrow B$ is a fuzzy α-pre nbd as well as fuzzy α-preopen nbd of $a_{0.3}$. But there does not exist any fuzzy open set U in (X, τ) such that $a_{0.3} \in U \leq B$. So B is not a fuzzy nbd and fuzzy open nbd of $a_{0.3}$.
Next $a_{0.45} q C \leq C \Rightarrow C$ is a fuzzy α-pre q-nbd as well as fuzzy α-preopen q-nbd of $a_{0.45}$. But there does not exist a fuzzy open set U in X with $a_{0.45} q U \leq C \Rightarrow C$ is

not a fuzzy q-nbd and fuzzy open q-nbd of $a_{0.45}$.

Theorem 3.12. *For any fuzzy set A in an fts (X,τ), $x_t \in \alpha pclA$ if and only if every fuzzy α-preopen q-nbd U of x_t, UqA.*

Proof. Let $x_t \in \alpha pclA$ for any fuzzy set A in an fts (X,τ). Let $U \in F\alpha PO(X)$ with x_tqU. Then $U(x)+\alpha > 1 \Rightarrow x_t \notin 1_X \setminus U \in F\alpha PC(X)$. Then by definition, $A \not\leq 1_X \setminus U \Rightarrow$ there exists $y \in X$ such that $A(y) > 1-U(y) \Rightarrow A(y)+U(y) > 1 \Rightarrow UqA$.

Conversely, let the given condition hold. Let $U \in F\alpha PC(X)$ with $A \leq U$ ... (1). We have to show that $x_t \in U$, i.e., $U(x) \geq t$. If possible, let $U(x) < t$. Then $1-U(x) > 1-t \Rightarrow x_tq(1_X \setminus U)$ where $1_X \setminus U \in F\alpha PO(X)$. By hypothesis, $(1_X \setminus U)qA \Rightarrow$ there exists $y \in X$ such that $1-U(y)+A(y) > 1 \Rightarrow A(y) > U(y)$, contradicts (1).

Theorem 3.13. *$\alpha pcl(\alpha pclA) = \alpha pclA$ for any fuzzy set A in an fts (X,τ).*

Proof. Let $A \in I^X$. Then

$$A \leq \alpha pclA \Rightarrow \alpha pclA \leq \alpha pcl(\alpha pclA). \tag{1}$$

Conversely, let $x_t \in \alpha pcl(\alpha pclA)$. If possible, let $x_t \notin \alpha pclA$. Then there exists $U \in F\alpha PO(X)$,

$$x_tqU, U \not{q} A \tag{2}$$

But as $x_t \in \alpha pcl(\alpha pclA)$, $Uq(\alpha pclA) \Rightarrow$ there exists $y \in X$ such that $U(y) + (\alpha pclA)(y) > 1 \Rightarrow U(y)+s > 1$ where $s = (\alpha pclA)(y)$. Then $y_s \in \alpha pclA$ and y_sqU where $U \in F\alpha PO(X)$. Then by definition, UqA, contradicts (2). So

$$\alpha pcl(\alpha pclA) \leq \alpha pclA \tag{3}$$

Combining (1) and (3), we get the result.

4. Fuzzy α-Precontinuous Function: Some Characterizations

In this section a new type of function is introduced and studied, the class of which is strictly larger than that of fuzzy continuous function [5] and fuzzy strong precontinuous function [7] but weaker than fuzzy pre-semi-continuous function [3].

Definition 4.1. *A function $f : X \to Y$ is said to be fuzzy α-precontinuous if for each fuzzy point x_t in X and every fuzzy nbd V of $f(x_t)$ in Y, $cl(f^{-1}(V))$ is a fuzzy α-nbd of x_t in X.*

Theorem 4.2. *For a function $f : X \to Y$, the following statements are equivalent:*
(a) f is fuzzy α-precontinuous,
(b) $f^{-1}(B) \leq \alpha int(cl(f^{-1}(B)))$, for all fuzzy open set B of Y,

(c) $f(\alpha clA) \leq cl(f(A))$, for all fuzzy open set A in X.

Proof. (a) $\Rightarrow$ (b). Let B be any fuzzy open set in Y and $x_t \in f^{-1}(B)$. Then $f(x_t) \in B \Rightarrow B$ is a fuzzy nbd of $f(x_t)$ in Y. By (a), $cl(f^{-1}(B))$ is a fuzzy α-nbd of x_t in X. So $x_t \in \alpha int(cl(f^{-1}(B)))$. Since x_t is taken arbitrarily, $f^{-1}(B) \leq \alpha int(cl(f^{-1}(B)))$.
(b) $\Rightarrow$ (a). Let x_t be a fuzzy point in X and B be a fuzzy nbd of $f(x_t)$ in Y. Then $x_t \in f^{-1}(B) \leq \alpha int(cl(f^{-1}(B)))$ (by (b)) $\leq cl(f^{-1}(B))$. So $cl(f^{-1}(B))$ is a fuzzy α-nbd of x_t in X.
(b) $\Rightarrow$ (c). Let A be a fuzzy open set in X. Then $1_Y \setminus cl(f(A))$ is a fuzzy open set in Y. By (b), $f^{-1}(1_Y \setminus cl(f(A))) \leq \alpha int(cl(f^{-1}(1_Y \setminus cl(f(A))))) = \alpha int(cl(1_X \setminus f^{-1}(cl(f(A))))) \leq \alpha int(cl(1_X \setminus f^{-1}(f(A)))) \leq \alpha int(cl(1_X \setminus A)) = \alpha int(1_X \setminus A) = 1_X \setminus \alpha clA$. Then $\alpha clA \leq 1_X \setminus f^{-1}(1_Y \setminus cl(f(A))) = f^{-1}(cl(f(A)))$. So $f(\alpha clA) \leq cl(f(A))$.
(c) $\Rightarrow$ (b). Let B be any fuzzy open set in Y. Then $int(f^{-1}(1_Y \setminus B))$ is a fuzzy open set in X. By (c), $f(\alpha cl(int(f^{-1}(1_Y \setminus B)))) \leq cl(f(int(f^{-1}(1_Y \setminus B)))) \leq cl(f(f^{-1}(1_Y \setminus B))) \leq cl(1_Y \setminus B) = 1_Y \setminus B \Rightarrow B \leq 1_Y \setminus f(\alpha cl(int(f^{-1}(1_Y \setminus B))))$. Then $f^{-1}(B) \leq f^{-1}(1_Y \setminus f(\alpha cl(int(f^{-1}(1_Y \setminus B))))) = 1_X \setminus f^{-1}(f(\alpha cl(int(f^{-1}(1_Y \setminus B))))) \leq 1_X \setminus \alpha cl(int(f^{-1}(1_Y \setminus B))) = 1_X \setminus \alpha cl(int(1_X \setminus f^{-1}(B))) = \alpha int(cl(f^{-1}(B)))$.

Note 4.3. It is clear from above theorem that the inverse image under fuzzy α-precontinuous function of any fuzzy open set is fuzzy α-preopen.

Theorem 4.4. *For a function $f : X \to Y$, the following statements are equivalent:*
(a) f is fuzzy α-precontinuous,
(b) $f^{-1}(B) \leq \alpha int(cl(f^{-1}(B)))$, for all fuzzy open set B of Y,
(c) for each fuzzy point x_t in X and each fuzzy open nbd V of $f(x_t)$ in Y, there exists $U \in F\alpha PO(X)$ containing x_t such that $f(U) \leq V$,
(d) $f^{-1}(F) \in F\alpha PC(X)$, for all fuzzy closed sets F in Y,
(e) for each fuzzy point x_t in X, the inverse image under f of every fuzzy nbd of $f(x_t)$ is a fuzzy α-pre nbd of x_t in X,
(f) $f(\alpha pclA) \leq cl(f(A))$, for all fuzzy set A in X,
(g) $\alpha pcl(f^{-1}(B)) \leq f^{-1}(clB)$, for all fuzzy set B in Y,
(h) $f^{-1}(intB) \leq \alpha pint(f^{-1}(B))$, for all fuzzy set B in Y,
(i) for every basic open fuzzy set V in Y, $f^{-1}(V) \in F\alpha PO(X)$.

Proof. (a) $\Leftrightarrow$ (b). Follows from Theorem 4.2 (a) $\Leftrightarrow$ (b).
(b) $\Rightarrow$ (c). Let x_t be a fuzzy point in X and V be a fuzzy open nbd of $f(x_t)$ in Y. By (b), $f^{-1}(V) \leq \alpha int(cl(f^{-1}(V)))$ ... (1). Now $f(x_t) \in V \Rightarrow x_t \in f^{-1}(V)$ ($= U$, say). Then $x_t \in U$ and by (1), $U(= f^{-1}(V)) \in F\alpha PO(X)$ and $f(U) = f(f^{-1}(V)) \leq V$.
(c) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and let $x_t \in f^{-1}(V)$. Then $f(x_t) \in$

$V \Rightarrow V$ is a fuzzy open nbd of $f(x_t)$ in Y. By (c), there exists $U \in F\alpha PO(X)$ containing x_t such that $f(U) \leq V$. Then $x_t \in U \leq f^{-1}(V)$. Now $U \leq \alpha int(clU)$. Then $U \leq \alpha int(clU) \leq \alpha int(cl(f^{-1}(V))) \Rightarrow x_t \in U \leq \alpha int(cl(f^{-1}(V)))$. Since x_t is taken arbitrarily, $f^{-1}(V) \leq \alpha int(cl(f^{-1}(V)))$.
(b) $\Leftrightarrow$ (d). Obvious.
(b) $\Rightarrow$ (e). Let W be a fuzzy nbd of $f(x_t)$ in Y. Then there exists a fuzzy open set V in Y such that $f(x_t) \in V \leq W \Rightarrow V$ is a fuzzy open nbd of $f(x_t)$ in Y. Then by (b), $f^{-1}(V) \in F\alpha PO(X)$ and $x_t \in f^{-1}(V) \leq f^{-1}(W) \Rightarrow f^{-1}(W)$ is a fuzzy α-pre nbd of x_t in X.
(e) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and $x_t \in f^{-1}(V)$. Then $f(x_t) \in V$. Then V is a fuzzy open nbd of $f(x_t)$ in Y. By (e), there exists $U \in F\alpha PO(X)$ containing x_t such that $U \leq f^{-1}(V) \Rightarrow x_t \in U \leq \alpha int(clU) \leq \alpha int(cl(f^{-1}(V)))$. Since x_t is taken arbitrarily, $f^{-1}(V) \leq \alpha int(cl(f^{-1}(V)))$.
(d) $\Rightarrow$ (f). Let $A \in I^X$. Then $cl(f(A))$ is a fuzzy closed set in Y. By (d), $f^{-1}(cl(f(A))) \in F\alpha PC(X)$ containing A. Therefore, $\alpha pclA \leq \alpha pcl(f^{-1}(cl(f(A))))$ $= f^{-1}(cl(f(A))) \Rightarrow f(\alpha pclA) \leq cl(f(A))$.
(f) $\Rightarrow$ (d). Let B be a fuzzy closed set in Y. Then $f^{-1}(B) \in I^X$. By (f), $f(\alpha pcl(f^{-1}(B))) \leq cl(f(f^{-1}(B))) \leq clB = B \Rightarrow \alpha pcl(f^{-1}(B)) \leq f^{-1}(B) \Rightarrow f^{-1}(B) \in F\alpha PC(X)$.
(f) $\Rightarrow$ (g). Let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. By (f), $f(\alpha pcl(f^{-1}(B))) \leq cl(f(f^{-1}(B))) \leq clB \Rightarrow \alpha pcl(f^{-1}(B)) \leq f^{-1}(clB)$.
(g) $\Rightarrow$ (f). Let $A \in I^X$. Let $B = f(A)$. Then $B \in I^Y$. By (g), $\alpha pcl(f^{-1}(f(A))) \leq f^{-1}(cl(f(A))) \Rightarrow \alpha pclA \leq f^{-1}(cl(f(A))) \Rightarrow f(\alpha pclA) \leq cl(f(A))$.
(b) $\Rightarrow$ (h). Let $B \in I^Y$. Then $intB$ is a fuzzy open set in Y. By (b), $f^{-1}(intB) \leq \alpha int(cl(f^{-1}(intB))) \Rightarrow f^{-1}(intB) \in F\alpha PO(X) \Rightarrow f^{-1}(intB) = \alpha pint(f^{-1}(intB))$ $\leq \alpha pint(f^{-1}(B))$.
(h) $\Rightarrow$ (b). Let A be any fuzzy open set in Y. Then $f^{-1}(A) = f^{-1}(intA) \leq \alpha pint(f^{-1}(A))$ (by (h)) $\Rightarrow f^{-1}(A) \in F\alpha PO(X)$.
(b) $\Rightarrow$ (i). Obvious.
(i) $\Rightarrow$ (b). Let W be any fuzzy open set in Y. Then there exists a collection $\{W_\alpha : \alpha \in \Lambda\}$ of fuzzy basic open sets in Y such that $W = \bigvee_{\alpha\in\Lambda} W_\alpha$. Now $f^{-1}(W) = f^{-1}(\bigvee_{\alpha\in\Lambda} W_\alpha) = \bigvee_{\alpha\in\Lambda} f^{-1}(W_\alpha) \in F\alpha PO(X)$ (by (i) and by Result 3.2). Hence (b) follows.

Theorem 4.5. *A function $f : X \to Y$ is fuzzy α-precontinuous if and only if for each fuzzy point x_t in X and each fuzzy open q-nbd V of $f(x_t)$ in Y, there exists a fuzzy α-pre q-nbd W of x_t in X such that $f(W) \leq V$.*

Proof. Let f be fuzzy α-precontinuous function and x_t be a fuzzy point in X and V be a fuzzy open q-nbd of $f(x_t)$ in Y. Then $f(x_t)qV$. Let $f(x) = y$. Then $V(y)+t > 1 \Rightarrow V(y) > 1-t \Rightarrow V(y) > \beta > 1-t$, for some real number β. Then V is a fuzzy open nbd of y_β. By Theorem 4.4 (a)$\Rightarrow$(c), there exists $W \in F\alpha PO(X)$ containing x_β, i.e., $W(x) \geq \beta$ such that $f(W) \leq V$. Then $W(x) \geq \beta > 1-t \Rightarrow x_tqW$ and $f(W) \leq V$.
Conversely, let the given condition hold and let V be a fuzzy open set in Y. Put $W = f^{-1}(V)$. If $W = 0_X$, then we are done. Suppose $W \neq 0_X$. Then for any $x \in W_0$, let $y = f(x)$. Then $W(x) = [f^{-1}(V)](x) = V(f(x)) = V(y)$. Let us choose $m \in \mathcal{N}$ where $\mathcal{N}$ is the set of all natural numbers such that $1/m \leq W(x)$. Put $\alpha_n = 1 + 1/n - W(x)$, for all $n \in \mathcal{N}$. Then for $n \in \mathcal{N}$ and $n \geq m$, $1/n \leq 1/m \Rightarrow 1+1/n \leq 1+1/m \Rightarrow \alpha_n = 1+1/n-W(x) \leq 1+1/m-W(x) \leq 1$. Again $\alpha_n > 0$, for all $n \in \mathcal{N} \Rightarrow 0 < \alpha_n \leq 1$ so that $V(y) + \alpha_n > 1 \Rightarrow y_{\alpha_n}qV \Rightarrow V$ is a fuzzy open q-nbd of y_{α_n}. By the given condition, there exists $U_n^x \in F\alpha PO(X)$ such that $x_{\alpha_n}qU_n^x$ and $f(U_n^x) \leq V$, for all $n \geq m$. Let $U^x = \bigvee\{U_n^x : n \in \mathcal{N}, n \geq m\}$. Then $U^x \in F\alpha PO(X)$ (by Result 3.2) and $f(U^x) \leq V$. Again $n \geq m \Rightarrow U_n^x(x) + \alpha_n > 1 \Rightarrow U_n^x(x) + 1 + 1/n - W(x) > 1 \Rightarrow U_n^x(x) > W(x) - 1/n \Rightarrow U_n^x(x) \geq W(x)$, for each $x \in W_0$. Then $W \leq U_n^x$, for all $n \geq m$ and for all $x \in W_0 \Rightarrow W \leq U^x$, for all $x \in W_0 \Rightarrow W \leq \bigvee_{x\in W_0} U^x = U$ (say) ... (1) and $f(U^x) \leq V$, for all $x \in W_0 \Rightarrow f(U) \leq V \Rightarrow U \leq f^{-1}(f(U)) \leq f^{-1}(V) = W$ ... (2). By (1) and (2), $U = W = f^{-1}(V) \Rightarrow f^{-1}(V) \in F\alpha PO(X)$. Hence by Theorem 4.2, f is fuzzy α-precontinuous function.

Remark 4.6. *Composition of two fuzzy α-precontinuous functions need not be so, follows from the following example.*

Example 4.7. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.4$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are fuzzy α-precontinuous functions. Now $B \in \tau_3$. $(i_2 \circ i_1)^{-1}(B) = B \not\leq \alpha int_{\tau_1}(cl_{\tau_1}B) = 0_X \Rightarrow B \notin F\alpha PO(X, \tau_1) \Rightarrow i_2 \circ i_1$ is not fuzzy α-precontinuous function.

Let us now recall the following definitions from [5] for ready references.

Definition 4.8. *A function $f : X \to Y$ is called fuzzy continuous function* [5] *(resp., fuzzy strong precontinuous function* [7]*, fuzzy pre-semi-continuous function* [3]*) if the inverse image of every fuzzy open set in Y is fuzzy open (resp., fuzzy strongly preopen, fuzzy pre-semiopen) set in X.*

Note 4.9. It is clear from above definitions that
(i) fuzzy continuous function is fuzzy α-precontinuous function,
(ii) fuzzy strong precontinuity $\Rightarrow$ fuzzy α-precontinuity $\Rightarrow$ fuzzy pre-semicontinuity.
But the converses are not necessarily true follow from the next examples.

Example 4.10. (i) Fuzzy α-precontinuity $\not\Rightarrow$ fuzzy continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5 = A(b)$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $A \in \tau_2, i^{-1}(A) = A \notin \tau_1$. Clearly i is not fuzzy continuous function. Now every fuzzy set in (X, τ_1) is fuzzy α-preopen in $(X, \tau_1) \Rightarrow i$ is fuzzy α-precontinuous function.
(ii) Fuzzy α-precontinuity $\not\Rightarrow$ fuzzy strong precontinuity
Let $X = \{a, b\}, \tau_1 = \{0_X, 1_X, A, B\}, \tau_2 = \{0_X, 1_X, C\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.55, C(a) = C(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $C \in \tau_2, i^{-1}(C) = C \leq \alpha int_{\tau_1}(cl_{\tau_1}C) = B \Rightarrow i$ is fuzzy α-precontinuous function. But $C \not\leq int_{\tau_1}(pcl_{\tau_1}C) = A \Rightarrow i$ is not fuzzy strong precontinuous function.
(iii) Fuzzy pre-semi-continuity $\not\Rightarrow$ fuzzy α-precontinuity
Let $X = \{a, b\}, \tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = B(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $B \in \tau_2, i^{-1}(B) = B \leq sint_{\tau_1}(cl_{\tau_1}B) = 1_X \setminus A \Rightarrow i$ is fuzzy pre-semi-continuous function. But $\alpha int_{\tau_1}(cl_{\tau_1}B) = A \not\geq B \Rightarrow i$ is not fuzzy α-precontinuous function.

Lemma 4.11. [2] *Let Z, X, Y be fts's and $f_1 : Z \to X$ and $f_2 : Z \to Y$ be functions. Let $f : Z \to X \times Y$ be defined by $f(z) = (f_1(z), f_2(z))$ for $z \in Z$, where $X \times Y$ is provided with the product fuzzy topology. Then if B, U_1, U_2 are fuzzy sets in Z, X, Y respectively such that $f(B) \leq U_1 \times U_2$, then $f_1(B) \leq U_1$ and $f_2(B) \leq U_2$.*

Theorem 4.12. *Let Z, X, Y be fts's. For any functions $f_1 : Z \to X, f_2 : Z \to Y$, if $f : Z \to X \times Y$, defined by $f(x) = (f_1(x), f_2(x))$, for all $x \in Z$, is fuzzy α-precontinuous function, so are f_1 and f_2.*
Proof. Let U_1 be any fuzzy open q-nbd of $f_1(x_t)$ in X for any fuzzy point x_t in Z. Then $U_1 \times 1_Y$ is a fuzzy open q-nbd of $f(x_t)$, i.e., $(f(x))_t$ in $X \times Y$. Since f is fuzzy α-precontinuous function, there exists $V \in F\alpha PO(Z)$ with $x_t q V$ such that $f(V) \leq U_1 \times 1_Y$. By Lemma 4.11, $f_1(V) \leq U_1$, $f_2(V) \leq 1_Y$. Consequently, f_1 is fuzzy α-precontinuous function.
Similarly, f_2 is fuzzy α-precontinuous function.

Lemma 4.13. [1] *Let X, Y be fts's and let $g : X \to X \times Y$ be the graph of* *a function $f : X \to Y$. Then if A, B are fuzzy sets in X and Y respectively, $g^{-1}(A \times B) = A \bigwedge f^{-1}(B)$.*

Theorem 4.14. *Let $f : X \to Y$ be a function from an fts X to an fts Y and $g : X \to X \times Y$ be the graph function of f. If g is fuzzy α-precontinuous function, then f is so.*

Proof. Let g be fuzzy α-precontinuous function and B be a fuzzy set in Y. Then by Lemma 4.13, $f^{-1}(B) = 1_X \bigwedge f^{-1}(B) = g^{-1}(1_X \times B)$. Now if B is fuzzy open in Y, then $1_X \times B$ is fuzzy open in $X \times Y$. Again, $g^{-1}(1_X \times B) = f^{-1}(B) \in F\alpha PO(X)$ as g is fuzzy α-precontinuous function. Hence f is fuzzy α-precontinuous function.

5. Fuzzy α-Preirresolute Function: Some Properties

In this section we introduce a new type of function, viz., fuzzy α-preirresolute function, the class of which is coarser than that of fuzzy α-precontinuous function.

Definition 5.1. *A function $f : X \to Y$ is called fuzzy α-preirresolute if the inverse image of every fuzzy α-preopen set in Y is fuzzy α-preopen in X.*

Theorem 5.2. *For a function $f : X \to Y$, the following statements are equivalent:*
(a) f is fuzzy α-preirresolute,
(b) for each fuzzy point x_t in X and each fuzzy α-preopen nbd V of $f(x_t)$ in Y, there exists a fuzzy α-preopen nbd U of x_t in X and $f(U) \leq V$,
(c) $f^{-1}(F) \in F\alpha PC(X)$, for all $F \in F\alpha PC(Y)$,
(d) for each fuzzy point x_t in X, the inverse image under f of every fuzzy α-preopen nbd of $f(x_t)$ is a fuzzy α-preopen nbd of x_t in X,
(e) $f(\alpha pclA) \leq \alpha pcl(f(A))$, for all $A \in I^X$,
(f) $\alpha pcl(f^{-1}(B)) \leq f^{-1}(\alpha pclB)$, for all $B \in I^Y$,
(g) $f^{-1}(\alpha pintB) \leq \alpha pint(f^{-1}(B))$, for all $B \in I^Y$.

Proof. The proof is similar to that of Theorem 4.4 and hence is omitted.

Theorem 5.3. *A function $f : X \to Y$ is fuzzy α-preirresolute if and only if for each fuzzy point x_t in X and corresponding to any fuzzy α-preopen q-nbd V of $f(x_t)$ in Y, there exists a fuzzy α-preopen q-nbd W of x_t in X such that $f(W) \leq V$.*

Proof. The proof is similar to that of Theorem 4.5 and hence is omitted.

Note 5.4. Composition of two fuzzy α-preirresolute functions is also so.

Theorem 5.5. *If $f : X \to Y$ is fuzzy α-preirresolute and $g : Y \to Z$ is fuzzy α-precontinuous (resp., fuzzy continuous), then $g \circ f : X \to Z$ is fuzzy α-precontinuous.*

Proof. Obvious.

Remark 5.6. *Every fuzzy α-preirresolute function is fuzzy α-precontinuous, but* *the converse is not true, in general, follows from the following example.*

Example 5.7. Fuzzy α-precontinuous function $\not\Rightarrow$ fuzzy α-preirresolute function Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Clearly i is fuzzy α-precontinuous function. Now every fuzzy set in (X, τ_1) is fuzzy α-preopen set in (X, τ_1). Consider the fuzzy set B defined by $B(a) = B(b) = 0.4$. Then $B \in F\alpha PO(X, \tau_1)$. Now $i^{-1}(B) = B \nleq \alpha int_\tau(cl_\tau B) = 0_X \Rightarrow B \notin F\alpha PO(X, \tau) \Rightarrow i$ is not fuzzy α-preirresolute function.

6. Fuzzy α-Preregular Space

In this section we introduce fuzzy α-preregular space in which space fuzzy α-preopen set and fuzzy open set coincide.

Definition 6.1. *An fts (X, τ) is said to be fuzzy α-preregular space if for each fuzzy α-preclosed set F in X and each fuzzy point x_t in X with $x_t \notin F$, there exist a fuzzy open set U in X and a fuzzy α-preopen set V in X such that $x_t qU$, $F \leq V$ and $U \not{q} V$.*

Theorem 6.2. *For an fts (X, τ), the following statements are equivalent:*
(a) X is fuzzy α-preregular,
(b) for each fuzzy point x_t in X and each fuzzy α-preopen set U in X with $x_t qU$, there exists a fuzzy open set V in X such that $x_t qV \leq \alpha pclV \leq U$,
(c) for each fuzzy α-preclosed set F in X, $\bigwedge\{clV : F \leq V, V \in F\alpha PO(X)\} = F$,
(d) for each fuzzy set G in X and each fuzzy α-preopen set U in X such that GqU, there exists a fuzzy open set V in X such that GqV and $\alpha pclV \leq U$.
Proof. (a)$\Rightarrow$(b). Let x_t be a fuzzy point in X and U, a fuzzy α-preopen set in X with $x_t qU$. Then $x_t \notin 1_X \setminus U \in F\alpha PC(X)$. By (a), there exist a fuzzy open set V and a fuzzy α-preopen set W in X such that $x_t qV$, $1_X \setminus U \leq W$, $V \not{q} W$. Then $x_t qV \leq 1_X \setminus W \leq U \Rightarrow x_t qV \leq \alpha pclV \leq \alpha pcl(1_X \setminus W) = 1_X \setminus W \leq U$.
(b)$\Rightarrow$(a). Let F be a fuzzy α-preclosed set in X and x_t be a fuzzy point in X with $x_t \notin F$. Then $x_t q(1_X \setminus F) \in F\alpha PO(X)$. By (b), there exists a fuzzy open set V in X such that $x_t qV \leq \alpha pclV \leq 1_X \setminus F$. Put $U = 1_X \setminus \alpha pclV$. Then $U \in F\alpha PO(X)$ and $x_t qV$, $F \leq U$ and $U \not{q} V$.
(b)$\Rightarrow$(c). Let F be fuzzy α-preclosed set in X. Then $F \leq \bigwedge\{clV : F \leq V, V \in F\alpha PO(X)\}$.
Conversely, let $x_t \notin F \in F\alpha PC(X)$. Then $F(x) < t \Rightarrow x_t q(1_X \setminus F)$ where $1_X \setminus F \in F\alpha PO(X)$. By (b), there exists a fuzzy open set U in X such that $x_t qU \leq \alpha pclU \leq 1_X \setminus F$. Put $V = 1_X \setminus \alpha pclU$. Then $F \leq V$ and $U \not{q} V \Rightarrow x_t \notin clV \Rightarrow \bigwedge\{clV : F \leq V, V \in F\alpha PO(X)\} \leq F$.

(c)⇒(b). Let V be any fuzzy α-preopen set in X and x_t be any fuzzy point in X with x_tqV. Then $V(x)+t>1 \Rightarrow x_t \notin (1_X \setminus V)$ where $1_X \setminus V \in F\alpha PC(X)$. By (c), there exists $G \in F\alpha PO(X)$ such that $1_X \setminus V \leq G$ and $x_t \notin clG$. Then there exists a fuzzy open set U in X with x_tqU, $U \not{q} G \Rightarrow U \leq 1_X \setminus G \leq V$ $\Rightarrow x_tqU \leq \alpha pclU \leq \alpha pcl(1_X \setminus G) = 1_X \setminus G \leq V$.
(c)⇒(d). Let G be any fuzzy set in X and U be any fuzzy α-preopen set in X with GqU. Then there exists $x \in X$ such that $G(x)+U(x)>1$. Let $G(x)=t$. Then $x_tqU \Rightarrow x_t \notin 1_X \setminus U$ where $1_X \setminus U \in F\alpha PC(X)$. By (c), there exists $W \in F\alpha PO(X)$ such that $1_X \setminus U \leq W$ and $x_t \notin clW \Rightarrow (clW)(x) < t \Rightarrow x_tq(1_X \setminus clW)$. Let $V = 1_X \setminus clW$. Then V is fuzzy open set in X and $V(x)+t>1 \Rightarrow V(x)+G(x)>1 \Rightarrow VqG$ and $\alpha pclV = \alpha pcl(1_X \setminus clW) \leq \alpha pcl(1_X \setminus W) = 1_X \setminus W \leq U$.
(d)⇒(b). Obvious.

Note 6.3. It is clear from Theorem 6.2 that in a fuzzy α-preregular space, every fuzzy α-preclosed set is fuzzy closed and hence every fuzzy α-preopen set is fuzzy open. As a result, in a fuzzy α-preregular space, the collection of all fuzzy closed (resp., fuzzy open) sets and fuzzy α-preclosed (resp., fuzzy α-preopen) sets coincide.

Theorem 6.4. *If $f : X \to Y$ is fuzzy α-precontinuous function where Y is fuzzy α-preregular space, then f is fuzzy α-preirresolute function.*
Proof. Let x_t be a fuzzy point in X and V be any fuzzy α-preopen q-nbd of $f(x_t)$ in Y where Y is fuzzy α-preregular space. By Theorem 6.2 (a)⇒(b), there exists a fuzzy open set W in Y such that $f(x_t)qW \leq \alpha pclW \leq V$. Since f is fuzzy α-precontinuous function, by Theorem 4.5, there exists $U \in F\alpha PO(X)$ with x_tqU and $f(U) \leq W \leq V$. By Theorem 5.3, f is fuzzy α-preirresolute function.
Let us now recall following definitions from [5, 6] for ready references.

Definition 6.5. [5] *A collection $\mathcal{U}$ of fuzzy sets in an fts X is said to be a fuzzy cover of X if $\bigcup \mathcal{U} = 1_X$. If, in addition, every member of $\mathcal{U}$ is fuzzy open, then $\mathcal{U}$ is called a fuzzy open cover of X.*

Definition 6.6. [5] *A fuzzy cover $\mathcal{U}$ of an fts X is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 = 1_X$.*

Definition 6.7. [6] *An fts (X, τ) is said to be fuzzy almost compact if every fuzzy open cover $\mathcal{U}$ of X has a finite proximate subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{clU : U \in \mathcal{U}_0\}$ is again a fuzzy cover of X.*

Theorem 6.8. *If $f : X \to Y$ is a fuzzy α-precontinuous, surjective function where X is fuzzy α-preregular and almost compact space, then Y is fuzzy almost compact space.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of Y. Then as f is fuzzy α-precontinuous function, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy cover of X by fuzzy α-preopen and hence by fuzzy open sets of X as X is fuzzy α-preregular space (by Note 6.3). Since X is fuzzy almost compact, there are finitely many members $U_1, U_2, ..., U_n$ of $\mathcal{U}$ such that $\bigcup\limits_{i=1}^{n} cl(f^{-1}(U_i)) = 1_X$. Since X is fuzzy α-preregular, by Note 6.3, $clA = \alpha pclA$ for all $A \in I^X$. So $1_X = \bigcup\limits_{i=1}^{n} \alpha pcl(f^{-1}(U_i)) \Rightarrow 1_Y = f(\bigcup\limits_{i=1}^{n} \alpha pcl(f^{-1}(U_i))) = \bigcup\limits_{i=1}^{n} f(\alpha pcl(f^{-1}(U_i))) \leq \bigcup\limits_{i=1}^{n} cl(f(f^{-1}(U_i)))$ (by Theorem 4.4 (a)$\Rightarrow$(f)) $\leq \bigcup\limits_{i=1}^{n} cl(U_i) \Rightarrow \bigcup\limits_{i=1}^{n} cl(U_i) = 1_Y \Rightarrow Y$ is fuzzy almost compact space.

7. Conclusion

I think this types of works make a bridge between different types of fuzzy open-like set. I am willing to proceed further to achieve more such interrelations between previous works already done.

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