South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 1 (2024), pp. 129-136

DOI: 10.56827/SEAJMMS.2024.2001.10 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

COMPLEX SADIK INTEGRAL TRANSFORM OF LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF SECOND KIND

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(**Received:** *Feb. 13, 2024* **Accepted:** *Apr. 16, 2024* **Published:** *Apr. 30, 2024)*

Abstract: When we transform an initial value problem into an integral equation, Volterra integral equations form. Compared to the starting value problem, the Volterra integral equation is much simpler to solve as a significant subclass of integral equations, numerous academics and mathematicians focus on Volterra integral equations to provide approximate or precise solutions. Recently, mathematics scientists have turned their attention to using the integral transform to solve issues in a variety of scientific domains, including engineering and mathematical sciences. The fundamental problem is reduced to a simpler algebraic equation using the complex transform. This paper utilizes the complex Sadik transform to track down the specific answer for the second kind of linear Volterra integro-differential equation. After that, the solution to this fundamental problem can be found by solving this equation and using the complex Sadik transform's inverse. Through some numeric examples, complex Sadik integral transform's effectiveness and capacity to deliver a precise solution with the least number of calculations are shown.

Keywords and Phrases: Complex Sadik integral transform, Volterra integro-differential equation, inverse complex Sadik integral transform.

2020 Mathematics Subject Classification: 45D05, 44A10, 44A20.

1. Introduction

The theory of Functionals and of Integral and Integro-Differential Equation has been discussed by Vito Volterra [11]. The potential for differential as well as integral operators to show up in integral equations that are both nonlinear and linear has been studied by Abdul Majid-Wazwaz [12]. Volterra integro-differential equations have been generally utilized in different fields, including as material science, space and numerous others; examples of these applications include the population growth studied by Bappa Ghosh and Jugal Mohapatra [5]. The mathematical community established as many techniques as they could because of the immense significance of Volterra integro-differential equations (see for detailed study Baharum et al. [4] and Al-Shimmary et al. [3]). Because integral transforms can provide an algebraic solution to integral equations, thus they stand out as a suitable method for solving Volterra equations. The second kind linear Volterra integral transform, the subject of that investigation, has been resolved by numerous integral transforms (see Mehndi et al. [7], D. Rani and V. Mishra [8], Shams A. Ahmed and Tarig. M. Elzaki and [2] and Aggarwal et al. [1]).

Definition 1.1. *The representation of the Volterra integro-differential equation is provided as follows:*

$$u^{(n)}(x) = f(x) + \lambda \int_0^x u(t)k(x,t)dt \tag{1.1}$$

where $u^{(n)}(x) = \frac{d^n u}{dx^n}$*. As the resulting equation in (1.1) contains both differential and integral operators, it is crucial to establish initial conditions* $u(0), u'(0), \ldots, u^{(n-1)}(0)$ *to acquire the specific solution* $u(x)$ *of the Volterra integro-differential equation.*

Definition 1.2 *From the very first, a novel Sadik integral transformation defined as follows was introduced in 2018 by researcher Sadik L. Sheikh* [9]*:*

$$S_\alpha[f(t)] = F(v^\alpha, \beta) = \frac{1}{v^\beta}\int_0^\infty f(t)e^{-v^\alpha t}dt, \quad t \geq 0 \tag{1.2}$$

where $v \in \mathbb{C}$, $\alpha \in \mathbb{R}^*$, *and* $\beta \in \mathbb{R}$.

Definition 1.3. *The "Complex Sadik" integral transform was firstly presented in 2022 by Saed M. Turq et al.* [10]. *It has applications in fields including engineering, applied physics, and signal processing and is employed to resolve ordinary differential equations by Emand A. Kuffi* [6]. *The operator* $S^C_\alpha\{.\}$ *is used to denote the complex Sadik integral transform and represented as:*

$$S^C_\alpha[f(t)] = F^C(v^\alpha, \beta) = \frac{1}{v^\beta}\int_0^\infty f(t)e^{-iv^\alpha t}dt, \quad t \geq 0 \tag{1.3}$$

where $v \in \mathbb{C}$, $\alpha \in \mathbb{R}^*$, *and* $\beta \in \mathbb{R}$.
Thus, the precise solution to the linear second class Volterra integro-differential equation will be found in this paper using the Complex Sadik integral transform, which will be demonstrated through several examples.

"The Complex Sadik Integral Transform" of Some Elementary Functions
The complex Sadik integral transform of some core functions given by Saed M. Turq and Emad A. Kuffi [10] as follows:

$$S_\alpha^C[C] = \frac{-ic}{v^{\alpha+\beta}}, \quad Re(v) > 0$$

$$S_\alpha^C[t^n] = (-i)^{n+1}\frac{n!}{v^{n\alpha+(\alpha+\beta)}}, \quad Re(v) > 0$$

$$S_\alpha^C[e^{at}] = \frac{-1}{v^\beta}\left[\frac{a}{(v^{2\alpha}+a^2)} + \frac{iv^\alpha}{(v^{2\alpha}+a^2)}\right], \quad Re(v) > a$$

$$S_\alpha^C[\sin(at)] = \frac{-a}{v^\beta(v^{2\alpha}-a^2)}, \quad Re(v) > |a|$$

$$S_\alpha^C[\cos(at)] = \frac{-iv^\alpha}{v^\beta(v^{2\alpha}-a^2)}, \quad Re(v) > |a|$$

$$S_\alpha^C[\sinh(at)] = \frac{-a}{v^\beta(v^{2\alpha}+a^2)}, \quad Re(v) > 0$$

$$S_\alpha^C[\cosh(at)] = \frac{-iv^\alpha}{v^\beta(v^{2\alpha}+a^2)}, \quad Re(v) > 0$$

The derivatives of the function u(x) have a complex Sadik transform, is as follows:

$$\begin{aligned} S_\alpha^C[f^{(n)}(t)] = {} & (iv^\alpha)^n F^C(v^\alpha,\beta) \\ & - \frac{f^{(n-1)}(0)}{v^\beta} - \frac{(iv^\alpha f^{(n-2)}(0))}{v^\beta} - \frac{(iv^\alpha)^2 f^{(n-3)}(0)}{v^\beta} - \dots \\ & - \frac{(iv^\alpha)^{n-2} f'(0)}{v^\beta} - \frac{(iv^\alpha)^{n-1} f(0)}{v^\beta} \end{aligned}$$

The complex Sadik integral transform convolution property

$$\text{If } S_\alpha^C[g(t)] = F_1^C(v^\alpha,\beta) \text{ and } S_\alpha^C[h(t)] = F_2^C(v^\alpha,\beta),$$
$$\text{then } S_\alpha^C[g(t) * h(t)] = v^\beta F_1^C(v^\alpha,\beta) \cdot F_2^C(v^\alpha,\beta)$$

The complex Sadik integral transform changing of scale property

$$\text{Since, } S_\alpha^C[f(t)] = F^C(v^\alpha,\beta) = \frac{1}{v^\beta}\int_0^\infty f(t)e^{-iv^\alpha t}dt, \quad t \geq 0$$

$$\text{Thus, } S_\alpha^C[f(at)] = \frac{1}{a}F^C\left(\frac{v^\alpha}{a}, \beta\right)$$

The Inverse Complex Sadik Integral Transform of Some Basic Functions The inverse of several simple functions' complex Sadik integral transform is provided as: [see Saed M. Turq and Emad A. Kuffi [10]]

$$(S_\alpha^C)^{-1}\left[(-i)^{n+1}\frac{n!}{v^{n\alpha+(\alpha+\beta)}}\right] = t^n$$

$$(S_\alpha^C)^{-1}\left[\frac{-1}{v^\beta}\left[\frac{a}{(v^{2\alpha}+a^2)} + \frac{iv^\alpha}{(v^{2\alpha}+a^2)}\right]\right] = e^{at}$$

$$(S_\alpha^C)^{-1}\left[\frac{-a}{v^\beta(v^{2\alpha}-a^2)}\right] = \sin(at)$$

$$(S_\alpha^C)^{-1}\left[\frac{-iv^\alpha}{v^\beta(v^{2\alpha}-a^2)}\right] = \cos(at)$$

$$(S_\alpha^C)^{-1}\left[\frac{-a}{v^\beta(v^{2\alpha}+a^2)}\right] = \sinh(at)$$

$$(S_\alpha^C)^{-1}\left[\frac{-iv^\alpha}{v^\beta(v^{2\alpha}+a^2)}\right] = \cosh(at).$$

2. Solution of Volterra Integro-differential Equation of second kind by Using Complex Sadik Integral Transform

When an initial value problem is converted by the Leibnitz rule into an integral equation, Volterra integro-differential equations are created as stated by Vitto Volterra [11]. The second kind of the linear Volterra integro-differential equation studied by Mehndi et al. [7] is specified as follows:

$$u^{(n)}(x) = f(x) + \lambda\int_0^x u(t)k(x,t)dt \tag{2.1}$$

Where $u^{(n)}(x) = \frac{d^n u}{dx^n}$ is the n-th order derivative of function $u(x)$ with respect to the variable x. Hence, equation (2.1) contains both integration and differentiation, thus for the determination of the precise solution of $u(x)$, it is crucial to specify the initial conditions $u(0), u'(0), \ldots, u^{(n-1)}(0)$.

The linear second class Volterra integro-differential equation can be expressed as follows, assuming that the kernel of Eq. (2.1) is a difference kernel, which may be written by difference $(x-t)$:

$$u^{(n)}(x) = f(x) + \lambda\int_{t=0}^x u(t)k(x-t)dt \tag{2.2}$$

With conditions $u(0) = a_0, u'(0) = a_1, \ldots, u^{(n-1)}(0) = a_{n-1}$ To evaluate the precise solution of equation (2.2), apply complex Sadik integral transform on each sides of equation (2.2) as:

$$\begin{aligned}\Rightarrow (iv^{\alpha})^n S^C_{\alpha}[u(x)] - \frac{u^{(n-1)}(0)}{v^{\beta}} - \frac{(iv^{\alpha}u^{(n-2)}(0))}{v^{\beta}} - \frac{(iv^{\alpha})^2 u^{(n-3)}(0)}{v^{\beta}} - \ldots - \frac{(iv^{\alpha})^{n-1}u(0)}{v^{\beta}} \\ = S^C_{\alpha}[f(x)] + \lambda S^C_{\alpha}[\int_{t=0}^{x} u(t)k(x,t)dt] \\ \Rightarrow (iv^{\alpha})^n S^C_{\alpha}[u(x)] = \frac{a_{n-1}}{v^{\beta}} + \frac{(iv^{\alpha}a_{n-2})}{v^{\beta}} + \frac{(iv^{\alpha})^2 a_{n-3}}{v^{\beta}} - \ldots + \frac{(iv^{\alpha})^{n-2}a_1}{v^{\beta}} + \frac{(iv^{\alpha})^{n-1}a_0}{v^{\beta}} \\ + S^C_{\alpha}[f(x)] + \lambda S^C_{\alpha}[\int_{t=0}^{x} u(t)k(x,t)dt]\end{aligned}$$

After stimulating the complex Sadik transform convolution property on above equation:

$$\begin{aligned}S^C_{\alpha}[u(x)] = \frac{a_{n-1}}{v^{\beta}(iv^{\alpha})^n} + \frac{iv^{\alpha}a_{n-2}}{v^{\beta}(iv^{\alpha})^n} + \frac{(iv^{\alpha})^2 a_{n-3}}{v^{\beta}(iv^{\alpha})^n} - \ldots \\ + \frac{(iv^{\alpha})^{n-2}a_1}{v^{\beta}(iv^{\alpha})^n} + \frac{(iv^{\alpha})^{n-1}a_0}{v^{\beta}(iv^{\alpha})^n} + \frac{S^C_{\alpha}[f(x)]}{(iv^{\alpha})^n} + \frac{\lambda}{(iv^{\alpha})^n} v^{\beta} S^C_{\alpha}\{k(x)\} \cdot S^C_{\alpha}\{u(x)\}\end{aligned}$$

Now, by using the inverse complex Sadik integral transform on both sides of above equation, we get:

$$\begin{aligned}u(x) = a_{n-1}\frac{x^{n-1}}{(n-1)!} + a_{n-2}\frac{x^{n-2}}{(n-2)!} + \ldots + a_0 \\ + (S^C_{\alpha})^{-1}\left\{\frac{S^C_{\alpha}[f(x)]}{(iv^{\alpha})^n}\right\} + \lambda (S^C_{\alpha})^{-1}\left\{\frac{v^{\beta}}{(iv^{\alpha})^n} S^C_{\alpha}\{k(x)\} \cdot S^C_{\alpha}\{u(x)\}\right\}\end{aligned} \tag{2.3}$$

The exact solution of the linear volterra integro-differential equation of second kind is represented by equation (2.3).

3. Numerical Examples

The viability of the complex Sadik transform for acquiring the exact answer for the linear Volterra integrodifferential equations of second class is featured through a few Numerical problems Studied by Elzaki [2] and Aggarwal et al. [1] in this segment.

Example 3.1. Consider the linear Volterra integro-differential of second kind equation under the following conditions:

$$u'(x) = 2 + \int_{t=0}^{x} u(t)dt, \quad \text{with } u(0) = 2. \tag{3.1}$$

Solution. After applying the complex Sadik Integral transform on each side of above equation (3.1) and also utilizing provided conditions, we get:

$$(-u(0))/v^{\beta} + iv^{\alpha} S_{\alpha}^{C}[u(x)] = (-2i)/v^{\alpha+\beta} + S_{\alpha}^{C}\left[\int_{t=0}^{x} u(t)dt\right]$$

$$\Rightarrow iv^{\alpha} S_{\alpha}^{C}[u(x)] = \frac{2}{v^{\beta}} + \left(-\frac{2i}{v^{\alpha+\beta}}\right) + S_{\alpha}^{C}\left[\int_{t=0}^{x} u(t)dt\right] \quad (3.2)$$

Now, by using the complex Sadik integral transform convolution property to above equation (3.2), provides:

$$iv^{\alpha} S_{\alpha}^{C}[u(x)] = \frac{2}{v^{\beta}} + \left(-\frac{2i}{v^{\alpha+\beta}}\right) + v^{\beta}\left(-\frac{i}{v^{\alpha+\beta}}\right) S_{\alpha}^{C}[u(x)]$$

$$\Rightarrow \left(\frac{iv^{2\alpha}+i}{v^{\alpha}}\right) S_{\alpha}^{C}[u(x)] = \frac{2}{v^{\beta}} + \left(-\frac{2i}{v^{\alpha+\beta}}\right)$$

$$S_{\alpha}^{C}[u(x)] = \frac{2v^{\alpha}}{v^{\beta}(iv^{2\alpha}+i)} + \left(-\frac{2iv^{\alpha}}{v^{\alpha+\beta}(iv^{2\alpha}+i)}\right)$$

$$S_{\alpha}^{C}[u(x)] = \frac{2v^{\alpha}-2i}{v^{\beta}(iv^{2\alpha}+i)} = -\frac{2}{v^{\beta}}\left[\frac{1}{v^{2\alpha}+1} + \frac{iv}{v^{2\alpha}+1}\right] \quad (3.3)$$

On both sides of Eq. (3.3), the inverse complex Sadik transform yields:

$$u(x) = 2e^{x} \quad (3.4)$$

This is the precise solution of equation (3.1).

Example 3.2. Consider the linear Volterra integro-differential equation of the second kind with the following conditions:

$$u''(x) = 1 + \int_{t=0}^{x} u(t)(x-t)dt, \quad \text{with } u(0) = 1 \text{ and } u'(0) = 0. \quad (3.5)$$

Solution. When both sides of equation (3.5) are subjected to the complex Sadik integral transform under the given initial conditions:

$$(iv^{\alpha})^{2} S_{\alpha}^{C}[u(x)] - \frac{u'(0)}{v^{\beta}} - \frac{iv^{\alpha}u(0)}{v^{\beta}} = S_{\alpha}^{C}[1] + S_{\alpha}^{C}\left[\int_{t=0}^{x} u(t)(x-t)dt\right]$$

$$-v^{2\alpha} S_{\alpha}^{C}[u(x)] - \frac{iv^{\alpha}}{v^{\beta}} = -\frac{i}{v^{\alpha+\beta}} + S_{\alpha}^{C}\left[\int_{t=0}^{x} u(t)(x-t)dt\right]$$

$$\Rightarrow v^{2\alpha} S_{\alpha}^{C}[u(x)] = \frac{i}{v^{\alpha+\beta}} - \frac{i}{v^{-\alpha+\beta}} - S_{\alpha}^{C}\left[\int_{t=0}^{x} u(t)(x-t)dt\right] \quad (3.6)$$

Now using the complex Sadik integral transform convolution property into the above equation (3.6), gives:

$$S_\alpha^C[u(x)] = \frac{i}{v^{3\alpha+\beta}} - \frac{i}{v^{\alpha+\beta}} - \frac{v^\beta}{v^{2\alpha}}\left(-\frac{1}{v^{2\alpha+\beta}}\right)$$

$$\left(\frac{v^{4\alpha}-1}{v^{4\alpha}}\right) S_\alpha^C[u(x)] = \frac{i}{v^{3\alpha+\beta}} - \frac{i}{v^{\alpha+\beta}}$$

$$S_\alpha^C[u(x)] = \frac{iv^{4\alpha}}{v^{3\alpha+\beta}(v^{4\alpha}-1)} - \frac{iv^{4\alpha}}{v^{\alpha+\beta}(v^{4\alpha}-1)}$$

$$S_\alpha^C[u(x)] = -\frac{iv^\alpha}{v^\beta(v^{2\alpha}+1)} \tag{3.7}$$

Equation (3.7), when both sides are subjected to the inverse complex Sadik integral transform, yields:

$$(S_\alpha^C)^{-1}[S_\alpha^C[u(x)]] = (S_\alpha^C)^{-1}\left[-\frac{iv^\alpha}{v^\beta(v^{2\alpha}+1)}\right]$$

$$u(x) = \cosh(x). \tag{3.8}$$

Thus, the concluded result equation (3.8) is the precise solution of equation (3.5).

4. Conclusion

The linear Volterra integro-differential problem of the second class, which is one of the most significant integro-differential equations, was introduced in this study as a good and efficient solution approach. The numericals discussed in the work's numerical example section have shown the extraordinary capacity of the complex Sadik integral transform to identify the precise solution to the second class linear Volterra integro-differential problem. This ability may be utilized as a benchmark for future applications of complex Sadik integral transform to the general problem of solving integro-differential equations.

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