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A CERTAIN CLASS OF STATISTICAL PRODUCT SUMMABLILITY MEAN AND KOROVKIN-TYPE THEOREMS

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Abstract: Statistical convergence is more extensive than the classical convergence and has recently drawn the recognition of many researchers. The Korovkin-type approximation theorems are usually based on the convergence analysis of sequences of positive linear operators. Gradually, such approximation theorems are extended over more general sequence spaces with several settings via different kinds of statistical summability techniques. In this paper, we introduce presumably a new statistical Riesz-Euler product summability technique to prove a Korovkin-type approximation theorem. Moreover, we demonstrate another result for the rate of statistical convergence under our proposed summability technique.

Keywords and Phrases: Statistical convergence; Korovkin's theorem; positive linear operator; (E, q) mean; $(\overline{N}, p_n, q_n)$ mean.

2020 Mathematics Subject Classification: 41A24, 41A25, 42B05, 42B08.

1. Introduction, Definitions and Preliminaries

The perception of statistical convergence for real sequences was first familiarized by Fast [7] in 1951. But previously the idea of statistical convergence was given by Zygmund [31] in 1935. Using real and complex sequences it was further investigated by Schoenberg [24] independently. Continuously several researchers are doing research in this statistical convergence and it plays an important role in different areas of mathematics such as Fourier analysis, approximation theory, Number theory and Functional analysis. Later on, statistical convergence was explored from the sequence point of view and connected with summability theory by Conner [4] and Fridy [8] and became a versatile field of research in last decades. Moricz [17] introduced statistical $(C, 1)$- summability. Through statistical $(C, 1)$- summability Mohiuddine *et al.* [18], Mohiuddine [19] and Mohiuddine and Alotaibi [20] proved Korovkin-type approximation theorems by using the test functions $1, \sin x, \cos x$ and $1, e^{-x}, e^{-2x}$, respectively. Belen *et al.* [3] proved approximation theorems by generalized statistical convergence. Acar and Mohiuddine [18] announced the statistical $(C, 1)(E, 1)$- summability and its applications to Korovkin's theory. Recently, Baliarsingh *et al.* [2] introduced and deliberated the notion of advance version of uncertain sequences via statistical deferred A-convergence and proved some inclusion theorems. Again, in that year Saini *et al.* [23] also studied the results on equi-statistical convergence via the deferred Cesàro and deferred Euler summability product means with associated Korovkin-type theorems. Also, Saini *et al.* [22] again studied deferred Riesz statistical convergence of a complex uncertain sequences with its applications and also in that year, Sharma *et al.* [25] demonstrated the implementations of statistical deferred Cesàro convergence of fuzzy number valued sequences of order (ξ, ω). In the year 2018, Srivastava *et al.* [30] studied and investigated the idea of sequences which converge equi-statistically based on the deferred Nörlund mean. Subsequently, Parida *et al.* [21] proposed some results for sequences that converge equi-statistically via the deferred Cesàro mean and accordingly demonstrated the Korovkin-type theorems. More recently, Demirci *et al.* [5] investigated the perception of sequences which converge equi-statistically under the power-series technique and proved some approximation results.

Let the set of positive integers be $\mathbb{N}$, and let $K \subseteq \mathbb{N}$ and $K_n = \{i : i \leq n \text{ and } i \in K\}$. Then the natural (asymptotic) density of K is defined as

$$\delta(K) = \lim_{n \to \infty} \frac{|K_n|}{n},$$

provided the limit exists.
A sequence of real numbers $u = (u_n)$ is said to be statistically convergent to l, if for each $\varepsilon \geq 0$, the set

$$\{k : k \in \mathbb{N} \text{ and } |u_k - l| \geq \varepsilon\}$$

ensures the natural density zero. That is for every$\varepsilon \geq 0$, we have

$$\delta(k : k \leq n \ \text{ and } |u_k - l| \geq \varepsilon) = 0$$

The statistical convergence is quite more general than the classical convergence, that is to say, a sequence can be convergent in statistical means even if it is not convergent in classical sense. Also, it has been seen that statistical convergence is very closely associated with the concept of convergence in probability. Let us now define here statistical product $(\overline{N}, p_n, q_n)(E, q)$- summability mean for the present study.

Suppose that, (p_n) and (q_n) be two non-negative real sequences with

$$P_n = p_1 + p_2 + ... + p_n, \quad P_{-1} = p_{-1} = 0, \ \texttt{and}$$

$$Q_n = q_1 + q_2 + ... + q_n, \quad Q_{-1} = q_{-1} = 0.$$

Product of above two sequences are defined as

$$R_n = \sum_{k=0}^{n} p_k q_k.$$

If the sequence to sequence transformation

$$t_n^{\overline{N}} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k u_k,$$

converges to l as $n \to \infty$, then the (u_n) sequence is summable to l by $(\overline{N}, p_n, q_n)$-summability.

We say that the sequence (u_n) is statistically summable to l by $(\overline{N}, p_n, q_n)$- summability generated by the sequences (p_n) and (q_n) or (u_n) is statistically $(\overline{N}, p_n, q_n)$ summable to l if,

$$st \lim_{n\to\infty} t_n^{\overline{N}} = l.$$

We also write this as

$$(\overline{N}_{p_n q_n})(st) \lim_{n\to\infty} u_n = l.$$

If the sequence to sequence transformation

$$E_n^q = \frac{1}{(1+q)^n} \sum_{r=0}^{k} \binom{k}{r} u_r \ \text{ for } q > 0$$

converges to l as $n \to \infty$, then the sequence (u_n) is summable to l by (E, q)- summability. The sequence (u_n) is statistically summable to l by (E, q)- summability or statistically (E, q) summable to l if,

$$st \lim_{n\to\infty} E_n^q = l.$$

We can also write this as

$$(E^q)(st) \lim_{n\to\infty} u_n = l.$$

Now we define a composite transformation, that is the $(\overline{N}, p_n, q_n)$ transformation over (E, q) transformation as

$$(\overline{N}_{p_n q_n}.E^q) = T_n^{\overline{N}E} = \frac{1}{R_n}\sum_{k=0}^{n} p_{nk} q_k \left\{\frac{1}{(1+q)^n}\sum_{r=0}^{k}\binom{k}{r} q^{k-r} u_r\right\}. \tag{1.1}$$

If $T_n^{\overline{N}E} \to l$ as $n \to \infty$, we say (u_n) is summable to l by $(\overline{N}, p_n, q_n)(E, q)$ summability.

The sequence (u_n) is said to be statistically summable to l by $(\overline{N}, p_n, q_n)(E, q)$-summability or the sequence (u_n) is $(\overline{N}, p_n, q_n)(E, q)$ summable to l if,

$$st \lim_{n\to\infty} T_n^{\overline{N}E} = l. \tag{1.2}$$

We can also write this as

$$(\overline{N}_{p_n q_n}.E^q)(st) \lim_{n} u_n = l.$$

Remark 1.1. *If we put* $p_n = 1$, $q_n = 1$ *and* $R_n = n + 1$ *in* (1.1), *then we have*

$$T_n^{\overline{N}E} = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+q)^n}\sum_{r=0}^{k}\binom{k}{r} q^{k-r} u_r\right\} \tag{1.3}$$

$$= \frac{1}{n+1}\sum_{k=0}^{n} \left\{\frac{1}{(1+q)^n}\sum_{r=0}^{k}\binom{k}{r} q^{k-r} u_r\right\}, \tag{1.4}$$

which is the Cesàro-Euler summation. Moreover, if we put $p_n = 1$ *and* $q_n = 1$ *in* (1.1), *the generalized* $(\overline{N}, p_n, q_n)(E, q)$- *summability method reduces to Cesàro-Euler summability method.*

Example 1.1. Let us consider $p_k = 1$ and $q_k = \frac{1}{k}$ for all $k \in \mathbb{N}$. Also, consider the following sequence

$$u_k = (-2)^k, \ k \in \mathbb{N}.$$

This sequence (u_k) is not statistically summable but $(\overline{N}, p_n, q_n)(E, q)$ summable. If we put $p_k = 1$ and $q_k = \frac{1}{k}$ in (1.1), then the equation becomes

$$\frac{1}{n+1}\sum_{k=0}^{n} 1 \frac{1}{(1+q)^k}\left\{\sum_{v=0}^{k}\binom{k}{v} q^{k-v}(-2)^v\right\} = \frac{1}{n+1}\sum_{k=0}^{n}\frac{(q-2)^k}{(q+2)^k}. \tag{1.5}$$

Motivated essentially by the above-mentioned discussions and results, in this paper we introduce a new statistical $(\overline{N}, p_n, q_n)(E, q)$- summablity method by considering the product of $(\overline{N}, p_n, q_n)$ and (E, q) means to prove a Korovkin-type approximation theorem for sequence of positive linear operators. Subsequently, we investigate the rate of statistical $(\overline{N}, p_n, q_n)(E, q)$ summable sequences and demonstrate another new result based on the proposed method.

Here, we begin with some basic definitions concerning statistical convergence.

2. Korovkin-type Theorem

In 1960, Korovkin [16] established the foundational Korovkin-type theorem by demonstrating the uniform convergence of a sequence of positive linear operators $(\mathfrak{L}_m)$ in the space of continuous functions. Subsequently, numerous mathematicians embarked on extending Korovkin's findings to various contexts, including function spaces, Banach spaces, and beyond. These developments gave rise to the theory now recognized as Korovkin-type theorems, which have found elegant applications in Advanced Analysis, Fourier series, and Summability theory. However, the Korovkin-type hypothesis is still in its early stages of research, particularly in the areas where it deals with limit operators other than the identity operator.

Recently, Korovkin-type results have been investigated and studied under various notions of statistical convergence techniques (see [1], [6], [14], [9], [10], [11], [12], [13], [26], [27], [28] and [29]). In this section,Korovkin-type approximation theorem for sequence of positive linear operators we prove a Korovkin-type theorem via statistical $(\overline{N}, p_n, q_n)(E, q)$- summability mean.

Let $\mathbb{C}[a, b]$ be the class (space) of all continuous real-valued functions f defined over $[a, b]$, and $(\mathcal{A})$ be a positive linear operators mapped from $\mathbb{C}[a, b]$ into $\mathbb{C}[a, b]$. And, the space is equipped well with the norm

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|.$$

Following [15], the classical Korovkin approximation theorem is stated as follows.

Theorem 2.1. *Let (T_n) be a sequence of positive linear operators from $\mathbb{C}[a, b]$ into* *itself. Then,*

$$\lim_{n\to\infty} \|T_n(f;x) - f(x)\|_\infty = 0 \tag{2.1}$$

for all $\mathbb{C}[a,b]$ *if and only if*

$$\lim_{n\to\infty} \|T_n(f_i;x) - f_i(x)\|_\infty = 0 \tag{2.2}$$

where $f_i(x) = x^i$ *and i=0,1,2.*

The main purpose of this investigation is to prove the following theorem by using our proposed summability mean.

Theorem 2.2. *Let* $(\mathcal{A})$ *be a positive linear operator which maps* $\mathbb{C}[a,b]$ *into* $\mathbb{C}[a,b]$. *Then for all* $f \in \mathbb{C}[a,b]$ *bounded on the whole real line,*

$$(\overline{N}_{p_nq_n}.E^q)(st) - \lim_{k\to\infty} \|\mathcal{A}_k(f;x) - f(x)\|_\infty = 0, \tag{2.3}$$

if and only if

$$(\overline{N}_{p_nq_n}.E^q)(st) - \lim_{k\to\infty} \|\mathcal{A}_k(1;x) - 1\|_\infty = 0, \tag{2.4}$$

$$(\overline{N}_{p_nq_n}.E^q)(st) - \lim_{k\to\infty} \|\mathcal{A}_k(t;x) - x\|_\infty = 0, \tag{2.5}$$

$$(\overline{N}_{p_nq_n}.E^q)(st) - \lim_{k\to\infty} \|\mathcal{A}_k(t^2;x) - x^2\|_\infty = 0. \tag{2.6}$$

Proof. The equations (2.4) to (2.6) immediately follows from (2.3) because of each of the algebraic test functions 1, x and x^2 belongs to $\mathcal{C}[a,b]$. Next we prove its converse part, i.e., if the conditions (2.4) to (2.6) holds true, then (2.3) is valid. Let $f \in \mathbb{C}[a,b]$, then there exist a constant $C > 0$ such that

$$|f(x)| \leq C \quad \text{for} \quad \text{all } x \in (-\infty,\infty), \tag{2.7}$$

and therefore

$$|f(t) - f(x)| \leq 2C, \quad -\infty < t, x < \infty. \tag{2.8}$$

We may write, for every $\varepsilon > 0$, than there exists a number $\delta > 0$ such that

$$|f(t) - f(x)| \leq \varepsilon, \text{for} \quad \text{all } |t - x| \leq \delta. \tag{2.9}$$

Combining (2.8) and (2.9) and substituting $\phi(t) = (t-x)^2$, we get

$$|f(t) - f(x)| < \varepsilon + \frac{2C}{\delta^2}\phi(t). \tag{2.10}$$

Now (2.10) can be written as

$$-\varepsilon - \frac{2C}{\delta^2}\omega(t) \leq f(t) - f(x) < \varepsilon + \frac{2C}{\delta^2}\phi(t). \tag{2.11}$$

As the operators are positive, operating $\mathcal{A}_k(1;x)$ to (2.11), we get

$$|\mathcal{A}_k(1;x)\left(-\varepsilon - \frac{2C}{\delta^2}\phi(t)\right) \leq \mathcal{A}_k(1;x)(f(t)-f(x)) \leq \mathcal{A}_k(1;x)\left(\varepsilon + \frac{2C}{\delta^2}\phi(t)\right). \tag{2.12}$$

We know x is fixed, so $f(x)$ is a constant number and using the linearity property of $(\mathcal{A}_k)$, we have

$$-\varepsilon\mathcal{A}_k(1;x) - \frac{2C}{\delta^2}\mathcal{A}_k(\phi(t);x) \leq \mathcal{A}_k(f;x) - f(x)\mathcal{A}_k(1;x) \leq \varepsilon\mathcal{A}_k(1;x) + \frac{2C}{\delta^2}\mathcal{A}_k(\phi(t);x). \tag{2.13}$$

The term $\mathcal{A}_k(f;x) - f(x)$ can be written as

$$\mathcal{A}_k(f;x) - f(x) = \mathcal{A}_k(f;x) - f(x)\mathcal{A}_k(1;x) + f(x)[\mathcal{A}_k(1;x) - 1]. \tag{2.14}$$

Now considering the inequality (2.13) and equality (2.14), we get

$$\mathcal{A}_k(f;x) - f(x) < \varepsilon\mathcal{A}_k(1;x) + \frac{2C}{\delta^2}\mathcal{A}_k(\phi(t);x) + f(x)[\mathcal{A}_k(1;x) - 1]. \tag{2.15}$$

The term $\mathcal{A}_k(\phi(t);x)$ can be written as follows:

$$\mathcal{A}_k(\phi(t);x) = \left[\mathcal{A}_k(t^2;x) - x^2\right] - 2x[\mathcal{A}_k(t;x) - x] + [\mathcal{A}_k(1;x) - 1]. \tag{2.16}$$

Now putting the value of $\mathcal{A}_k(\phi(t);x)$ in (2.16), we get

$$\begin{aligned}\mathcal{A}_k(f;x) - f(x) < \mathcal{A}_k(1;x) + \frac{2C}{\delta^2}\left\{\left[\mathcal{A}_k(t^2;x) - x^2\right] - 2x[\mathcal{A}_k(t;x) - x] + \right.\\ \left.[\mathcal{A}_k(1;x) - 1]\right\} + f(x)[\mathcal{A}_k(1;x) - 1]\\ = \varepsilon[\mathcal{A}_k(1;x) - 1] + \varepsilon + f(x)[\mathcal{A}_k(1;x) - 1]\\ + \frac{2C}{\delta^2}\left\{\left[\mathcal{A}_k(t^2;x) - x^2\right] - 2x[\mathcal{A}_k(t;x) - x] + [\mathcal{A}_k(1;x) - 1]\right\}.\end{aligned} \tag{2.17}$$

If we take $h = \max(|x|)$, then (2.17) becomes

$$\begin{aligned}|\mathcal{A}_k(f;x) - f(x)| < & \left(\varepsilon + \frac{2Ch^2}{\delta^2}|\mathcal{A}_k(1;x) - 1|\right) \\ & + \frac{4Ch}{\delta^2}|\mathcal{A}_k(t;x) - x| \\ & + \frac{2C}{\delta^2}|\mathcal{A}_k(t^2;x) - x|. \end{aligned} \tag{2.18}$$

Now taking suprimum over all $x \in [a, b]$, we have

$$\begin{aligned}\|\mathcal{A}_k(f;x) - f(x)\|_\infty < & \left(\varepsilon + \frac{2Ch^2}{\delta^2}\|\mathcal{A}_k(1;x) - 1\|_\infty\right) \\ & + \frac{4Ch}{\delta^2}\|\mathcal{A}_k(t;x) - x\|_\infty \\ & + \frac{2C}{\delta^2}\|\mathcal{A}_k(t^2;x) - x\|_\infty \\ & \leq H\left(\|\mathcal{A}_k(1;x) - 1\|_\infty + \|\mathcal{A}_k(t;x) - x\|_\infty + \|\mathcal{A}_k(t^2;x) - x\|_\infty\right), \end{aligned} \tag{2.19}$$

where

$$H = \max\{\varepsilon + \frac{2Ch^2}{\delta^2} + C, \frac{4Ch}{\delta^2}, \frac{2C}{\delta^2}\}.$$

We have,

$$\mathcal{L}_n(.;x) = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+q)^k}\sum_{r=0}^{k}\binom{k}{r} q^{k-r}\mathcal{A}_r(.;x)\right\}. \tag{2.20}$$

Now replacing $\mathcal{A}_k(.;x)$ by $\mathcal{L}_n(.;x)$ in both sides of (2.19), we get

$$\|\mathcal{L}_k(f;x) - f(x)\|_\infty < H[\|\mathcal{L}_k(1;x) - 1\|_\infty + \|\mathcal{L}_k(t;x) - x\|_\infty + \|\mathcal{L}_k(t^2;x) - x\|_\infty].$$

For $\varepsilon' > 0$, let

$$\underline{E} = \left\{n \in \mathbb{N} : \|\mathcal{L}_k(1;x) - 1\|_\infty + \|\mathcal{L}_k(t;x) - x\|_\infty + \|\mathcal{L}_k(t^2;x) - x\|_\infty \geq \frac{\varepsilon'}{H}\right\};$$

$$\underline{E}_1 = \left\{n \in \mathbb{N} : \|\mathcal{L}_n(1;x) - 1\|_\infty \geq \frac{\varepsilon'}{3H}\right\};$$

$$\underline{E}_2 = \left\{n \in \mathbb{N} : \|\mathcal{L}_n(t;x) - x\|_\infty \geq \frac{\varepsilon'}{3H}\right\};$$

$$\underline{E}_3 = \left\{n \in \mathbb{N} : \|\mathcal{L}_n(t^2;x) - x^2\|_\infty \geq \frac{\varepsilon'}{3H}\right\}.$$

Clearly,

$$E \subseteq E_1 \cup E_2 \cup E_3. \tag{2.21}$$

This implies,

$$\delta(E) \subseteq \delta(E_1) \cup \delta(E_2) \cup \delta(E_3). \tag{2.22}$$

Now by using conditions (2.4) to (2.6), we get

$$(\overline{N}_{p_n q_n}.E^q)(st) \ \lim_{k\to\infty} \|\mathcal{A}_k(f;x) - f(x)\|_\infty = 0. \tag{2.23}$$

This completes the proof of the theorem.

3. Order of the statistical $(\overline{N}, p_n, q_n)(E, q)$- summability

In this section, we study the rate of statistical- $(\overline{N}, p_n, q_n)(E, q)$ - summability for a sequence of positive linear operators $(\mathcal{A}_k)$ defined on $\mathbb{C}[a, b]$. We begin by presenting the following definition.

Definition 3.1. *Let (a_m) be a positive increasing sequence. Then the sequence $u = (u_k)$ is said to be statistically $(\overline{N}, p_n, q_n)(E, q)$ summable to μ with the rate $o(a_m)$, if for every $\varepsilon > 0$*

$$\lim_{m\to\infty} \frac{1}{a_m} \left| \left\{ n \le m : |T_n^{\overline{N}E} - \mu| \ge \varepsilon \right\} \right| = 0. \tag{3.1}$$

In this case, we write

$$u_k - \mu = (\overline{N}_{p_n q_n}.E^q)(st) - o(a_m).$$

Based on the above definition, we state the following lemma.

Lemma 3.1. [29] *Let (a_m) and (b_m) be two positive non-increasing sequences. Let $u = (u_k)$ and $v = (v_k)$ be two sequences such that $u_k - \mu_1 = (\overline{N}_{p_n q_n}.E^q)(st) - o(a_m)$ and $v_k - \mu_2 = (\overline{N}_{p_n q_n}.E^q)(st) - o(b_m)$. Then,*

(i) $\alpha(u_k - \mu_1) = (\overline{N}_{p_n q_n}.E^q)(st) - o(a_m)$, *for any scalar* α.

(ii) $(u_k - \mu_1) \pm (v_k - \mu_2) = (\overline{N}_{p_n q_n}.E^q)(st) - o(c_m)$, *where* $c_m = \max\{a_m, b_m\}$.

(iii) $(u_k - \mu_1).(v_k - \mu_2) = (\overline{N}_{p_n q_n}.E^q)(st) - o(a_m b_m)$.

Now we recall the notions of modulus of continuity of f in $\mathcal{C}[a,b]$ as

$$\omega(f,\delta) = \sup\{|f(t) - f(y)| : t, y \in [a,b], |t-y| < \delta\}.$$

Thus, it yields

$$\|f(t) - f(y)\| \leq \omega(f,\delta)\left\{\frac{|t-y|}{\delta} + 1\right\}. \tag{3.2}$$

Next, we prove the following theorem.

Theorem 3.1. *Let $(\mathcal{A}_k)$ be a sequence of non-negative linear operators such that $\mathcal{A}_k : \mathbb{C}[a,b] \to \mathbb{C}[a,b]$ satisfies the following conditions*

(i) $\|\mathcal{A}_k(1;x) - 1\|_\infty = (\overline{N}_{p_n q_n}.E^q)(st) - o(a_m)$ *and*

(ii) $\omega(f,\delta_k) = (\overline{N}_{p_n q_n}.E^q)(st) - o(b_m)$, *where* $\delta_k(x) = \sqrt{(\mathcal{A}_k(\varphi_x;x))}$ *and* $\varphi(x) = (t-y)^2$,

where (a_m) and (b_m) are positive non-increasing sequences. Then for all $f \in \mathbb{C}[a,b]$ and $c_m = \max\{a_m, b_m\}$, we have

$$\|\mathcal{A}_k(f;x) - f(x)\|_\infty = (\overline{N}_{p_n q_n}.E^q)(st) - o(c_m). \tag{3.3}$$

Proof. Let $f \in \mathbb{C}[a,b]$, for all $x \in [a,b]$. Then the equation (3.2) can be reformed into

$$\begin{aligned}
|\mathcal{A}_k(f;x) - f(x)| &\leq \mathcal{A}_k\left(|f(t) - f(x)|, x\right) + |f(x)||\mathcal{A}_k(1;x) - 1| \\
&\leq \mathcal{A}_k\left(1 + \frac{|t-x|}{\delta}; x\right)\omega(f,\delta) + |f(x)||\mathcal{A}_k(1;x) - 1| \\
&\leq \mathcal{A}_k\left(1 + \frac{(t-x)^2}{\delta^2}; x\right)\omega(f,\delta) + |f(x)||\mathcal{A}_k(1;x) - 1| \\
&\leq \left(\mathcal{A}_k(1;x) + \frac{\mathcal{A}_k(\varphi^2;x)}{\delta^2}\right)\omega(f,\delta) + |f(x)||\mathcal{A}_k(1;x) - 1| \\
&\leq \omega(f,\delta)|\mathcal{A}_k(1;x) - 1| + |f(x)||\mathcal{A}_k(1;x) - 1| + \omega(f,\delta) \\
&\quad + \omega(f,\delta)\frac{1}{\delta^2}\mathcal{A}_k(\varphi^2;x).
\end{aligned} \tag{3.4}$$

If we choose $\delta = \delta_k = \sqrt{(\mathcal{A}_k(\varphi_x;x))}$, the it yields

$$\begin{aligned}
\|\mathcal{A}_k(f;x) - f(x)\|_\infty &\leq \|f\|_\infty\|\mathcal{A}_k(1;x) - 1\|_\infty + \omega(f,\delta_k)\|\mathcal{A}_k(1;x) - 1\|_\infty + 2\omega(f,\delta_k) \\
&\leq \lambda\left\{\|\mathcal{A}_k(f;x) - x\|_\infty + \omega(f,\delta_k)\|\mathcal{A}_k(1;x) - 1\|_\infty + \omega(f,\delta_k)\right\},
\end{aligned} \tag{3.5}$$

where $\lambda = \max\{\|f\|_\infty, 2\}$. Now replacing $\mathcal{A}_k(.;x)$ by $\mathcal{A}_n(.;x)$ in (3.5), we get

$$\|\mathcal{A}_n(f;x) - f(x)\|_\infty \leq \lambda\{\|\mathcal{A}_n(1;x) - 1\|_\infty + \omega(f,\delta_k) + \omega(f,\delta_k)\mathcal{A}_k(1;x) - 1\|_\infty\}. \tag{3.6}$$

Now by using Definition 3.1, and conditions (i) and (ii) of Lemma 3.1, we get

$$\|\mathcal{A}_k(f;x) - f(x)\|_\infty = (\overline{N}_{p_n q_n}.E^q)(st) - o(c_m). \tag{3.7}$$

This completes the proof of theorem.

4. Conclusion

Through this study, we have preluded the conception of statistical product summability technique and established some fundamental concepts and proved a Korovkin-type approximation theorem. Moreover, we have demonstrated a new result for the rate of statistical convergence under our proposed summability technique.

Many researchers have considered different summability means on the sequence spaces to prove several approximation results. A list of some articles has been mentioned in the references. Furthermore, combining the existing ideas and direction of the sequence spaces associated with our proposed mean, many new Korvokin-type approximation theorems can be proved under different settings of algebraic and trigonometric functions. Also, in certain special cases our result generalizes some existing previous results. In particular, if we put $p_n = 1$, $q_n = 1$ and $q > 0$, then $(\overline{N}, p_n, q_n)(E, q)$- summability method reduces to $(C, 1)(E, 1)$-summability method defined by Belen and Mohiuddine [3].

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