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ON UNIQUENESS OF MEROMORPHIC FUNCTIONS IGNORING MULTIPLICITY CONCERNING A QUESTION OF YI

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Abstract: Let $S = \{z \in \mathbb{C} : P(z) = z^n + az^{n-1} + b = 0\}$, where $a, b \in \mathbb{C}$ be nonzero constants satisfying $\frac{b}{a^n} \neq \frac{(-1)^n(n-1)^{n-1}}{n^n}$. The uniqueness of meromorphic functions sharing S counting multiplicity(resp. with weight 2) has been studied by Yi ([18]) (resp. Lahiri, Banerjee ([12])). In this paper, we consider the uniqueness of meromorphic functions sharing S ignoring multiplicity. We first obtain the analog of Yi's Theorem 2 ([18]). Next, we show that S is a unique range set for the class of meromorphic functions ignoring multiplicity of higher multiplicities of either zeros or poles, which different from S. Mallick - D. Sarkar's ([13]). We discuss some applications of the main result. Our results are inspired by a work of Yi ([18]) and Khoai ([11]).

Keywords and Phrases: Uniqueness, ignoring multiplicity, multiplicities of zeros, poles of meromorphic functions.

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1. Introduction and Main Results

Let f be a meromorphic function in $\mathbb{C}$, $a \in \mathbb{C} \cup \{\infty\}$, and k be a nonnegative integer or infinity. We assume that the reader is familiar with the notations of Nevanlinna theory (see, for example ([6]), ([8])): $T(r, f)$, $N(r, f)$, $m(r, f)$, $\Theta(\infty, f), \ldots$.

Denote by $E_f(a)$ the set of all $a-$ points of f where an $a-$ point is counted with its multiplicity, and by $\overline{E}_f(a)$ where an $a-$ point is counted only one time, and by $E_f(a,k)$ the set of all $a-$ points of f where an $a-$ point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$.

For a nonempty subset $S \subset \mathbb{C} \cup \{\infty\}$, define $E_f(S) = \cup_{a \in S} E_f(a)$, and similarly for $\overline{E}_f(S)$, $E_f(S,k)$. Let $\mathcal{F}$ be a nonempty subset of $\mathcal{M}(\mathbb{C})$. Two meromorphic functions f, g of $\mathcal{F}$ are said to *share* S, *counting multiplicity*, (share S CM), if $E_f(S) = E_g(S)$, and to *share* S, *ignoring multiplicity*, (share S IM), if $\overline{E}_f(S) = \overline{E}_g(S)$, and to *share* S *with weight* k if $E_f(S,k) = E_g(S,k)$.

If the condition $E_f(S) = E_g(S)($ resp. $\overline{E}_f(S) = \overline{E}_g(S))$ implies $f = g$ for any two nonconstant meromorphic (entire) functions f, g of $\mathcal{F}$, then S is called a unique range set for meromorphic (entire) functions of $\mathcal{F}$ counting multiplicity(resp.ignoring multiplicity), and similarly for unique range set for meromorphic (entire) functions of $\mathcal{F}$ with weight k. Clearly $E_f(S) = E_f(S,\infty)$, and $\overline{E}_f(S) = \overline{E}_f(S,0)$.

In 1976 Gross ([7]) proved that there exist three finite sets S_j $(j = 1,2,3)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$, $j = 1,2,3$ must be identical. In the same paper Gross ([7]) posed the following question:

Question A. *Can one find two (or possible even one) finite set* S_j $(j = 1,2)$ *such that any two entire functions* f *and* g *satisfying* $E_f(S_j) = E_g(S_j)$ $(j = 1,2)$ *must be identical?*

Yi first gave an affirmative answer to Question A for entire functions(see ([17]), ([18])).

In response to Question A Yi ([18]) proved for meromorphic functions the following result.

Theorem A. ([18]) *Let* n *and* m *be two positive integers such that* $n > 2m + 8$, $m \geq 2$ *and* $(n,m) = 1$. *Let* $a, b \in \mathbb{C}$ *be nonzero constants such that the* $P(z) = z^n + az^{n-m} + b$ *has no multiple zeros and* $S = \{z : P(z) = 0\}$. *Then for any two nonconstant meromorphic functions* f *and* g, *the condition* $E_f(S) = E_g(S)$ *implies* $f = g$.

In the same paper Yi ([18]) asked:

Question B. *What can be said if* $m = 1$ *in Theorem A ?*

In this connection Yi ([18]) proved the following theorem.

Theorem B. ([18]) *Let* n *and* m *be two positive integers such that* $n \geq 11$. *Let* $a, b \in \mathbb{C}$ *be nonzero constants such that the* $P(z) = z^n + az^{n-1} + b$ *has no multiple zeros and* $S = \{z : P(z) = 0\}$. *If* f *and* g *are nonconstant meromorphic functions satisfying*

$E_f(S) = E_g(S)$ *then either* $f = g$ *or* $f = -\dfrac{ah(h^{n-1}-1)}{h^n - 1}$, $g = -\dfrac{a(h^{n-1}-1)}{h^n-1}$, *where* $h = \dfrac{f}{g}$.

Remark.

In response to Question B Lahiri, Banerjee ([12]) proved that:

If f and g are nonconstant meromorphic functions satisfying $E_f(S,2) = E_g(S,2)$ and $\Theta(\infty, f) + \Theta(\infty, g) > \dfrac{4}{n-1}$ and $n \geq 9$, then $f = g$.

In 2018 Ha Huy Khoai, Vu Hoai An and Nguyen Xuan Lai ([11]) investigated the uniqueness problems for meromorphic functions with higher multiplicities of zeros and poles. Denote by $\mathcal{F}_{sl}$ the class of meromorphic functions whose zeros and poles are of multiplicities at least s, l, respectively, and denote by S_K the set S in Theorem 1.2 ([11]). From Corollary 1.1 and Part 1 of Theorem 1.2 ([11]) it clearly implies the following:

i) If $s = 1, l = 2$, then there exist unique range sets S_K of 9 elements for $\mathcal{F}_{12}$ counting multiplicity.

ii) If $s = 1, l = 3$, then there exist unique range sets S_K of 8 elements for $\mathcal{F}_{13}$ counting multiplicity.

iii) If $s = 2, l = 2$, then there exist unique range sets S_K of 7 elements for $\mathcal{F}_{22}$ counting multiplicity.

iv) If $s = 2, l = 3$, then there exist unique range sets S_K of 6 elements for $\mathcal{F}_{13}$ counting multiplicity.

In 2018 S. Mallick, D. Sarkar studied unique range sets for powers of meromorphic functions ([13]). Denote by $\mathcal{M}^d(\mathbb{C})$ to be the collection of all such meromorphic functions which are powers of some meromorphic functions of power at least d, where d is a positive integer, and denote by S_M the set S in Theorem 1 ([13]). Clearly, $\mathcal{M}^1(\mathbb{C}) = \mathcal{M}(\mathbb{C}) \neq \mathcal{F}_{11}$ and $\mathcal{M}^d(\mathbb{C}) \neq \mathcal{F}_{sl}$ if $d \geq 2$ and if s or l is not a multiple of d.

Example 1. We have $u = \dfrac{(z-1)^s}{(z-2)^l} \in \mathcal{F}_{sl}$. Clear $u \notin \mathcal{M}^d(\mathbb{C})$ if $d \geq 2$ and if s or l is not a multiple of d.

From Part (i) of Theorem 1 ([13]) it clearly implies the following:

i) If $d = 1$, then there exist unique range sets S_M of 17 (10) elements for meromorphic(entire) functions ignoring multiplicity.

ii) If $d = 2$, then there exist unique range sets S_M of 10 (6) elements for meromorphic (entire) functions ignoring multiplicity.

iii) If $d = 3$, then there exist unique range sets S_M of 7 (5) elements for meromorphic (entire) functions ignoring multiplicity.

iv) If $d = 4$, then there exist unique range sets S_M of 6 (4) elements for meromorphic (entire) functions ignoring multiplicity.

v) If $d = 5$, then there exist unique range sets S_M of 5 (4) elements for meromorphic (entire) functions ignoring multiplicity.

vi) If $d = 8$, then there exist unique range sets S_M of 4 (4) elements for meromorphic (entire) functions ignoring multiplicity.

Note that $S \neq S_K$, $S \neq S_M$ and $S_K \neq S_M$ (see ([11]) and ([13])).

Regarding Theorems A and B and Remark it is natural to ask the following question which motivates us to write this paper.

Question 1. *What will happen if we replace CM sharing by IM sharing in Theorem B ?*

In this paper, we apply the arguments used in ([11]) and ([1]) to answer to Question 1.

We shall prove the following main theorem.

Theorem 1. *Let $S = \{z \in \mathbb{C} : P(z) = z^n + az^{n-1} + b = 0\}$, where $a, b \in \mathbb{C}$ be nonzero constants satisfying $\frac{b}{a^n} \neq \frac{(-1)^n(n-1)^{n-1}}{n^n}$. Then for any two nonconstant meromorphic functions f and g, the condition $\overline{E}_f(S) = \overline{E}_g(S)$ implies:*

Either $f = g$ or $f = -\frac{ah(h^{n-1}-1)}{h^n-1}$, $g = -\frac{a(h^{n-1}-1)}{h^n-1}$, where $h = \frac{f}{g}$, if $n \geq 15$.

If all zeros and poles of f and g have multiplicity at least s, l respectively and if either $l \geq 2$ or $s \geq 2$, and $n \geq 5 + \frac{6}{l} + \frac{4}{s}$, then $f = g$.

Following example shows that the condition either $l \geq 2$ or $s \geq 2$ is sharp.

Example 2. Let $h = \frac{z-2}{z}$ and let $g = -\frac{a(h^{n-1}-1)}{h^n-1}$, and let $f = gh$, and S be as in Theorem 1. Then $\overline{E}_f(S) = \overline{E}_g(S)$. Since $h - r, r \neq 1$, has only simple zeros and simple poles. Therefore all zeros and poles of f and g have multiplicity 1. Clear $f \neq g$.

Applications. We discuss some applications of Theorem 1.

Let $S = \{z \in \mathbb{C} : P(z) = z^n + az^{n-1} + b = 0\}$, where $a, b \in \mathbb{C}$ be nonzero constants satisfying $\frac{b}{a^n} \neq \frac{(-1)^n(n-1)^{n-1}}{n^n}$, $n \geq 2$. From Example 2 it clearly implies:

For every $n \geq 2$, S is not a unique range set for $\mathcal{M}(\mathbb{C})$ ignoring multiplicities.

Theorem 2. *There exist the sets of* 10 *elements such that for arbitrary two meromorphic functions* f, g *and for an integer* $d \geq 2$*, the condition* $\overline{E}_{f^d}(S) = \overline{E}_{g^d}(S)$ *implies* $f = tg$*, where* t *is a root of unity of degree* d*.*

Indeed, for two meromorphic functions f, g and $d \geq 2$, if f, g have zeros and poles. Because all zeros and poles of the functions f and g have multiplicity at least 2, if we take $s = l = 2$ and $P(z)$ as in Theorem 1, then the hypothesis of Theorem 1 is satisfied. In this case the zero set of $P(z)$ has 10 elements.Then, the condition $\overline{E}_{f^d}(S) = \overline{E}_{g^d}(S)$ implies $f^d = g^d$ and therefore $f = tg$, where t is a root of unity of degree d. If f, g have only zeros. Then proceeding similarly as in the line of proof of Theorem 1 and note that $\overline{N}(r, f) = 0$, $\overline{N}(r, g) = 0$, we get $f = g$ from $n \geq 7$ when $s = 2$. By using similar arguments as in this case we obtain the proof of other cases.

Giving specific values for s, l in Theorem 1, we can get the following interesting cases:

i) *If* $s = 1, l = 2$ (resp.$s = 1, l = 3$), *then* S *is a unique range set for* $\mathcal{F}_{12}$ (resp. $\mathcal{F}_{13}$) *ignoring multiplicities for* $n \geq 12$(resp. $n \geq 11$).

ii) *If* $s = 1, l = 6$ (resp. $s = 2, l = 2$), *then* S *is a unique range set for* $\mathcal{F}_{16}$ (resp. $\mathcal{F}_{22}$) *ignoring multiplicities for* $n \geq 10$.

iii) *If* $s = 2, l = 3$ (resp. $s = 2, l = 6$), *then* S *is a unique range set for* $\mathcal{F}_{23}$ (resp. $\mathcal{F}_{26}$) *ignoring multiplicities for* $n \geq 9$(resp. $n \geq 8$).

iv) *If* $s = 3, l = 9$ (resp. $s = 11, l = 10$), *then* S *is a unique range set for* $\mathcal{F}_{39}$ (resp. $\mathcal{F}_{1110}$) *ignoring multiplicities for* $n \geq 7$(resp.$n \geq 6$).

Let f is a Weierstrass elliptic function. Note that each pole of Weierstrass elliptic function has order equals 2. As a consequence of Theorem 1 and i), we see that there exist the sets S with 12 elements such that for any meromorphic functions g, the condition $g^{-1}(S) = f^{-1}(S)$ implies that g is the Weierstrass elliptic function.

2. Lemmas and Definitions

We have a another form of two Fundamental Theorems of the Nevanlinna theory:

As an immediate consequence of Nevanlinna's First fundamental theorem (([8], Theorem 1.2, p.5)) we have

(A another form of the First Fundamental Theorem (see [16], Theorem 1.2, p.8)). *Let* $f(z)$ *be a nonconstant meromorphic function and for a* $a \in \mathbb{C}$*. Then*

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1),$$

where $O(1)$ *is a bounded quantity depending on* a*.*

(A another form of the Second Fundamental Theorem (see [16], Theorem 1.6', p.22)). Let f be a nonconstant meromorphic function on $\mathbb{C}$ and let $a_1, a_2, ...,$ a_q be distinct points of $\mathbb{C}$. Then

$$(q-1)T(r,f) \leq \overline{N}(r,f) + \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f-a_i}) - N_0(r, \frac{1}{f'}) + S(r,f),$$

where $N_0(r, \frac{1}{f'})$ is the counting function of those zeros of f', which are not zeros of function $(f-a_1)...(f-a_q)$, and $S(r,f) = o(T(r,f))$ for all r, except for a set of finite Lebesgue measure.

We need some lemmas.

Lemma 2.1. ([6]) *For any nonconstant meromorphic function f, we have*

i) $T(r, f^{(k)}) \leq (k+1)T(r,f) + S(r,f)$;

ii) $S(r, f^{(k)}) = S(r,f)$.

([19]) For any nonconstant meromorphic function f,

$$N(r, \frac{1}{f'}) \leq N(r, \frac{1}{f}) + \overline{N}(r,f) + S(r,f).$$

Definition. *Let f be a nonconstant meromorphic function, and k be a positive integer. We denote by $\overline{N}_{(k}(r,f)$ the counting function of the poles of order $\geq k$ of f, where each pole is counted only once. If z is a zero of f, denote by $\nu_f(z)$ its multiplicity. We denote by $\overline{N}(r, \frac{1}{f'}; f \neq 0)$ the counting function of the zeros z of f' satisfying $f(z) \neq 0$, where each zero is counted only once.*

Let be given two nonconstant meromorphic functions f and g. For simplicity, denote by $\nu_1(z) = \nu_f(z)$ (resp.,$\nu_2(z) = \nu_g(z)$), if z is a zero of f(resp.,g). Let $f^{-1}(0) = g^{-1}(0)$. We denote by $N(r, \frac{1}{f}; \nu_1 = \nu_2 = 1)$(resp., $\overline{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1)$) the counting function of the common zeros z, satisfying $\nu_1(z) = \nu_2(z) = 1$(resp., $\nu_1(z) > \nu_2(z) \geq 1$, where each zero is counted only once), and by $N(r, \frac{1}{f}; \nu_1 \geq 2)$ the counting function of the zeros z of f, satisfying $\nu_1(z) \geq 2$. Similarly, we define the counting functions $\overline{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1)$, $N(r, \frac{1}{g}; \nu_2 \geq 2)$.

Lemma 2.2. ([[1], Lemma 2.3]) *Let f, g be two nonconstant meromorphic functions such that $f^{-1}(0) = g^{-1}(0)$. Set*

$$F = \frac{1}{f},\ G = \frac{1}{g},\ L = \frac{F''}{F'} - \frac{G''}{G'}.$$

Suppose that $L \not\equiv 0$. *Then*

$$1)\ N(r,L) \le \overline{N}_{(2}(r,f) + \overline{N}_{(2}(r,g) +$$
$$\overline{N}(r,\frac{1}{f};\nu_1 > \nu_2 \ge 1) + \overline{N}(r,\frac{1}{g};\nu_2 > \nu_1 \ge 1) + \overline{N}(r,\frac{1}{f'};f \ne 0) +$$
$$\overline{N}(r,\frac{1}{g'};g \ne 0).$$

Moreover, if a *is a common simple zero of* f *and* g, *then* $L(a) = 0$.

$$2)\ \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{f};\nu_1 > \nu_2 \ge 1) + \overline{N}(r,\frac{1}{g};\nu_2 > \nu_1 \ge 1)$$
$$\le N(r,L) + \frac{1}{2}(N(r,\frac{1}{f}) + N(r,\frac{1}{g})) + N(r,\frac{1}{f};\nu_1 \ge 2) + N(r,\frac{1}{g};\nu_2 \ge 2)$$
$$+S(r,f) + S(r,g).$$

Lemma 2.3. ([12], Lemma 1) *Let* f, g *be two nonconstant meromorphic functions. Then* $(f^n + af^{n-1})(g^n + ag^{n-1}) \not\equiv b$, *where* a, b *are nonzero finite constants and* $n(\ge 5)$ *is an integer.*

Lemma 2.4. *Let* f, g *be two non-constant meromorphic functions and* a *is nonzero finite constant. If* f *and* g *share* a *CM, then one of the following three cases holds:*

i)

$$T(r,f) \le \overline{N}(r,f) + \overline{N}_{(2}(r,f) + \overline{N}(r,g) + \overline{N}_{(2}(r,g) + \overline{N}(r,\frac{1}{f})$$
$$+\overline{N}_{(2}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g}) + \overline{N}_{(2}(r,\frac{1}{g}) + S(r,f) + S(r,g),$$

the same inequality holding for T(r, g);

ii)

$$fg = a^2;$$

iii)

$$f = g.$$

Proof. Set $F = \dfrac{f}{a}$, $= \dfrac{g}{a}$. Then $\overline{N}(r,F) = \overline{N}(r,f)$, $\overline{N}_{(2}(r,F) = \overline{N}_{(2}(r,f)$, $\overline{N}(r,G) = \overline{N}(r,g)$, $\overline{N}_{(2}(r,G) = \overline{N}_{(2}(r,g)$, $\overline{N}(r,\frac{1}{F}) = \overline{N}(r,\frac{1}{f})$, $\overline{N}_{(2}(r,\frac{1}{F}) = \overline{N}_{(2}(r,\frac{1}{f})$, $\overline{N}(r,\frac{1}{G}) = \overline{N}(r,\frac{1}{g})$, $\overline{N}_{(2}(r,\frac{1}{G}) = \overline{N}_{(2}(r,\frac{1}{g})$. From this and ([15], Lemma 3) we obtain Lemma 2.4.

Lemma 2.5. *Let $P(z) = z^n + az^{n-1} + b$, where $a, b \in \mathbb{C}$ be nonzero constants and $n \geq 2$. If $\frac{b}{a^n} \neq \frac{(-1)^n(n-1)^{n-1}}{n^n}$, then $P(z)$ has no multiple zeros.*

For the proof, see ([14], Lemma 2.7, Part (ii)).

3. Proof of Theorem 1

Note that $P(z)$ has no multiple zeros from Lemma 2.5, and therefore $P(z)$ has n distinct simple roots. Then $P(z) = (z - a_1)...(z - a_n)$, and recall that $P'(z) = nz^{n-2}(z - a')$, where $a' = -\frac{n-1}{n}a$, $S = \{z \in \mathbb{C} : P(z) = 0\}$. Set

$$F = \frac{1}{P(f)},\ G = \frac{1}{P(g)}, L = \frac{F''}{F'} - \frac{G''}{G'},$$

$$T(r) = T(r, f) + T(r, g), S(r) = S(r, f) + S(r, g).$$

Then $T(r, P(f)) = nT(r, f)+S(r, f)$ and $T(r, P(g)) = nT(r, g)+S(r, g)$, and hence $S(r, P(f)) = S(r, f)$ and $S(r, P(g)) = S(r, g)$.

We prove following.

Lemma 3.1. *Let $\overline{E}_f(S) = \overline{E}_g(S)$. Then $P(f) = P(g)$ if one of the following conditions is satisfied:*

i) All zeros and poles of f and g have multiplicity at least s, l, respectively, and $n \geq 5 + \frac{6}{l} + \frac{4}{s}$, where $l, s \geq 1$;

ii) $n \geq 15$.

Proof. We put $S(r) = S(r, f) + S(r, g)$ and $T(r) = T(r, f) + T(r, g)$.

Assume that condition i) is satisfied. We first prove that:

If $n \geq 5 + \frac{6}{l} + \frac{4}{s}$, then $L \equiv 0$, where $l, s \geq 1$.

Suppose $L \not\equiv 0$.

Claim 1. We have

$$nT(r) \leq (\frac{1}{l}+\frac{1}{s})T(r)+\overline{N}(r, \frac{1}{P(f)})+\overline{N}(r, \frac{1}{P(g)})-N_0(r, \frac{1}{f'})-N_0(r, \frac{1}{g'})+S(r), \quad (3.1)$$

where $N_0(r, \frac{1}{f'})$ ($N_0(r, \frac{1}{g'})$) is the counting function of those zeros of f', which are not zeros of the function $(f - a_1)...(f - a_n)f(f - a')((g - a_1)...(g - a_n)g(g - a'))$. Indeed, applying a another form of two Fundamental Theorems to the functions f, g and the values $a_1, a_2, ..., a_n, 0, a'$, and noting that

$$\sum_{i=1}^{n} \overline{N}(r, \frac{1}{f - a_i}) = \overline{N}(r, \frac{1}{P(f)}),\ \sum_{i=1}^{n} \overline{N}(r, \frac{1}{g - a_i}) = \overline{N}(r, \frac{1}{P(g)}),$$

we obtain

$$(n+1)T(r) \leq \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}(r,\frac{1}{P(f)}) + \overline{N}(r,\frac{1}{P(g)}) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g})$$

$$+\overline{N}(r,\frac{1}{f-a'}) + \overline{N}(r,\frac{1}{g-a'}) - N_0(r,\frac{1}{f'}) - N_0(r,\frac{1}{g'}) + S(r). \tag{3.2}$$

On the other hand,

$$\overline{N}(r,f) + \overline{N}(r,g) \leq \frac{1}{l}(T(r,f)+T(r,g)) + S(r) = \frac{1}{l}T(r) + S(r),$$

$$\overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g}) \leq \frac{1}{s}(T(r,f)+T(r,g)) + S(r) = \frac{1}{s}T(r) + S(r),$$

$$\overline{N}(r,\frac{1}{f-a'}) + \overline{N}(r,\frac{1}{g-a'}) \leq (T(r,f)+T(r,g)) + S(r) = T(r) + S(r).$$

From this and (3.2) we obtain (3.1).

Claim 2. We have

$$\overline{N}(r,\frac{1}{P(f)}) + \overline{N}(r,\frac{1}{P(g)}) \leq$$

$$(\frac{n}{2}+1+\frac{2}{l})T(r) + \overline{N}(r,\frac{1}{[P(f)]'};P(f)\neq 0) + \overline{N}(r,\frac{1}{[P(g)]'};P(g)\neq 0) + S(r).$$

Indeed, by $\overline{E}_f(S) = \overline{E}_g(S)$ we get $(P(f))^{-1}(0) = (P(g))^{-1}(0)$. For simplicity, we set $\nu_1 = \nu_1(z)$, $\nu_2 = \nu_2(z)$, where $\nu_1(z) = \nu_{P(f)}(z)$, $\nu_2(z) = \nu_{P(g)}(z)$. Note that

$$\overline{N}_{(2}(r,P(f)) = \overline{N}(r,f),\ \overline{N}_{(2}(r,P(g)) = \overline{N}(r,g),$$

$$S(r,P(f)) = S(r,f),\ S(r,P(g)) = S(r,g), S(r) = S(r,f) + S(r,g).$$

Applying the Lemma 2.2 to the functions $P(f), P(g)$. Then we obtain

$$N(r,L) \leq \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}(r,\frac{1}{P(f)};\nu_1>\nu_2\geq 1) + \overline{N}(r,\frac{1}{P(g)};\nu_2>\nu_1\geq 1)$$

$$+\overline{N}(r,\frac{1}{[P(f)]'};P(f)\neq 0) + \overline{N}(r,\frac{1}{[P(g)]'};P(g)\neq 0), \tag{3.3}$$

and

$$\overline{N}(r,\frac{1}{P(f)}) + \overline{N}(r,\frac{1}{P(g)}) + \overline{N}(r,\frac{1}{P(f)};\nu_1>\nu_2\geq 1)+$$

$$\overline{N}(r,\frac{1}{P(g)};\nu_2>\nu_1\geq 1)\leq N(r,L)+\frac{1}{2}(N(r,\frac{1}{P(f)})+N(r,\frac{1}{P(g)})+$$
$$N(r,\frac{1}{P(f)};\nu_1\geq 2)+N(r,\frac{1}{P(g)};\nu_2\geq 2))+S(r). \quad (3.4)$$

Morover,

$$\overline{N}(r,f)+\overline{N}(r,g)\leq \frac{1}{l}T(r)+S(r). \quad (3.5)$$

Obviously,

$$N(r,\frac{1}{P(f)})\leq nT(r,f)+S(r,f); N(r,\frac{1}{P(g)})\leq nT(r,g)+S(r,g),$$

$$N(r,\frac{1}{P(f)})+N(r,\frac{1}{P(g)})\leq nT(r)+S(r). \quad (3.6)$$

On the other hand, from $P(f)=(f-a_1)...(f-a_n)$ it follows that if z_0 is a zero of $P(f)$ with multiplicity ≥ 2, then z_0 is a zero of $f-a_i$ with multiplicity ≥ 2 for some $i\in\{1,2,...,n\}$, and therefore, it is a zero of $f^{'}$, so we have $N(r,\frac{1}{P(f)};\nu_1\geq 2)\leq N(r,\frac{1}{f'})$. From this and Lemma 2.1 we obtain

$$N(r,\frac{1}{P(f)};\nu_1\geq 2)\leq N(r,\frac{1}{f'})\leq$$

$$N(r,\frac{1}{f})+\overline{N}(r,f)+S(r,f)\leq T(r,f)+\frac{1}{l}T(r,f)+S(r,f).$$

Similarly, we have

$$N(r,\frac{1}{P(g)};\nu_2\geq 2)\leq N(r,\frac{1}{g'})\leq T(r,g)+\frac{1}{l}T(r,g)+S(r,g).$$

Therefore,

$$N(r,\frac{1}{P(f)};\nu_1\geq 2)+N(r,\frac{1}{P(g)};\nu_2\geq 2)\leq (1+\frac{1}{l})T(r)+S(r). \quad (3.7)$$

Combining (3.1)-(3.7) we get

$$\overline{N}(r,\frac{1}{P(f)})+\overline{N}(r,\frac{1}{P(g)})\leq$$

$$(\frac{n}{2}+1+\frac{2}{l})T(r)+\overline{N}(r,\frac{1}{[P(f)]'};P(f)\neq 0)+\overline{N}(r,\frac{1}{[P(g)]'};P(g)\neq 0)+S(r).$$

Claim 2 is proved.
Claim 3. We have

$$\overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) \leq (1 + \frac{1}{s})T(r) + N_0(r, \frac{1}{f'}) + N_0(r, \frac{1}{g'}) + S(r).$$

We have

$$\overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) = \overline{N}(r, \frac{1}{f^{n-2}(f-a')f'}; P(f) \neq 0) \leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-a'}) + \overline{N}_0(r, \frac{1}{f'}) \leq (1 + \frac{1}{s})T(r, f) + \overline{N}_0(r, \frac{1}{f'}) + S(r, f). \tag{3.8}$$

Similarly,

$$\overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) \leq (1 + \frac{1}{s})T(r, g) + \overline{N}_0(r, \frac{1}{g'}) + S(r, g). \tag{3.9}$$

Inequalities (3.8) and (3.9) give us

$$\overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) \leq (1 + \frac{1}{s})T(r) + \overline{N}_0(r, \frac{1}{f'}) + \overline{N}_0(r, \frac{1}{g'}) + S(r).$$

Claim 3 is proved.

Claim 1, 2, 3 give us:

$$nT(r) \leq (\frac{n}{2} + 2 + \frac{3}{l} + \frac{2}{s})T(r) + S(r). \text{ So } (n - 4 - \frac{6}{l} - \frac{4}{s})T(r) \leq S(r).$$

This is a contradiction to the assumption that $n \geq 5 + \frac{6}{l} + \frac{4}{s}$. So $L \equiv 0$. Then, we have $\frac{1}{P(f)} = \frac{c}{P(g)} + c_1$ for some constants $c \neq 0$ and c_1. Now we return the proof of the Lemma 3.1. Set $F = P(f) - b$, $G = P(g) - b$. Therefore $F = f^{n-1}(f + a)$, $G = g^{n-1}(g + a)$ and $\frac{1}{F+b} = \frac{c}{G+b} + c_1$. From this it follows F and G share $-b$ CM. Then applying Lemma 2.4 to the functions F, G we get:
Case 1.

$$T(r, F) \leq \overline{N}(r, F) + \overline{N}_{(2}(r, F) + \overline{N}(r, G) + \overline{N}_{(2}(r, G) + \overline{N}(r, \frac{1}{F})$$

$$+\overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + \overline{N}_{(2}(r, \frac{1}{G}) + S(r, F) + S(r, G), \tag{3.10}$$

the same inequality holding for $T(r, G)$. Note that $\overline{N}(r, F) = \overline{N}(r, f) = \overline{N}_{(2}(r, F)$, $\overline{N}(r, G) = \overline{N}(r, g) = \overline{N}_{(2}(r, G)$, $\overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f+a})$, $\overline{N}_{(2}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{f}) + \overline{N}_{(2}(r, \frac{1}{f+a})$, $\overline{N}(r, \frac{1}{G}) = \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{g+a})$, $\overline{N}_{(2}(r, \frac{1}{G}) = \overline{N}(r, \frac{1}{g}) + \overline{N}_{(2}(r, \frac{1}{g+a})$. Moreover $\overline{N}(r, f) + \overline{N}(r, g) \leq \frac{1}{l}T(r) + O(1)$, $\overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) \leq \frac{1}{s}T(r) + O(1)$, $\overline{N}(r, \frac{1}{f+a}) + \overline{N}(r, \frac{1}{g+a}) \leq T(r) + O(1)$, $\overline{N}_{(2}(r, \frac{1}{f+a}) + \overline{N}_{(2}(r, \frac{1}{g+a}) \leq T(r) + O(1)$, $S(r, F) = S(r, f)$, $S(r, G) = S(r, g)$. From this and (3.10) we get

$$T(r, F) = nT(r, f) + O(1) \leq (2 + \frac{2}{l} + \frac{2}{s})T(r) + S(r),$$

$$T(r, G) = nT(r, g) + O(1) \leq (2 + \frac{2}{l} + \frac{2}{s})T(r) + S(r).$$

Adding the side with the side of the two inequalities above we obtain

$$nT(r) \leq (4 + \frac{4}{l} + \frac{4}{s})T(r) + S(r),\ (n - 4 - \frac{4}{l} - \frac{4}{s})T(r) \leq S(r).$$

This is a contradiction to the assumption that $n \geq 5 + \frac{6}{l} + \frac{4}{s}$.

Case 2. $F.G = b^2$. Then $(f^n + af^{n-1})(g^n + ag^{n-1}) = b^2$. Because $n \geq 5 + \frac{6}{l} + \frac{4}{s} > 5$, and then we have $(f^n + af^{n-1})(g^n + ag^{n-1}) \not\equiv b^2$ since Lemma 2.3. A contradiction.

Case 3. $F = G$. Therefore $P(f) = P(g)$. Thanks to Lemma 2.3.

Now if condition ii) is satisfied: $n \geq 15$, then by a similar argument as in i) with $l = s = 1$ we get $P(f) = P(g)$.

Lemma 3.2. *Let f and g be two nonconstant meromorphic functions, and assume that all zeros and poles of f and g have multiplicity at least s, l, respectively, and if either $l \geq 2$ or $s \geq 2$ such that $n \geq 5 + \frac{6}{l} + \frac{4}{s}$, and let $P(f) = P(g)$. Then $f = g$.*

Proof. We have $P(z) = z^n + az^{n-1} + b$. Set $h = \frac{f}{g}$. Since $P(f) = P(g)$ we obtain

$$f^n + af^{n-1} = g^n + ag^{n-1}. \tag{3.11}$$

It implies

$$g = -a\frac{h^{n-1} - 1}{h^n - 1}. \tag{3.12}$$

Suppose that h is not a constant. Let $t_1, t_2, ..., t_{n-2},$ (resp.,$r_1, r_2, ..., r_{n-1},$) be the roots of unity of degree $n-1$, (resp., n), $t_i \neq 1,\ i = 1, 2, ..., n-2$ (resp.,$r_i \neq 1,\ i = 1, 2, ..., n-1$). From (3.12) we get

$$T(r,g) = T(r, \frac{h^{n-1}-1}{h^n - 1}) = (n-1)T(r,h) + O(1),\ T(r,h) = \frac{1}{n-1}T(r,g) + O(1). \tag{3.13}$$

By (3.16) and g is a nonconstant meromorphic function it implies that so is h , and $S(r,h) = S(r,g)$. Then, applying a another form of two Fundamental Theorems to $t_1, t_2, ..., t_{n-2}, r_1, r_2, ..., r_{n-1}$, since (3.12)- (3.13) we obtain

$$(2n-4)T(r,h) \leq \overline{N}(r,h) + \sum_{i=1}^{n-2} \overline{N}(r, \frac{1}{h - t_i}) + \sum_{i=1}^{n-1} \overline{N}(r, \frac{1}{h - r_i}) + S(r,h),$$

$$\frac{2n-4}{n-1}T(r,g) \leq \overline{N}(r,g) + \overline{N}(r, \frac{1}{g}) + S(r,g),\ \frac{2n-4}{n-1}T(r,g) \leq (\frac{1}{l} + \frac{1}{s})T(r,g) + S(r,g),$$

$$(2 - \frac{2}{n-1} - \frac{1}{l} - \frac{1}{s})T(r,g) \leq S(r,g).$$

A contradiction, since $n \geq 5 + \frac{6}{l} + \frac{4}{s} > 5$, and either $l \geq 2$ or $s \geq 2$.
So h is a constant. Then from (3.11) it implies $h^n = 1$ and $h^{n-1} = 1$, because g is not a constant. Therefore $h = 1$ and $f = g$.
Now we return the proof of Theorem 1.
Applying Lemma 3.1 with $n \geq 15$ we get $P(f) = P(g),\ f^n + af^{n-1} = g^n + ag^{n-1}$. Then proceeding similarly as in the line of proof of [p.80 -p. 81, Case II, Theorem 2, ([18])], we get either $f = g$ or $f = -\frac{ah(h^{n-1}-1)}{h^n - 1},\ g = -\frac{a(h^{n-1}-1)}{h^n - 1}$, where $h = \frac{f}{g}$.
Applying Lemma 3.1, Lemma 3.2 with either $l \geq 2$ or $s \geq 2$ and $n \geq 5 + \frac{6}{l} + \frac{4}{s}$ we get $f = g$.
Theorem 1 is proved.

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