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A CERTAIN SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH CHEBYSHEV POLYNOMIALS

P. Nandini and S. Latha

Department of Mathematics,
JSS Academy of Technical Education,
Srinivaspura, Bengaluru - 560060, Karnataka, INDIA

E-mail : pnandinimaths@gmail.com, drlatha@gmail.com

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Abstract: In the present work, we investigate a new subclass of bi-univalent functions by applying the q-derivative operator associated with Chebyshev polynomials. We find estimates for the general Taylor-Maclaurin coefficients of the functions in this class and also obtain an estimation for the Fekete-Szegö problem for this class.

Keywords and Phrases: Analytic functions, Univalent and Bi-univalent functions, Fekete-Szegö inequality, Chebyshev polynomials and q-derivative operator.

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1. Introduction

We indicate by $\mathcal{A}$ the collection of functions, which are analytic in the open unit disk $\mathbb{D}$ given by

$$\mathbb{D} = \{z \in \mathbb{C} \quad and \quad |z| < 1\}$$

and have the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We denote by $\mathcal{S}$ the sub-collection of the set $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{D}$. The Koebe one-quarter theorem [5] asserts that the image of $\mathbb{D}$ under each univalent function f in $\mathcal{S}$ contains a disk of radius 1/4. According to this, every function $f \in \mathcal{S}$ has an inverse map f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ... \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and its inverse f^{-1} are univalent in $\mathbb{D}$. Let Σ denote the class of bi-univalent functions in $\mathbb{D}$ given by (1.1). For more basic results, one may refer Srivastava et al. [14] and references therein [12, 13].

Next, we recall the definition of subordination between analytic functions. For two functions $f, g \in \mathcal{A}$, we say that f is subordinate to g in $\mathbb{D}$, written as $f \prec g$, provided there is an analytic function w in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. It follows from Schwarz Lemma that

$$f(z) \prec g(z) \iff f(0) = g(0) \quad and \quad f(\mathbb{D}) \subset g(\mathbb{D}), \quad z \in \mathbb{D}.$$

For $q \in (0, 1)$, the Jackson q-derivative of a function $f \in \mathcal{A}$ is given by (see [8, 9])

$$D_qf(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z} & (z \neq 0), \\ f'(0) & (z = 0). \end{cases} \quad (1.3)$$

Thus from (1.3), we have

$$D_qf(z) = 1 + \sum_{n=2}^{\infty}[n]_q a_n z^{n-1} \quad (1.4)$$

where

$$[n]_q = \frac{1-q^n}{1-q},$$

and, as $q \to 1^-$, $[n]_q \to n$.

One of the important tools in numerical analysis, from both theoretical and practical points of view, is Chebyshev polynomials. Out of four kinds of Chebyshev polynomials, many researchers dealing with orthogonal polynomials of Chebyshev.

For a brief history of Chebyshev polynomials of first and second kinds $T_n(t)$ and $U_n(t)$ and their applications, one can refer [4, 6, 10, 11].

The Chebyshev polynomials of the first and second kinds are orthogonal for $t \in [-1, 1]$ and defined as follows:

Definition 1.1. *The Chebyshev polynomials of the first kind are defined by the following three-terms recurrence relation:*

$$T_0(t) = 1,$$

$$T_1(t) = t,$$

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t).$$

The first few of the Chebyshev polynomials of the first kind are

$$T_2(t) = 2t^2 - 1, \quad T_3(t) = 4t^3 - 3t, \quad T_4(t) = 8t^4 - 8t^2 + 1, ... \tag{1.5}$$

The generating function for the Chebyshev polynomials of the first kind, $T_n(t)$, is given by:

$$F(z,t) = \sum_{n=0}^{\infty} T_n(t)z^n = \frac{1 - tz}{1 - 2tz + z^2}, \quad (z \in \mathbb{D}).$$

Definition 1.2. *The Chebyshev polynomials of the second kind are defined by the following three-terms recurrence relation:*

$$U_0(t) = 1,$$

$$U_1(t) = 2t,$$

$$U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t).$$

The first few of the Chebyshev polynomials of the second kind are

$$U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, ... \tag{1.6}$$

The generating function for the Chebyshev polynomials of the second kind, $U_n(t)$, is given by:

$$H(z,t) = \sum_{n=0}^{\infty} U_n(t)z^n = \frac{1}{1 - 2tz + z^2}, \quad (z \in \mathbb{D}).$$

The Chebyshev polynomials of the first and second kinds are connected by the following relations:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); \quad T_n(t) = U_n(t) - tU_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

Motivated by the work of Dziok et al. [6], Altinkaya, and Yalcin [2, 1], we construct, in the next section, a new subclass of bi-univalent functions governed by the Chebyshev polynomials. Then, we investigate the optimal bounds for the Taylor- Maclaurin coefficients $|a_2|$ and $|a_3|$ and Furthermore we establish Fekete-Szegö inequality for the class $\mathfrak{B}_\Sigma(\alpha, t, q)$.

2. Main Results

Definition 2.1. *For $0 \leq \alpha \leq 1$, $t \in (1/2, 1]$ and $0 < q < 1$, a function $f \in \Sigma$ is said to be in the class $\mathfrak{B}_\Sigma(\alpha, t, q)$ if the following subordination holds for all $z, w \in \mathbb{D}$:*

$$\frac{(1-\alpha)zD_q(f(z)) + \alpha zD_q(zD_q(f(z)))}{(1-\alpha)f(z) + \alpha zD_q(f(z))} \prec H(z,t) := \frac{1}{1-2tz+z^2}$$

and

$$\frac{(1-\alpha)wD_q(g(w)) + \alpha wD_q(wD_q(g(w)))}{(1-\alpha)g(w) + \alpha wD_q(g(w))} \prec H(w,t) := \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is given by (1.2).

Example 1. For $\alpha = 0$, $t \in (1/2, 1]$ and $0 < q < 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{S}^*_\Sigma(t, q)$ if the following subordination holds for all $z, w \in \mathbb{D}$:

$$\frac{zD_q(f(z))}{f(z)} \prec H(z,t) := \frac{1}{1-2tz+z^2}$$

and

$$\frac{wD_q(g(w))}{g(w)} \prec H(w,t) := \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is given by (1.2).

Example 2. For $\alpha = 1$, $t \in (1/2, 1]$ and $0 < q < 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{K}_\Sigma(t, q)$ if the following subordination holds for all $z, w \in \mathbb{D}$:

$$\frac{D_q(zD_q(f(z)))}{D_q(f(z))} \prec H(z,t) := \frac{1}{1-2tz+z^2}$$

and

$$\frac{D_q(wD_q(g(w)))}{D_q(g(w))} \prec H(w,t) := \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is given by (1.2).

Remark 2.2. *By giving specific values to the parameter α in the class $\mathfrak{B}_\Sigma(\alpha, t, q)$, we get the following subclasses:*

1. *For $q \to 1$ and $\alpha = 0$, we get*

$$\frac{zf'(z)}{f(z)} \prec H(z,t) := \frac{1}{1-2tz+z^2},$$

and

$$\frac{wg'(w)}{g(w)} \prec H(w,t) := \frac{1}{1-2tw+w^2},$$

it reduces to the special case from the class $\mathcal{B}_{\Sigma}^{\mu}(\lambda,t)$, which was introduced by Bulut, Magesh, and Abirami [3].

2. *For $q \to 1$ and $\alpha = 1$, we get*

$$\frac{(zf'(z))'}{f'(z)} \prec H(z,t) := \frac{1}{1-2tz+z^2},$$

and

$$\frac{(wg'(z))'}{g'(w)} \prec H(w,t) := \frac{1}{1-2tw+w^2},$$

it reduces to the special case from the class $\mathcal{M}_{\Sigma}^{k}(\lambda,\phi(z,t))$, which was introduced by Güney, Murugusundaramoorthy, and Vijaya [7].

Theorem 2.3. *For $0 \leq \alpha \leq 1$, $t \in (1/2, 1]$ and $0 < q < 1$, let $f \in \mathcal{A}$ be in the class $\mathfrak{B}_{\Sigma}(\alpha,t,q)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|\{([3]_q-1)[1+\alpha([3]_q-1)]-[2]_q([2]_q-1)[1+\alpha([2]_q-1)^2]\}4t^2+([2]_q-1)^2[1+\alpha([2]_q-1)]^2\right|}}$$

and

$$|a_3| \leq \frac{2t}{([3]_q-1)[1+\alpha([3]_q-1)]} + \frac{4t^2}{([2]_q-1)^2[1+\alpha([2]_q-1)]^2}.$$

Proof. Let $f \in \mathfrak{B}_{\Sigma}(\alpha,t,q)$. Then there are two analytic functions $u, v : \mathbb{D} \to \mathbb{D}$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + ... \qquad (z \in \mathbb{D}) \tag{2.1}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + ... \qquad (w \in \mathbb{D}), \tag{2.2}$$

with $u(0) = v(0) = 0$ and $\max\{|u(z)|, |v(w)|\} < 1$ $(z, w \in \mathbb{D})$, such that

$$\frac{(1-\alpha)zD_q(f(z)) + \alpha zD_q(zD_q(f(z)))}{(1-\alpha)f(z) + \alpha zD_q(f(z))} = H(u(z),t)$$

and

$$\frac{(1-\alpha)wD_q(g(w))+\alpha wD_q(zD_q(g(w)))}{(1-\alpha)g(w)+\alpha wD_q(g(w))}=H(v(w),t)$$

or, equivalently, that

$$\frac{(1-\alpha)zD_q(f(z))+\alpha zD_q(zD_q(f(z)))}{(1-\alpha)f(z)+\alpha zD_q(f(z))}=1+U_1(t)u(z)+U_2(t)u^2(z)+... \quad (2.3)$$

$$\frac{(1-\alpha)wD_q(g(w))+\alpha wD_q(zD_q(g(w)))}{(1-\alpha)g(w)+\alpha wD_q(g(w))}=1+U_1(t)v(w)+U_2(t)v^2(w)+... \quad (2.4)$$

Combining (2.1), (2.2), (2.3), and (2.4), we find that

$$\frac{(1-\alpha)zD_q(f(z))+\alpha zD_q(zD_q(f(z)))}{(1-\alpha)f(z)+\alpha zD_q(f(z))}=1+U_1(t)u_1z+[U_1(t)u_2+U_2(t)u_1^2]z^2... \quad (2.5)$$

and

$$\frac{(1-\alpha)wD_q(g(w))+\alpha wD_q(zD_q(g(w)))}{(1-\alpha)g(w)+\alpha wD_q(g(w))}=1+U_1(t)v_1w+[U_1(t)v_2+U_2(t)v_1^2]w^2+... \quad (2.6)$$

It is well known that, if

$$\max\{|u(z)|,|v(w)|\}<1,(z,w\in\mathbb{D}),$$

then

$$|u_j|\le 1 \quad and \quad |v_j|\le 1 \quad (\forall j\in\mathbb{N}) \quad (2.7)$$

Now, by comparing the corresponding coefficients in (2.5) and (2.6) and after some simplification, we have

$$([2]_q-1)(1+\alpha([2]_q-1))a_2=U_1(t)u_1, \quad (2.8)$$

$$\begin{aligned}([3]_q-1)\left(1+\alpha([3]_q-1)\right)a_3-\left\{([2]_q-1)\left(1+\alpha([2]_q-1)\right)^2\right\}a_2^2\\ =U_1(t)u_2+U_2(t)u_1^2,\end{aligned} \quad (2.9)$$

$$-([2]_q-1)(1+\alpha([2]_q-1))a_2=U_1(t)v_1 \quad (2.10)$$

and

$$\begin{aligned}\left\{2([3]_q-1)\left(1+\alpha([3]_q-1)-([2]_q-1)(1+\alpha([2]_q-1))^2\right)\right\}a_2^2\\ -\left\{([3]_q-1)\left(1+\alpha([3]_q-1)\right)\right\}a_3=U_1(t)v_2+U_2(t)v_1^2.\end{aligned} \quad (2.11)$$

It follows from (2.8) and (2.10) that

$$u_1 = -v_1 \tag{2.12}$$

and

$$2\left([2]_q-1\right)^2\left(1+\alpha([2]_q-1)\right)^2 a_2^2 = (U_1(t))^2(u_1^2+v_1^2). \tag{2.13}$$

If we add (2.9) and (2.11), we find that

$$\begin{aligned} 2\left\{([3]_q-1)(1+\alpha([3]_q-1))-([2]_q-1)(1+\alpha([2]_q-1))^2\right\} a_2^2 \\ = U_1(t)(u_2+v_2)+U_2(t)(u_1^2+v_1^2) \end{aligned}. \tag{2.14}$$

Upon substituting the value of $u_1^2+v_1^2$ from (2.13) into the right-hand side of (2.14), we deduce that

$$a_2^2 = \frac{(U_1(t))^3(u_2+v_2)}{2\left\{[([3]_q-1)[1+\alpha([3]_q-1)]-([2]_q-1)[1+\alpha([2]_q-1)]^2](U_1(t))^2-([2]_q-1)^2[1+\alpha([2]_q-1)]^2U_2(t)\right\}}. \tag{2.15}$$

By further computations using (1.6), (2.7) and (2.15), we obtain

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|\left\{([3]_q-1)[1+\alpha([3]_q-1)]-[2]_q([2]_q-1)[1+\alpha([2]_q-1)^2]\right\}4t^2+([2]_q-1)^2[1+\alpha([2]_q-1)]^2\right|}}$$

Next, if we subtract (2.11) from (2.9), we can easily see that

$$2([3]_q-1)[1+\alpha([3]_q-1)](a_3-a_2^2) = U_1(t)(u_2-v_2)+U_2(t)(u_1^2-v_1^2). \tag{2.16}$$

In view of (2.12) and (2.13), we find from (2.16) that

$$a_3 = \frac{U_1(t)(u_2-v_2)}{2([3]_q-1)[1+\alpha([3]_q-1)]} + \frac{(U_1(t))^2(u_1^2+v_1^2)}{2([2]_q-1)^2[1+\alpha([2]_q-1)]^2}.$$

Thus, by applying (1.6), we obtain

$$|a_3| \leq \frac{2t}{([3]_q-1)[1+\alpha([3]_q-1)]} + \frac{4t^2}{([2]_q-1)^2[1+\alpha([2]_q-1)]^2}.$$

In the next theorem, we present the Fekete-Szegö inequality for the class $\mathfrak{B}_\Sigma(\alpha,t,q)$.

Theorem 2.4. *For $0 \leq \alpha \leq 1$, $t \in (1/2, 1]$, $0 < q < 1$ and $\mu \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathfrak{B}_\Sigma(\alpha,t,q)$. Then*

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{2t}{([3]_q-1)[1+\alpha([3]_q-1)]}; \\ \left(|\mu-1| \leq \frac{\left|([2]_q-1)^2[1+\alpha([2]_q-1)]^2/4t^2+([3]_q-1)[1+\alpha([3]_q-1)]-[2]_q([2]_q-1)[1+\alpha([2]_q-1)]^2\right|}{([3]_q-1)[1+\alpha([3]_q-1)]}\right) \\ \frac{8t^3|\mu-1|}{\left|\{([3]_q-1)[1+\alpha([3]_q-1)]-[2]_q([2]_q-1)[1+\alpha([2]_q-1)]^2\}4t^2+([2]_q-1)^2[1+\alpha([2]_q-1)]^2\right|}; \\ \left(|\mu-1| \geq \frac{\left|([2]_q-1)^2[1+\alpha([2]_q-1)]^2/4t^2+([3]_q-1)[1+\alpha([3]_q-1)]-[2]_q([2]_q-1)[1+\alpha([2]_q-1)]^2\right|}{([3]_q-1)[1+\alpha([3]_q-1)]}\right). \end{cases}$$

Proof. It follows from (2.15) and (2.16) that

$$a_3-\mu a_2^2=\frac{U_1(t)(u_2-v_2)}{2([3]_q-1)(1+\alpha([3_q-1]))}+(1-\mu)a_2^2$$

$$=\frac{U_1(t)(u_2-v_2)}{2([3]_q-1)[1+\alpha([3]_q-1)]}$$

$$+\frac{(U_1(t))^3(u_2+v_2)(1-\mu)}{2\left\{[([3]_q-1)[1+\alpha([3]_q-1)]-([2]_q-1)[1+\alpha([2]_q-1)]^2](U_1(t))^2-([2]_q-1)^2[1+\alpha([2]_q-1)]^2U_2(t)\right\}}$$

$$=\frac{U_1(t)}{2}\left[\left(\phi(\mu,t)+\frac{1}{([3]_q-1)[1+\alpha([3]_q-1)]}\right)u_2\right.$$
$$\left.+\left(\phi(\mu,t)-\frac{1}{([3]_q-1)[1+\alpha([3]_q-1)]}\right)v_2\right],$$

where

$$\phi(\mu,t)=\frac{(U_1(t))^2(1-\mu)}{\left\{([3]_q-1)[1+\alpha([3]_q-1)]-([2]_q-1)[1+\alpha([2]_q-1)]^2\right\}(U_1(t))^2-([2]_q-1)^2[1+\alpha([2]_q-1)]^2U_2(t)}.$$

Thus, according to (1.6), we have

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{2t}{([3]_q-1)[1+\alpha([3]_q-1)]} & 0\leq|\phi(\mu,t)|\leq\frac{1}{([3]_q-1)[1+\alpha([3]_q-1)]}\\ 2t|\phi(\mu,t)| & |\phi(\mu,t)|\geq\frac{1}{([3]_q-1)[1+\alpha([3]_q-1)]}.\end{cases}$$

Taking $\mu=1$ in Theorem 2.4, we led to the following corollary:

Corollary 2.5. *For $0\leq\alpha\leq1$, $t\in(1/2,1]$ and $0<q<1$, let $f\in\mathcal{A}$ be in the class $\mathfrak{B}_\Sigma(\alpha,t,q)$. Then*

$$|a_3-a_2^2|\leq\frac{2t}{([3]_q-1)[1+\alpha([3]_q-1)]}.$$

Remark 2.6. *By suitably specialising the parameter α, one can deduce the results for the subclasses $\mathcal{S}_\Sigma^*(t,q)$ and $\mathcal{K}_\Sigma(t,q)$, which are defined respectively in Examples 1 and 2 associated with the Chebyshev polynomials.*

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