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INEQUALITIES CONCERNING THE MODULUS OF RATIONAL FUNCTIONS WITH FIXED ZEROS

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Abstract: In this paper, we investigate the lower bound and the upper bound of the modulus of rational functions with fixed zeros. Some well-known results are generalized by our results.

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1. Introduction

Let k be any positive integer. The results will be presented using the notations listed below:

$$T_k = \{z \in \mathbb{C} : |z| = k\},$$

$$D_k^- = \{z \in \mathbb{C} : |z| < k\},$$

$$D_k^+ = \{z \in \mathbb{C} : |z| > k\}.$$

Let P_n be the class of complex polynomials $p(z)$ of degree at most n and $p'(z)$ be the derivative of $p(z)$. According to the famous result of Bernstein [4] if $p(z) \in P_n$, then

$$|p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

If $p(z)$ has all the zeros at the origin, then equality holds in (1.1).
In 1944, Lax [7] proved the Erdös conjectured that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)| \tag{1.2}$$

for polynomials $p(z)$ of degree n and having no zeros in D_1^-. Equality in (1.2) holds for $p(z) = \lambda z^n + \mu, |\lambda| = |\mu|$. Futhermore, if $p(z)$ is a polynomial of degree n having no zeros in D_1^+, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.3}$$

Inequality (1.3) was showed by Turán [11] and equality in (1.3) holds for polynomials which have all its zeros on $|z| = 1$.
From now on, we let

$$w(z) := \prod_{j=1}^{n} (z - a_j),$$

$$B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{a_j} z}{z - a_j} \right) = \frac{w^*(z)}{w(z)}$$

where $a_j \in \mathbb{C}$ for $j = 1, 2, 3, \ldots, n$ and

$$w^*(z) = z^n \overline{w\left(\frac{1}{\overline{z}}\right)}.$$

The product $B(z)$ is called Blaschke product with $|B(z)| = 1$ for $z \in T_1$.
Now, we define $R_{m,n}$ by

$$R_{m,n}(a_1, a_2, \ldots, a_n) := \left\{ \frac{p(z)}{w(z)} : p \in P_m \text{ and } m \leq n \right\}.$$

Then $R_{m,n}$ is the set of all rational functions with poles $a_1, a_2, \ldots, a_n$ at most and with finite limit at ∞.

Li et al. [8] extended inequality (1.1) to rational functions $r(z)$ by substituting polynomials $p(z)$ by rational functions $r(z)$ and z^n by Blaschke product $B(z)$. They proved the results as follows.

Theorem 1.1. *If $r(z) \in R_{m,n}$, then*

$$|r'(z)| \leq |B'(z)||r(z)|$$

for $z \in T_1$. The equality holds for $r(z) = \lambda B(z)$ with $|\lambda| = 1$.

Theorem 1.2. *If $r(z) \in R_{m,n}$ has all its zeros in $T_1 \cup D_1^+$, then*

$$|r'(z)| \leq \frac{1}{2}|B'(z)| \max_{z \in T_1} |r(z)|$$

for $z \in T_1$. The equality holds for $r(z) = \eta B(z) + \lambda$ with $|\eta| = |\lambda| = 1$.
On the other hand, Li et al. [8] proved the following results.

Theorem 1.3. *If $r(z) \in R_{m,n}$ has all its zeros in $T_1 \cup D_1^-$, then*

$$|r'(z)| \geq \frac{1}{2}|B'(z)| \max_{z \in T_1} |r(z)|$$

for $z \in T_1$. The equality holds for $r(z) = \eta B(z) + \lambda$ with $|\eta| = |\lambda| = 1$.

Theorem 1.4. *If $r(z) \in R_{m,n}$, where $r(z)$ has n poles $a_1, a_2, \ldots, a_n$ and all the zeros of $r(z)$ lie in $T_1 \cup D_1^-$, then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}(|B'(z)| - (n-m))|r(z)|$$

where m is the number of zeros of $r(z)$.
In 2004, Aziz and Shah [1] extended Theorem 1.4 to the class of rational functions $R_{m,n}$ having all its zeros in $T_k \cup D_k^-, k \leq 1$.

Theorem 1.5. *If $r(z) \in R_{m,n}$, where $r(z)$ has n poles $a_1, a_2, \ldots, a_n$ and all the zeros of $r(z)$ lie in $T_k \cup D_k^-, k \leq 1$, then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}(|B'(z)| + \frac{2m - n(1+k)}{1+k})|r(z)|$$

where m is the number of zeros of $r(z)$.
In 2021, Gulzar et al. [5] demonstrated the following interesting result.

Theorem 1.6. *If $r(z) = \frac{p(z)}{w(z)} \in R_{m,n}$ and $c_1, c_2, \ldots, c_m$ are the zeros of $r(z)$ all lying in $T_k \cup D_k^-, k \leq 1$. Then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| + \sum_{j=1}^{m} \frac{2}{1+|c_j|} - n\right]|r(z)|$$

where m is the number of zeros of $r(z)$.
In 2022, Gupta et al. [6] proved that if $r(z) \in R_{m,n}$ has $t-$folds zeros at the origin and the remaining $m-t$ in $T_1 \cup D_1^+$, then for $z \in T_1$

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - (n-m-t)\right]|r(z)|. \tag{1.4}$$

There are other refinements and generalizations of the above results that can be found in the literature [3, 9, 10].

2. Main Results

We start by showing the following lemmas that we apply to support our theorems. The first lemma was shown by Aziz and Zargar [2].

Lemma 2.1. *If $|z| = 1$, then*

$$Re\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - |B'(z)|}{2}.$$

The next lemmas was proved by Li et al. [8].

Lemma 2.2. *If $r(z) \in R_{m,n}$ and $r^* = B(z)\overline{r(\frac{1}{\bar{z}})}$, then*

$$|(r^*(z))'| + |r'(z)| \leq |B'(z)|\ |r(z)|$$

for $z \in T_1$.

Lemma 2.3. *Let z be a complex number. Then*

$$Re(z) \leq \frac{1}{2} \Leftrightarrow |z| \leq |z-1|.$$

Additionally, the statement holds when $\leq$ is changed to $<$ at each occurrence. In this paper, we prove some refinements and generalizations of Theorem 1.2, Theorem 1.4, Theorem 1.5 and Theorem 1.6.

Theorem 2.4. *Let $r(z) = \frac{(z-z_0)^t s(z)}{w(z)} \in R_{m,n}$ with $|z_0| > k$ and $c_1, c_2, \ldots, c_{m-t}$ be the zeros of $s(z)$ all lying in $T_k \cup D_k^-, k \leq 1$. Then we obtain*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| - n + 2\sum_{j=1}^{m-t}\frac{1}{1+|c_j|} + \frac{2t}{1+|z_0|}\right]|r(z)|$$

for $z \in T_1$.
Proof. Suppose $r(z) = \frac{(z-z_0)^t s(z)}{w(z)}$ with $|z_0| > k$ and $c_1, c_2, \ldots, c_{m-t}$ are the zeros of

$s(z)$ all lying in $T_k \cup D_k^-, k \leq 1$. It's easy to see that

$$\frac{zr'(z)}{r(z)} = \frac{zs'(z)}{s(z)} + \frac{tz}{z - z_0} - \frac{zw'(z)}{w(z)}.$$

Since

$$\frac{zs'(z)}{s(z)} = \sum_{j=1}^{m-t} \frac{z}{z - c_j},$$

it follows that

$$Re\left(\frac{zr'(z)}{r(z)}\right) = Re\left(\sum_{j=1}^{m-t} \frac{z}{z - c_j}\right) - Re\left(\frac{zw'(z)}{w(z)}\right) + Re\left(\frac{tz}{z - z_0}\right).$$

Applying Lemma 2.1 provides

$$Re\left(\frac{zr'(z)}{r(z)}\right) \geq \sum_{j=1}^{m-t} \frac{1}{1 + |c_j|} - \left(\frac{n - |B'(z)|}{2}\right) + Re\left(\frac{tz}{z - z_0}\right)$$

for $z \in T_1$. Therefore

$$Re\left(\frac{zr'(z)}{r(z)}\right) \geq \frac{|B'(z)|}{2} + \sum_{j=1}^{m-t} \frac{1}{1 + |c_j|} - \frac{n}{2} + \frac{t}{1 + |z_0|}$$

for $z \in T_1$. Consequently,

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| - n + 2\sum_{j=1}^{m-t} \frac{1}{1 + |c_j|} + \frac{2t}{1 + |z_0|}\right] |r(z)|$$

for $z \in T_1$. The proof is complete.

Remark 2.5. *By letting $t = 0$ in Theorem 2.4, we get that if $r(z) = \frac{s(z)}{w(z)}$ and $c_1, c_2, \ldots, c_m$ are the zeros of $s(z)$ all lying in $T_k \cup D_k^-, k \leq 1$, then we obtain*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| - n + \sum_{j=1}^{m} \frac{2}{1 + |c_j|}\right] |r(z)|$$

for $z \in T_1$. That is, Theorem 2.4 reduces to Theorem 1.6.

Corollary 2.6. *Since $|c_j| \leq k \leq 1$ for $j = 1, 2, 3 \ldots, m-t$, we get*

$$\frac{1}{1+|c_j|} \geq \frac{1}{1+k}.$$

Theorem 2.4 implies that if $r(z) = \frac{(z-z_0)^t s(z)}{w(z)}$ with $|z_0| > k$ and $c_1, c_2, \ldots, c_{m-t}$ are the zeros of $s(z)$ all lying in $T_k \cup D_k^-, k \leq 1$, then

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| + \frac{2(m-t) - n(1+k)}{1+k} + \frac{2t}{1+|z_0|}\right]|r(z)|$$

for $z \in T_1$.

Remark 2.7. *By letting $t = 0$ in Corollary 2.6, we get that if $r(z) = \frac{s(z)}{w(z)}$ and $c_1, c_2, \ldots, c_m$ are the zeros of $s(z)$ all lying in $T_k \cup D_k^-, k \leq 1$, then*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| + \frac{2m - n(1+k)}{1+k}\right]|r(z)|$$

for $z \in T_1$. That is, Corollary 2.6 reduces to Theorem 1.5.
If $k = 1$ in Corollary 2.6, then the result is as follows.

Corollary 2.8. *If $r(z) = \frac{(z-z_0)^t s(z)}{w(z)} \in R_{m,n}$ with $|z_0| > 1$ and $c_1, c_2, \ldots, c_{m-t}$ are the zeros of $s(z)$ lie in $T_1 \cup D_1^-$, then*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| - n + m - t + \frac{2t}{1+|z_0|}\right]|r(z)|$$

for $z \in T_1$.

Remark 2.9. *By letting $t = 0$ in Corollary 2.8, we get that if $r(z) = \frac{s(z)}{w(z)} \in R_{m,n}$ and all of the zeros of $s(z)$ lie in $T_1 \cup D_1^-$, then*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| - n + m\right]|r(z)|$$

for $z \in T_1$. That is, Corollary 2.8 reduces to Theorem 1.4.

Theorem 2.10. *Let $r(z) = \frac{z^t s(z)}{w(z)} \in R_{m,n}$ and $c_1, c_2, \ldots, c_{m-t}$ be the zeros of $s(z)$ all lying in $T_k \cup D_k^+, k \geq 1$ except t-fold at the origin. Then we obtain*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| + \frac{2(m-t) - n(1+k)}{1+k} + 2t\right]|r(z)|$$

for $z \in T_1$.

Proof. Suppose $r(z) = \frac{z^t s(z)}{w(z)} \in R_{m,n}$ and $c_1, c_2, \ldots, c_{m-t}$ are the zeros of $s(z)$ all lying in $T_k \cup D_k^+, k \geq 1$ except t-fold at the origin. Differentiation with respect to z, this yields

$$\frac{zr'(z)}{r(z)} = \frac{zs'(z)}{s(z)} + t - \frac{zw'(z)}{w(z)}. \tag{2.1}$$

We have

$$\frac{zs'(z)}{s(z)} = \sum_{j=1}^{m-t} \frac{z}{z - c_j}.$$

Since $|c_j| \geq k \geq 1$ for $j = 1, 2, 3, \ldots, m-t$, we can see

$$|\frac{z}{z - c_j}| \leq |\frac{z}{z - c_j} - 1|$$

for $z \in T_1$ and $z \neq c_j$. Applying Lemma 2.3, we obtain

$$Re(\frac{z}{z - c_j}) \leq \frac{1}{1 + k}$$

for $j = 1, 2, 3, \ldots, m - t$. From (2.1) and Lemma 2.1, we also provides

$$Re\left(\frac{zr'(z)}{r(z)}\right) \leq \frac{m - t}{1 + k} - \left(\frac{n - |B'(z)|}{2}\right) + t$$

for $z \in T_1$. Moreover, we have ([8], p.529)

$$\begin{aligned} |\frac{z(r^*(z))'}{r(z)}|^2 =& \; |\, |B'(z)| - \frac{zr'(z)}{r(z)}|^2 \\ &= |B'(z)|^2 - 2|B'(z)||\frac{zr'(z)}{r(z)}| + |\frac{zr'(z)}{r(z)}|^2 \\ &\geq |B'(z)|^2 - 2|B'(z)|Re\left(\frac{zr'(z)}{r(z)}\right) + |\frac{zr'(z)}{r(z)}|^2 \\ &\geq |B'(z)|^2 + |\frac{zr'(z)}{r(z)}|^2 - 2|B'(z)|\left[\frac{m - t}{1 + k} - \left(\frac{n - |B'(z)|}{2}\right) + t\right] \end{aligned}$$

for $z \in T_1$. Afterward

$$|(r^*(z))'|^2 \geq |B'(z)|^2|r(z)|^2 + |r'(z)|^2 - 2|B'(z)||r(z)|^2\left[\frac{m - t}{1 + k} - \left(\frac{n - |B'(z)|}{2}\right) + t\right]$$

for $z \in T_1$. That is,

$$\left[|r'(z)|^2 + \left(n - \frac{2(m-t)}{1+k} - 2t\right)|B'(z)||r(z)|^2\right]^{\frac{1}{2}} \leq |(r^*(z))'|$$

for $z \in T_1$. Lemma 2.2 implies that

$$|r'(z)|^2 + \left(n - \frac{2(m-t)}{1+k} - 2t\right)|B'(z)||r(z)|^2 \leq (|B'(z)||r(z)| - |r'(z)|)^2$$

for $z \in T_1$. Continuing simplification, we conclude that

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| + \frac{2(m-t) - n(1+k)}{1+k} + 2t\right]|r(z)|$$

for $z \in T_1$. Consequently, the proof is complete.
If $t = 0$ in Theorem 2.10, then the results is as below.

Corollary 2.11. *If $r(z) \in R_{m,n}$ and the zeros of $r(z)$ lie in $T_k \cup D_k^+, k \geq 1$, then for $z \in T_1$*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| + \frac{2m - n(1+k)}{1+k}\right]|r(z)|.$$

Remark 2.12. *By letting $k = 1$ in Theorem 2.10, we get that if $r(z) = \frac{z^t s(z)}{w(z)} \in R_{m,n}$ and $c_1, c_2, \ldots, c_{m-t}$ are the zeros of s(z) all lying in $T_1 \cup D_1^+$, except t-fold at the origin, then we obtain*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - (n-m-t)\right]|r(z)|$$

for $z \in T_1$. That is, Theorem 2.10 generalize inequality (1.4).

Theorem 2.13. *Let $r(z) = \frac{(z-z_0)^t s(z)}{w(z)} \in R_{m,n}$ with $|z_0| < k$ and $c_1, c_2, \ldots, c_{m-t}$ be the zeros of s(z) all lying in $T_k \cup D_k^+, k \geq 1$. Then*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| + \frac{2(m-t) - n(1+k)}{1+k} + \frac{2t}{1-|z_0|}\right]|r(z)|$$

for $z \in T_1$.

Proof. Suppose $r(z) = \frac{(z-z_0)^t s(z)}{w(z)} \in R_{m,n}$ with $|z_0| < k$ and $c_1, c_2, \ldots, c_{m-t}$ are the zeros of $s(z)$ all lying in $T_k \cup D_k^+, k \geq 1$. Differentiation with respect to z, we get

$$\frac{zr'(z)}{r(z)} = \frac{zs'(z)}{s(z)} - \frac{zw'(z)}{w(z)} + \frac{tz}{z-z_0}.$$

This implies that

$$Re\left(\frac{zr'(z)}{r(z)}\right) = Re\left(\frac{zs'(z)}{s(z)}\right) - Re\left(\frac{zw'(z)}{w(z)}\right) + Re\left(\frac{tz}{z-z_0}\right).$$

We have

$$\frac{zs'(z)}{s(z)} = \sum_{j=1}^{m-t} \frac{z}{z-c_j}.$$

Since $|c_j| \geq k \geq 1$ for $j = 1, 2, 3, \dots, m-t$, we can see

$$|\frac{z}{z-c_j}| \leq |\frac{z}{z-c_j} - 1|$$

for $z \in T_1$ and $z \neq c_j$. By using Lemma 2.3, we get

$$Re(\frac{z}{z-c_j}) \leq \frac{1}{1+k}$$

for $j = 1, 2, 3, \dots, m-t$. Lemma 2.1 also provides

$$\begin{aligned} Re\left(\frac{zr'(z)}{r(z)}\right) &\leq \frac{m-t}{1+k} - \left(\frac{n-|B'(z)|}{2}\right) + Re\left(\frac{tz}{z-z_0}\right) \\ &\leq \frac{m-t}{1+k} - \left(\frac{n-|B'(z)|}{2}\right) + \frac{t}{1-|z_0|} \end{aligned}$$

for $z \in T_1$. We have ([8], p.529)

$$\begin{aligned} |\frac{z(r^*(z))'}{r(z)}|^2 = &\ |\ |B'(z)| - \frac{zr'(z)}{r(z)}|^2 \\ &\geq |B'(z)|^2 - 2|B'(z)|Re\left(\frac{zr'(z)}{r(z)}\right) + |\frac{zr'(z)}{r(z)}|^2 \\ &\geq |B'(z)|^2 + |\frac{zr'(z)}{r(z)}|^2 - 2|B'(z)|\left[\frac{m-t}{1+k} - \left(\frac{n-|B'(z)|}{2}\right) + \frac{t}{1-|z_0|}\right]. \end{aligned}$$

for $z \in T_1$. That is,

$$\left[|r'(z)|^2 + \left(n - \frac{2(m-t)}{1+k} - \frac{2t}{1-|z_0|}\right)|B'(z)||r(z)|^2\right]^{\frac{1}{2}} \leq |(r^*(z))'|$$

for $z \in T_1$. Applying Lemma 2.2, we get

$$|r'(z)|^2 + \left(n - \frac{2(m-t)}{1+k} - \frac{2t}{1-|z_0|}\right)|B'(z)||r(z)|^2 \le (|B'(z)||r(z)| - |r'(z)|)^2$$

for $z \in T_1$. Continuing simplification, we conclude that

$$|r'(z)| \le \frac{1}{2}\left[|B'(z)| + \frac{2(m-t)-n(1+k)}{1+k} + \frac{2t}{1-|z_0|}\right]|r(z)|$$

for $z \in T_1$. The proof is already complete.

Remark 2.14. *By letting $z_0 = 0$ in Theorem 2.13, we get that if $r(z) = \frac{(z)^t s(z)}{w(z)} \in R_{m,n}$ and $c_1, c_2, \ldots, c_{m-t}$ are the zeros of s(z) all lying in $T_k \cup D_k^+, k \ge 1$, then*

$$|r'(z)| \le \frac{1}{2}\left[|B'(z)| + \frac{2(m-t)-n(1+k)}{1+k} + 2t\right]|r(z)|$$

for $z \in T_1$. That is, Theorem 2.13 reduces to Theorem 2.10.
If $k = 1$ in Theorem 2.13, then the result is as follows.

Corollary 2.15. *If $r(z) = \frac{(z-z_0)^t s(z)}{w(z)}$ with $|z_0| < 1$ and $c_1, c_2, \ldots, c_{m-t}$ are the zeros of $s(z)$ all lying in $T_1 \cup D_1^+$, then*

$$|r'(z)| \le \frac{1}{2}\left[|B'(z)| - n + m - t + \frac{2t}{1-|z_0|}\right]|r(z)|$$

for $z \in T_1$.

Remark 2.16. *By letting $t = 0$ and $r(z) \in R_{m,n}$, we get that if $r(z) = \frac{s(z)}{w(z)}$ and $c_1, c_2, \ldots, c_m$ are the zeros of $s(z)$ all lying in $T_1 \cup D_1^+$, then*

$$|r'(z)| \le \frac{1}{2}\left[|B'(z)| - n + m\right]|r(z)|$$

for $z \in T_1$. That is, Corollary 2.15 reduces to Theorem 1.2.

3. Conclusions

In this paper, we generalize well-know inequalities for the modulus of derivative of rational functions. The results are as follows:
1. A lower bound of a modulus of the derivative of rational functions

$$r(z) = \frac{(z-z_0)^t s(z)}{w(z)} \in R_{m,n},$$

where $r(z)$ has the zero z_0 with $|z_0| > k$ and the remaining $m-t$ zeros lie in $T_k \cup D_k^-, k \leq 1$.
2. An upper bound of a modulus of the derivative of rational functions

$$r(z) = \frac{z^t s(z)}{w(z)} \in R_{m,n},$$

where $r(z)$ has exactly n poles and all the zeros of $r(z)$ lie in $T_k \cup D_k^+, k \geq 1$ except the zeros of order t lying in the origin.
3. An upper bound of a modulus of the derivative of rational functions

$$r(z) = \frac{(z-z_0)^t s(z)}{w(z)} \in R_{m,n},$$

where $r(z)$ has the zero z_0 with $|z_0| < k$ and the remaining $m-t$ zeros lie in $T_k \cup D_k^+, k \geq 1$.

4. Future scope of work

Future research could focus on extending known inequalities for the modulus of rational functions to composite functions. This could expand the scope of zeros within a variety of regions of the complex plane.

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References

[1] Aziz A., Shah W. M., Some properties of rational functions with prescribed poles and restricted zeros, Math. Balkanica, 19 (2004), 33-40.

[2] Aziz A., Zargar B. A., Some properties of rational functions with prescribed poles, Canad. Math. Bull., 42 (1999), 417-426.

[3] Ahanger U. M., Shah W. M., inequalities for the Derivatives of Rational Functions with Prescribed Poles and Restricted Zeros, Vestnik St. Petersb. Univ. Math., 56 (2023), 392-402.

[4] Bernstein S., Lecons sur les propriétés extrémales et la meilleure approximation des functions analytique d'une variable réelle, Gauthier Villars, Paris, 1926.

[5] Gulzar M. H., Zargar B. A., Akhter Rubia, Some inequalities for the rational functions with prescribed poles and restricted zeros, J. Anal., 30 (2021), 35-41.

[6] Gupta P., Hans S., Mir A., Generalizations of some inequalities for rational functions with prescribed and restricted zeros, Anal. Math. Phys., 12 (2022), 34.

[7] Lax P. D., Proof of a conjecture of P. Erd*ö*s on the derivative of a polynomial, Bull. Amer. Math. Soc. (N.S.), 50 (1944), 509-513.

[8] Li X., Mohapatra R. N., Rodgriguez R. S., Bernstein-type inequalities for rational functions with prescribed poles, J. Lond. Math. Soc., 51 (1995), 523-531.

[9] Mir M. Y., Wali S. L., Shah W. M., Improvement of some inequalities for rational functions, J. Math. Sci., 271 (2023), 546-554.

[10] Mir M. Y., Wali S. L., Shah W. M., Generalizations of certain well-know inequalities for rational functions, Prol. Anal. Issue Anal., 12 (2023), 25–33.

[11] Turán P., Über die Ableitung von Polynomen, Compos. Math., 7 (1939), 89-95.