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GENERALIZED DEGENERATE CHANGHEE-GENOCCHI NUMBERS AND POLYNOMIALS

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Abstract: The degenerate Changhee-Genocchi numbers (and also Changhee - Genocchi), which appear in analysis and combinatorial mathematics and play a significant role in the applications and theory of mathematics, are associated with the Daehee, Cauchy, and Stirling numbers with several extensions and have proven to be powerful tools in varied subjects in combinatorics and analysis. In combinatorics and analytic number theory, many special numbers such as degenerate Changhee-Genocchi numbers, Changhee-Genocchi numbers, derangement numbers, and Stirling numbers play an important role to solve and analyze problems of combinatorial, analytical, and many other disciplines. In this paper, we introduce generalized degenerate Changhee-Genocchi polynomials and analyze some properties by providing several relations and applications. We first attain diverse relations and formulas covering addition formulas, recurrence rules, implicit summation formulas and relations with the earlier polynomials in the literature. By using their generating function, we derive some new relations including the Stirling numbers of the first and second kinds. Moreover, we introduce generalized higher-order degenerate Changhee-Genocchi polynomials. We also derive some new identities and properties of this type of polynomials.

Keywords and Phrases: Genocchi polynomials and numbers; modified degenerate Changhee-Genocchi polynomials; higher-order modified degenerate Changhee-Genocchi polynomials and numbers.

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1. Introduction

The theory of special polynomials, several mathematicians have extensively studied the works and various generalizations of Bernoulli polynomials, Euler polynomials, Genocchi polynomials, and Cauchy polynomials (see [3, 4, 22-25] for more information). The importance of generalization of the special polynomials encompass a range of specialized polynomial families, offering a unified methodology for addressing a wide array of mathematical questions. They prove valuable not only in theoretical realms, but also in practical applications, enhancing our grasp of fundamental mathematical concepts and furnishing sophisticated resolutions to complex problems in disciplines like calculus, number theory, and physics. Moreover, recent years have witnessed a surge in research on various degenerate versions of special polynomials and numbers, reigniting the interest of mathematicians in diverse categories of special polynomials and numbers [8-12]. Notably, Alatawi and Khan [1, 2] Kim *et al.* [13, 14, 15] as well as Khan and Alatawi [16] revisited the Changhee-Genocchi functions in connection with degenerate functions, building upon the foundational work initiated by Kim *et al.* [17-21, 26-34].

The generalized Euler-Genocchi (EGP) polynomials, for r and m be integers with $r \geq 0$ and $m \geq 0$ as follows (see [5, 6, 7])

$$\sum_{\psi=0}^{\infty} \mathbb{A}_{\psi}^{(r,m)}(\rho)\frac{\varpi^{\psi}}{\psi!} = \left(\frac{2\varpi^{r}}{e^{\varpi}+1}\right)^{m} e^{\rho\varpi}, \mid \varpi \mid < \pi, \tag{1.1}$$

with $\mathbb{A}_{\psi}^{(r,0)}(\rho) = \rho^{\psi}$ for all $r \geq 0$ and $\mathbb{A}_{\psi}^{(r,m)}(\xi) = 0$ for all $\psi < rm$. Also, the corresponding the generalized Euler-Genocchi (EG) numbers of order m are determined by $\mathbb{A}_{\psi}^{(r,m)}(0) = \mathbb{A}_{\psi}^{(r,m)}$.

It is readily that $\mathbb{A}_{\psi}^{(r,1)}(\rho) = \mathbb{A}_{\psi}^{(r)}(\rho)$ and $\mathbb{A}_{0}^{(r,m)}(\rho) = 0$ for $r, m \geq 1$. Furthermore, the corresponding generalized Euler and Genocchi polynomials are determined by $\mathbb{A}_{\psi}^{(0,m)}(\rho) = \mathbb{E}_{\psi}^{(m)}(\rho)$ and $A_{\psi}^{(1,m)}(\rho) = \mathbb{G}_{\psi}^{(m)}(\rho)$ as follows (see [4, 12, 17]):

$$\sum_{\psi=0}^{\infty} \mathbb{E}_{\psi}^{(m)}(\rho)\frac{\varpi^{\psi}}{\psi!} = \left(\frac{2}{e^{\varpi}+1}\right)^{m} e^{\rho\varpi} \quad \mid \varpi \mid < \pi, \tag{1.2}$$

and

$$\sum_{\psi=0}^{\infty} \mathbb{G}_{\psi}^{(m)}(\rho)\frac{\varpi^{\psi}}{\psi!} = \left(\frac{2\omega}{e^{\varpi}+1}\right)^{m} e^{\rho\varpi} \quad \mid \varpi \mid < \pi. \tag{1.3}$$

The Stirling numbers of the first kind are given by (see [1-15])

$$\frac{1}{k!}(\log(1+\omega))^k = \sum_{\upsilon=k}^{\infty} S_1(\upsilon,k)\frac{\omega^{\upsilon}}{\upsilon!} \quad (k \geq 0). \tag{1.4}$$

The Stirling numbers of the second kind are given by (see [16-25])

$$\frac{1}{k!}(e^{\omega}-1)^k = \sum_{\upsilon=k}^{\infty} S_2(\upsilon,k)\frac{\omega^{\upsilon}}{\upsilon!} \quad (k \geq 0). \tag{1.5}$$

The Daehee polynomials are defined by (see [20])

$$\frac{\log(1+\omega)}{\omega}(1+\omega)^{\xi} = \sum_{\upsilon=0}^{\infty} \mathbb{D}_{\upsilon}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{1.6}$$

When $\xi = 0$, $\mathbb{D}_{\upsilon} = \mathbb{D}_{\upsilon}(0)$ are called the Daehee numbers. We find that

$$\mathbb{D}_{\upsilon} = (-1)^{\upsilon}\frac{\upsilon!}{\upsilon+1} \ (\upsilon \in \mathbb{N}_0).$$

The first few are

$$\mathbb{D}_0 = 1, \mathbb{D}_1 = -\frac{1}{2}, \mathbb{D}_2 = \frac{2}{3}, \mathbb{D}_3 = -\frac{3}{2}, \cdots.$$

The Changhee polynomials are defined by (see [15])

$$\frac{2}{2+\omega}(1+\omega)^{\xi} = \sum_{\upsilon=0}^{\infty} Ch_{\upsilon}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{1.7}$$

When $\xi = 0$, $Ch_{\upsilon} = Ch_{\upsilon}(0)$, $(\upsilon \in \mathbb{N}_0)$ are called the Changhee numbers.

The Changhee-Genocchi polynomials are defined by the generating function (see [17])

$$\frac{2\log(1+\omega)}{2+\omega}(1+\omega)^{\xi} = \sum_{\upsilon=0}^{\infty} CG_{\upsilon}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{1.8}$$

When $\xi = 0$, $CG_{\upsilon} = CG_{\upsilon}(0)$ are called the Changhee-Genocchi numbers.

Recently, Kim *et al.* [13] introduced modified Changhee-Genocchi polynomials defined by

$$\frac{2\omega}{2+\omega}(1+\omega)^{\xi}=\sum_{\upsilon=0}^{\infty}CG_{\upsilon}^{*}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{1.9}$$

When $\xi=0$, $CG_{\upsilon}^{*}=CG_{\upsilon}^{*}(0)$ are called the modified Changhee-Genocchi numbers.

The Bernoulli numbers of the second kind are defined by (see [21])

$$\frac{\omega}{\log(1+\omega)}=\sum_{\upsilon=0}^{\infty}b_{\upsilon}\frac{\omega^{\upsilon}}{\upsilon!}\ (\upsilon\in\mathbb{N}_0). \tag{1.10}$$

By (1.10), we see

$$\left(\frac{\omega}{\log(1+\omega)}\right)^{r}(1+\omega)^{\xi-1}=\sum_{\upsilon=0}^{\infty}\mathbb{B}_{\upsilon}^{(\upsilon-r+1)}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}, \tag{1.11}$$

where $\mathbb{B}_{\upsilon}^{(r)}(\xi)$ are the higher-order Bernoulli polynomials defined by

$$\left(\frac{\omega}{e^{\omega}-1}\right)^{r}e^{\xi\omega}=\sum_{\upsilon=0}^{\infty}\mathbb{B}_{\upsilon}^{(r)}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{1.12}$$

For $\xi=1$ and $r=1$ in (1.11) and (1.12), we get

$$b_{\upsilon}=\mathbb{B}_{\upsilon}^{(\upsilon)}(1).$$

The degenerate Changhee-Genocchi polynomials are defined by (see [13])

$$\frac{2\lambda\log\left(1+\frac{1}{\lambda}\log(1+\lambda\omega)\right)}{2\lambda+\log(1+\lambda\omega)}\left(1+\frac{1}{\lambda}\log(1+\lambda\omega)\right)^{\xi}=\sum_{\upsilon=0}^{\infty}CG_{\upsilon,\lambda}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{1.13}$$

By (1.8) and (1.13), we see

$$\lim_{\lambda\to 0}CG_{\upsilon,\lambda}(\xi)=CG_{\upsilon}(\xi)\ (\upsilon\geq 0).$$

The modified degenerate Changhee-Genocchi polynomials are defined by (see [13])

$$\frac{2\lambda\omega}{2\lambda+\log(1+\lambda\omega)}\left(1+\frac{1}{\lambda}\log(1+\lambda\omega)\right)^{\xi}=\sum_{\upsilon=0}^{\infty}CG_{\upsilon,\lambda}^{*}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{1.14}$$

From (1.9) and (1.14), we have

$$\lim_{\lambda\to 0} CG^{*}_{\upsilon,\lambda}(\xi) = CG^{*}_{\upsilon}(\xi)\ (\upsilon \geq 0).$$

Replacing ω by $\frac{1}{\lambda}(e^{\lambda\omega}-1)$ in (1.13), we get

$$\begin{aligned}\frac{2\log(1+\omega)}{2+\omega}(1+\omega)^{\xi} &= \sum_{\sigma=0}^{\infty} CG_{\sigma,\lambda}(\xi)\lambda^{-\sigma}\frac{1}{\sigma!}(e^{\lambda\omega}-1)^{\sigma}\\ &= \sum_{\sigma=0}^{\infty} CG_{\sigma,\lambda}\lambda^{-\sigma}\sum_{\upsilon=\sigma}^{\infty} S_2(\upsilon,\sigma)\lambda^{\upsilon}\frac{\omega^{\upsilon}}{\upsilon!}\\ &= \sum_{\upsilon=0}^{\infty}\left(\sum_{\sigma=0}^{\upsilon} CG_{\sigma,\lambda}\lambda^{\upsilon-\sigma}S_2(\upsilon,\sigma)\right)\frac{\omega^{\upsilon}}{\upsilon!}. \qquad (1.15)\end{aligned}$$

Thus, from (1.8) and (1.15), we get

$$CG_{\upsilon}(\xi) = \sum_{\sigma=0}^{\upsilon} CG_{\sigma,\lambda}\lambda^{\upsilon-\sigma}S_2(\upsilon,\sigma)\ \ (\upsilon \geq 0). \qquad (1.16)$$

The degenerate Changhee polynomials (or λ-Changhee polynomials) are defined by (see [21])

$$\frac{2\lambda}{2\lambda+\log(1+\lambda\omega)}\left(1+\frac{1}{\lambda}\log(1+\lambda\omega)\right)^{\xi} = \sum_{\upsilon=0}^{\infty} Ch_{\upsilon,\lambda}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \qquad (1.17)$$

When $\xi = 0$, $Ch_{\upsilon,\lambda} = Ch_{\upsilon,\lambda}(0)$, $(\upsilon \in \mathbb{N}_0)$ are called the degenerate Changhee numbers.

For $\lambda \in \mathbb{R}$ and $r \in \mathbb{Z}$ with $r \geq 0$, the generalized modified degenerate Changhee-Genocchi polynomials defined by means of the following generating function (see [3])

$$\frac{2(\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{r}}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi} = \sum_{\upsilon=0}^{\infty} CG^{*(r)}_{\upsilon,\lambda,2}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \qquad (1.18)$$

At the point $\xi = 0, CG^{*(r)}_{\upsilon,\lambda,2} = CG^{*(r)}_{\upsilon,\lambda,2}(0)$, $(\upsilon \in \mathbb{N}_0)$ are called the generalized modified degenerate Changhee-Genocchi numbers.

For $r \in \mathbb{Z}$ with $r \geq 0$, the generalized Changhee-Genocchi polynomials are defined by (see [10])

$$\frac{2(\log(1+\varpi))^{r}}{2+\varpi}(1+\varpi)^{\rho} = \sum_{\upsilon=0}^{\infty} C\mathbb{G}^{(r)}_{\upsilon}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \qquad (1.19)$$

Also, the corresponding Changhee-Genocchi (CG) numbers are determined by $C\mathbb{G}_{\upsilon}^{(r)} = C\mathbb{G}_{\upsilon}^{(r)}(0)$.

Inspired by the works of Khan *et al.* [10] and Kim *et al.* [13], in this paper, we define generalized degenerate Changhee-Genocchi numbers and polynomials and investigate some new properties of these numbers and polynomials and derive some new identities and relations between the generalized degenerate Changhee-Genocchi numbers and polynomials. We also derive the generalized higher-order degenerate Changhee-Genocchi polynomials and construct relations between some beautiful special polynomials and numbers.

2. Generalized degenerate Changhee-Genocchi numbers and polynomials

In this section, we introduce generalized degenerate Changhee-Genocchi polynomials and investigate some explicit expressions for generalized degenerate Changhee-Genocchi polynomials and numbers. We begin with the following definition as.

For $\lambda \in \mathbb{R}$ and $r \in \mathbb{Z}$ with $r \geq 0$, we consider the generalized degenerate Changhee-Genocchi polynomials defined by means of the following generating function

$$\frac{2(\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}}))^{r}}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi} = \sum_{\upsilon=0}^{\infty} CG_{\upsilon,\lambda}^{(r)}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{2.1}$$

Note that $CG_{0,\lambda}^{(r)}(\xi) = CG_{1,\lambda}^{(r)}(\xi) = \cdots = CG_{\upsilon-1,\lambda,2}^{(r)}(\xi) = 0$.

At the point $\xi = 0, CG_{\upsilon,\lambda}^{(r)} = CG_{\upsilon,\lambda}^{(r)}(0)$, $(\upsilon \in \mathbb{N}_0)$ are called the generalized degenerate Changhee-Genocchi numbers. Here the function $\log(1+\lambda\omega)^{\frac{1}{\lambda}}$ is called the degenerate function of ω.

$$CG_{\upsilon,\lambda}^{(0)}(\xi) = Ch_{\upsilon,\lambda}(\xi), \quad CG_{\upsilon,\lambda}^{(1)}(\xi) = CG_{\upsilon,\lambda}(\xi) \quad (\omega \geq 0).$$

We note that

$$\sum_{\upsilon=0}^{\infty} \lim_{\lambda\to 0} CG_{\upsilon,\lambda}^{(r)}(\xi)\frac{\omega^{\upsilon}}{\upsilon!} = \lim_{\lambda\to 0}\frac{2(\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}}))^{r}}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi}$$

$$= \frac{2(\log(1+\omega))^{r}}{2+\omega}(1+\omega)^{\xi} = \sum_{\upsilon=0}^{\infty} CG_{\upsilon}^{(r)}(\xi)\frac{\omega^{\upsilon}}{\upsilon!} \text{ (see [7]).} \tag{2.2}$$

From (2.1) and (2.2), we have

$$\lim_{\lambda\to 0} CG_{\upsilon,\lambda}^{(r)}(\xi) = CG_{\upsilon}^{(r)}(\xi) \quad (\upsilon \geq 0).$$

Theorem 2.1. *For $\upsilon \geq 0$, we have*

$$CG_{\upsilon}^{(r)}(\xi) = \sum_{\sigma=0}^{\upsilon} CG_{\sigma,\lambda}^{(r)}(\xi)\lambda^{\upsilon-\sigma} S_2(\upsilon,\sigma). \tag{2.3}$$

Proof. Replacing ω by $\frac{1}{\lambda}(e^{\lambda\omega}-1)$ in (2.1) and using (1.5), we get

$$\frac{2(\log(1+\omega))^r}{2+\omega}(1+\omega)^{\xi} = \sum_{\sigma=0}^{\infty} CG_{\sigma,\lambda}^{(r)}(\xi)\lambda^{-\sigma}\frac{1}{\sigma!}(e^{\lambda\omega}-1)^m$$

$$= \sum_{\sigma=0}^{\infty} CG_{\sigma,\lambda}^{(r)}(\xi)\lambda^{-\sigma}\sum_{\upsilon=\sigma}^{\infty} S_2(\upsilon,\sigma)\lambda^{\upsilon}\frac{\omega^{\upsilon}}{\upsilon!}$$

$$= \sum_{\upsilon=0}^{\infty}\left(\sum_{\sigma=0}^{\upsilon} CG_{\sigma,\lambda}^{(r)}(\xi)\lambda^{\upsilon-\sigma}S_2(\upsilon,\sigma)\right)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{2.4}$$

Therefore, by (2.4), we obtain the result.

Theorem 2.2. *For $\upsilon \geq 0$, we have*

$$CG_{\upsilon,\lambda}^{(r)}(\xi) = \sum_{\sigma=0}^{\upsilon} CG_{\sigma,\lambda,2}^{(r)}(\xi)S_1(\upsilon,\sigma)\lambda^{\upsilon-\sigma}. \tag{2.5}$$

Proof. By using (1.4) and (2.1), we see that

$$\frac{2(\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}}))^r}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi} = \sum_{\sigma=0}^{\infty} CG_{\sigma,\lambda}^{(r)}(\xi)\frac{1}{\sigma!}\left(\frac{1}{\lambda}\log(1+\lambda\omega)\right)^{\sigma}$$

$$= \sum_{\sigma=0}^{\infty} CG_{\sigma,\lambda}^{(r)}(\xi)\sum_{\upsilon=\sigma}^{\infty} S_1(\upsilon,\sigma)\lambda^{\upsilon-\sigma}\frac{\omega^{\upsilon}}{\upsilon!}$$

$$= \sum_{\upsilon=0}^{\infty}\left(\sum_{\sigma=0}^{\upsilon} CG_{\sigma,\lambda}^{(r)}(\xi)S_1(\upsilon,\sigma)\lambda^{\upsilon-\sigma}\right)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{2.6}$$

Therefore, by (2.1) and (2.6), we get the result.

Theorem 2.3. *For $\upsilon \geq 0$, we have*

$$CG_{\upsilon,\lambda}^{(r)}(\xi) = \sum_{\sigma=0}^{\upsilon}\sum_{\rho=0}^{\sigma}\binom{\upsilon}{\sigma}(\xi)_{\rho}\lambda^{\sigma-\rho}S_1(\sigma,\rho)CG_{\upsilon-\sigma,\lambda}^{(r)}. \tag{2.7}$$

Proof. By (1.4) and (2.1), we get

$$\sum_{\upsilon=0}^{\infty} CG^{*(r)}_{\upsilon,\lambda}(\xi)\frac{\omega^\upsilon}{\upsilon!} = \frac{2(\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}}))^r}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi}$$

$$= \sum_{\upsilon=0}^{\infty} CG^{(r)}_{\upsilon,\lambda}\frac{\omega^\upsilon}{\upsilon!}\sum_{\sigma=0}^{\infty}\binom{\xi}{\sigma}\left(\log(1+\lambda\omega)^{\frac{1}{\lambda}}\right)^{\sigma}$$

$$= \sum_{\upsilon=0}^{\infty} CG^{(r)}_{\upsilon,\lambda}\frac{\omega^\upsilon}{\upsilon!}\sum_{\sigma=0}^{\infty}(\xi)_\sigma\frac{1}{\sigma!}\lambda^{-\sigma}(\log(1+\lambda\omega))^{\sigma}$$

$$= \sum_{\upsilon=0}^{\infty} CG^{(r)}_{\upsilon,\lambda}\frac{\omega^\upsilon}{\upsilon!}\sum_{\sigma=0}^{\infty}(\xi)_\sigma\lambda^{-\sigma}\sum_{\sigma=\rho}^{\infty}S_1(\sigma,\rho)\lambda^{\sigma}\frac{\omega^\sigma}{\sigma!}$$

$$= \left(\sum_{\upsilon=0}^{\infty} CG^{(r)}_{\upsilon,\lambda}\frac{\omega^\upsilon}{\upsilon!}\right)\left(\sum_{\sigma=0}^{\infty}\sum_{\rho=0}^{\sigma}(\xi)_\rho\lambda^{\sigma-\rho}S_1(\sigma,\rho)\frac{\omega^\sigma}{\sigma!}\right)$$

$$= \sum_{\upsilon=0}^{\infty}\left(\sum_{\sigma=0}^{\upsilon}\sum_{\rho=0}^{\sigma}\binom{\upsilon}{\sigma}(\xi)_\rho\lambda^{\sigma-\rho}S_1(\sigma,\rho)CG^{(r)}_{\upsilon-\sigma,\lambda}\right)\frac{\omega^\upsilon}{\upsilon!}. \tag{2.8}$$

Therefore, by (2.1) and (2.8), we obtain at the required result.

Theorem 2.4. *For $\upsilon \geq 0$, we have*

$$CG^{(r)}_{\upsilon,\lambda}(\xi) = \sum_{\sigma=0}^{\upsilon} CG^{(r)}_{\sigma}(\xi)\lambda^{\upsilon-\sigma}S_1(\upsilon,\sigma). \tag{2.9}$$

Proof. Replacing ω by $\log(1+\lambda\omega)^{\frac{1}{\lambda}}$ in (2.2) and applying (1.4), we get

$$\frac{2(\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}}))^r}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\omega}\xi = \sum_{\sigma=0}^{\infty} CG^{(r)}_{\sigma}(\xi)\frac{1}{\sigma!}(\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\sigma}$$

$$= \sum_{\sigma=0}^{\infty} CG^{(r)}_{\sigma}(\xi)\lambda^{\upsilon-\sigma}\sum_{\upsilon=\sigma}^{\infty}S_1(\upsilon,\sigma)\frac{\omega^\upsilon}{\upsilon!}$$

$$= \sum_{\upsilon=0}^{\infty}\left(\sum_{\sigma=0}^{\upsilon} CG^{(r)}_{\sigma}(\xi)\lambda^{\upsilon-\sigma}S_1(\upsilon,\sigma)\right)\frac{\omega^\upsilon}{\upsilon!}. \tag{2.10}$$

By using (2.1) and (2.10), we acquire at the desired result.

Theorem 2.5. *For $\upsilon \geq 0$, we have*

$$CG_{\upsilon,\lambda}^{(r)}(\xi) = \sum_{\sigma=0}^{\upsilon} \binom{\upsilon}{\sigma} CG_{\upsilon-\sigma,\lambda}^{(r)}(\xi)\mathbb{D}_{\sigma,\lambda}^{(r)}. \tag{2.11}$$

Proof. From (1.14) and (2.1), we note that

$$\sum_{\upsilon=0}^{\infty} CG_{\upsilon,\lambda}^{(r)}(\xi)\frac{\omega^\upsilon}{\upsilon!} = \frac{2(\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}}))^r}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi}$$

$$= \frac{2\omega^r}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi}\left(\frac{\log(1+\log(1+\lambda\omega)}{\lambda\omega}\right)^r$$

$$= \sum_{\upsilon=0}^{\infty} CG_{\upsilon,\lambda}^{(r)}(\xi)\frac{\omega^\upsilon}{\upsilon!}\sum_{\sigma=0}^{\infty}\mathbb{D}_{\sigma,\lambda}^{(r)}\frac{\omega^\sigma}{\sigma!}$$

$$= \sum_{\upsilon=0}^{\infty}\left(\sum_{\sigma=0}^{\upsilon}\binom{\upsilon}{\sigma}CG_{\upsilon-\sigma,\lambda}^{(r)}(\xi)\mathbb{D}_{\sigma,\lambda}^{(r)}\right)\frac{\omega^\upsilon}{\upsilon!}. \tag{2.12}$$

Therefore, by (2.1) and (2.12), we obtain the result.

Theorem 2.6. *For $\upsilon \geq 0$, we have*

$$\mathbb{A}_{\upsilon}^{(r)}(\xi) = \sum_{l=0}^{k}\sum_{\sigma=0}^{l} CG_{\sigma,\lambda}^{(r)}(\xi)\lambda^{l-\sigma}S_2(l,\sigma)S_2(\upsilon,l). \tag{2.13}$$

Proof. Replacing ω by $e^\omega - 1$ in (2.4) and using equations (1.1), we get

$$\frac{2\omega^r}{e^\omega+1}e^{\xi\omega} = \sum_{l=0}^{\infty}\sum_{\sigma=0}^{l} CG_{\sigma,\lambda}^{(r)}(\xi)\lambda^{l-\sigma}S_2(l,\sigma)\frac{1}{l!}(e^\omega-1)^l$$

$$\frac{2\omega^r}{e^\omega+1}e^{\xi\omega} = \sum_{\upsilon=0}^{\infty}\sum_{l=0}^{k}\sum_{\sigma=0}^{l} CG_{\sigma,\lambda}^{(r)}(\xi)\lambda^{l-\sigma}S_2(l,\sigma)S_2(\upsilon,l)\frac{\omega^\upsilon}{\upsilon!}. \tag{2.14}$$

By (2.14), we get the result.

Theorem 2.7. *For $\upsilon \geq 0$, we have*

$$CG_{\upsilon}^{(r)}(\xi) = \sum_{l=0}^{\upsilon}\sum_{\sigma=0}^{l} CG_{\sigma,\lambda}^{(r)}(\xi)S_1(l,\sigma)\lambda^{\upsilon-\sigma}S_2(\upsilon,l). \tag{2.15}$$

Proof. Replacing ω by $\frac{1}{\lambda}(e^{\lambda\omega}-1)$ in (2.6), we get

$$\frac{2(\log(1+\omega))^r}{2+\omega}(1+\omega)^{\xi}=\sum_{l=0}^{\infty}\sum_{\sigma=0}^{l}CG_{\sigma,\lambda}^{(r)}(\xi)S_1(l,\sigma)\lambda^{l-\sigma}\lambda^{-l}\frac{1}{l!}(e^{\lambda\omega}-1)^l$$

$$=\sum_{l=0}^{\infty}\sum_{\sigma=0}^{l}CG_{\sigma,\lambda}^{(r)}(\xi)S_1(l,\sigma)\lambda^{l-\sigma}\lambda^{-l}\sum_{\upsilon=l}^{\infty}S_2(\upsilon,l)\lambda^{\upsilon}\frac{\omega^{\upsilon}}{\upsilon!}$$

$$=\sum_{\upsilon=0}^{\infty}\left(\sum_{l=0}^{\upsilon}\sum_{\sigma=0}^{l}CG_{\sigma,\lambda}^{(r)}(\xi)S_1(l,m)\lambda^{\upsilon-\sigma}S_2(\upsilon,l)\right)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{2.16}$$

Therefore, by (2.2) and (2.16), we get the result.

Theorem 2.8. *For $\upsilon\geq 0$, we have*

$$CG_{\upsilon,\lambda}^{(r)}(\xi)=\sum_{\sigma=0}^{\upsilon}\binom{\upsilon}{\sigma}CG_{\upsilon-\sigma,\lambda,2}^{*(r)}(\xi)d_{\sigma}^{(r)}\lambda^{\sigma}. \tag{2.17}$$

Proof. From (1.4) and (2.1), we have

$$\sum_{\upsilon=0}^{\infty}CG_{\upsilon,\lambda}^{(r)}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}=\frac{2(\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}}))^r}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi}$$

$$=\frac{2(\log(1+\lambda\omega)^{\frac{1}{\lambda}})^r}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi}\left(\frac{\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})}{\log(1+\lambda\omega)^{\frac{1}{\lambda}}}\right)^r$$

$$=\sum_{\upsilon=0}^{\infty}CG_{\upsilon,\lambda,2}^{*(r)}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}\sum_{\sigma=0}^{\infty}d_{\sigma}^{(r)}\lambda^{\sigma}\frac{\omega^{\sigma}}{\sigma!}$$

$$=\sum_{\upsilon=0}^{\infty}\left(\sum_{\sigma=0}^{\upsilon}\binom{\upsilon}{\sigma}CG_{\upsilon-\sigma,\lambda,2}^{*(r)}(\xi)d_{\sigma}^{(r)}\lambda^{\sigma}\right)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{2.18}$$

Therefore, by (2.18), we obtain the result.

Here, we now consider the generalized higher-order degenerate Changhee-Genocchi polynomials by the following definition.

Let $\alpha\in\mathbb{N}$. We consider the generalized higher-order degenerate Changhee-Genocchi polynomials of the second kind given by the following generating function

$$(\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}}))^r\left(\frac{2}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}\right)^{\alpha}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi}=\sum_{\upsilon=0}^{\infty}CG_{\upsilon,\lambda}^{(\alpha,r)}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}. \tag{2.19}$$

When $\xi = 0, CG_{\upsilon,\lambda}^{(\alpha,r)} = CG_{\upsilon,\lambda}^{(\alpha,r)}(0)$ are called the generalized higher-order degenerate Changhee-Genocchi numbers of the second kind.

It is worth noting that

$$\lim_{\lambda\to 0} CG_{\upsilon,\lambda}^{(\alpha,r)}(\xi) = CG_{\upsilon}^{(\alpha,r)}(\xi)\ (\upsilon \geq 0),$$

are called the generalized higher-order Changhee-Genocchi polynomials.

Theorem 2.9. *For $\upsilon \geq 0$, we have*

$$CG_{\upsilon+r,\lambda}^{(\alpha,r)}(\xi) = \frac{1}{(\upsilon+r)_r}\sum_{\sigma=0}^{\upsilon}\binom{\upsilon}{\sigma}\mathbb{D}_{\sigma,\lambda}^{(r)}Ch_{\upsilon-\sigma,\lambda}^{(\alpha)}(\xi). \tag{2.20}$$

Proof. From (1.6), (1.14) and (2.19), we note that

$$\begin{aligned}\sum_{\upsilon=0}^{\infty} CG_{\upsilon,\lambda,2}^{(\alpha,r)}(\xi)\frac{\omega^{\upsilon}}{\upsilon!} &= \left(\frac{\log(1+\log(1+\lambda\omega))}{\omega}\right)^{r}\left(\frac{2}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}\right)^{\alpha}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi}\\ &= \left(\sum_{\sigma=0}^{\infty}\mathbb{D}_{\sigma,\lambda}^{(r)}\frac{\omega^{\sigma}}{\sigma!}\right)\left(\sum_{\upsilon=0}^{\infty}Ch_{\upsilon,\lambda}^{(\alpha)}(\xi)\frac{\omega^{\upsilon}}{\upsilon!}\right)\\ &= \sum_{\upsilon=0}^{\infty}\left(\sum_{\sigma=0}^{\upsilon}\binom{\upsilon}{\sigma}\mathbb{D}_{\sigma,\lambda}^{(r)}Ch_{\upsilon-\sigma,\lambda}^{(\alpha)}(\xi)\right)\frac{\omega^{\upsilon}}{\upsilon!}.\end{aligned} \tag{2.21}$$

Other hand, we have

$$\sum_{\upsilon=0}^{\infty} CG_{\upsilon,\lambda}^{(\alpha,r)}(\xi)\frac{\omega^{\upsilon-r}}{\upsilon!} = \sum_{\upsilon=0}^{\infty} CG_{\upsilon+r,\lambda}^{(\alpha,r)}(\xi)(\upsilon+r)_r\frac{\omega^{\upsilon}}{\upsilon!}. \tag{2.22}$$

Therefore, by (2.21) and (2.22), we obtain the result.

Theorem 2.10. *For $\alpha, \beta \in \mathbb{N}$, with $\alpha > \beta$ and $\upsilon \geq 0$, we have*

$$CG_{\upsilon,\lambda}^{(\alpha,r)}(\xi) = \sum_{l=0}^{\upsilon}\binom{\upsilon}{l}CG_{l,\lambda}^{(\alpha-\beta,r)}Ch_{\upsilon-l,\lambda}^{(\beta)}(\xi). \tag{2.23}$$

Proof. By (2.19), we see that

$$(\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}}))^{r}\left(\frac{2}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}\right)^{\alpha}(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi}$$

$$= (\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}}))^r \left(\frac{2}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}\right)^{\alpha-\beta} \left(\frac{2}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}\right)^{\beta} (1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi}$$

$$= \left(\sum_{l=0}^{\infty} CG_{l,\lambda}^{(\alpha-\beta,r)} \frac{\omega^l}{l!}\right) \left(\sum_{\sigma=0}^{\infty} Ch_{\sigma,\lambda}^{(\beta)}(\xi) \frac{\omega^\sigma}{\sigma!}\right)$$

$$= \sum_{\upsilon=0}^{\infty} \left(\sum_{l=0}^{\upsilon} \binom{\upsilon}{l} CG_{l,\lambda}^{(\alpha-\beta,r)} Ch_{n-l,\lambda}^{(\beta)}(\xi)\right) \frac{\omega^\upsilon}{\upsilon!}. \tag{2.24}$$

Therefore, by (2.24), we get the result.

Theorem 2.11. *For $\upsilon \geq 0$, we have*

$$CG_{\upsilon,\lambda}^{(\alpha,r)}(\xi+\eta) = \sum_{k=0}^{\upsilon} \sum_{\sigma=0}^{k} \binom{\upsilon}{k} CG_{\upsilon-k,\lambda}^{(\alpha,r)}(\xi)(\eta)_\sigma \lambda^{k-\sigma} S_1(k,\sigma). \tag{2.25}$$

Proof. Now, we observe that

$$\sum_{\upsilon=0}^{\infty} CG_{\upsilon,\lambda}^{(\alpha,r)}(\xi+\eta) \frac{\omega^\upsilon}{\upsilon!} = (\log(1+\log(1+\lambda\omega)^{\frac{1}{\lambda}}))^r \left(\frac{2}{2+\log(1+\lambda\omega)^{\frac{1}{\lambda}}}\right)^{\alpha} (1+\log(1+\lambda\omega)^{\frac{1}{\lambda}})^{\xi+\eta}$$

$$= \left(\sum_{l=0}^{\infty} CG_{l,\lambda}^{(\alpha,r)}(\xi) \frac{\omega^l}{l!}\right) \left(\sum_{\sigma=0}^{\infty} (\eta)_\sigma \lambda^{-\sigma} \frac{(\log(1+\lambda\omega))^\sigma}{\sigma!}\right)$$

$$= \left(\sum_{\upsilon=0}^{\infty} CG_{\upsilon,\lambda}^{(\alpha,r)}(\xi) \frac{\omega^\upsilon}{\upsilon!}\right) \left(\sum_{k=0}^{\infty} \sum_{\sigma=0}^{k} (\eta)_\sigma \lambda^{k-\sigma} S_1(k,\sigma) \frac{\omega^k}{k!}\right)$$

$$= \sum_{\upsilon=0}^{\infty} \left(\sum_{k=0}^{\upsilon} \sum_{\sigma=0}^{k} \binom{\upsilon}{k} CG_{\upsilon-k,\lambda}^{(\alpha,r)}(\xi)(\eta)_\sigma \lambda^{k-\sigma} S_1(k,\sigma)\right) \frac{\omega^\upsilon}{\upsilon!}. \tag{2.26}$$

Equating the coefficients of ω^υ on both sides, we get the result.

3. Conclusion

In the present paper, we have introduced generalized degenerate Changhee-Genocchi polynomials and analyzed some properties and relations by using the generating function. Also, we have acquired several properties and formulas covering addition formulas, recurrence relations, implicit summation formulas, and relations with the earlier polynomials in the literature. Moreover, we have derived the generalized higher-order degenerate Changhee-Genocchi polynomials of the second kind and constructed relations between some special polynomials and numbers. In addition, advancing the purpose of this article, we will proceed with this idea in our next research studies in several directions.

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