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A NOTE ON CERTAIN TYPE OF GENERATING FUNCTIONS

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"(Dedicated to Prof. Oscar Chavoya Aceves (1957 - 2023))"

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Abstract: We show that certain type of generating functions allow to construct Cauchy convolutions for the Apostol-Euler and Apostol-Bernoulli polynomials and their connection with partial Bell Polynomials.

Keywords and Phrases: Apostol-Bernoulli polynomials, Apostol-Euler polynomials, Cauchy convolution, Generating functions, Partial Bell polynomials.

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1. Introduction

The property

$$\sum_{n=0}^{\infty} y^n t^n = \frac{1}{\sum_{m=0}^{\infty} z_m t^m}, \; y_0 = z_0 = 1, \tag{1}$$

is very interesting because automatically it implies the Cauchy convolution [13, 22]:

$$\sum_{k=0}^{n} z_k y_{n-k} = 0, \quad n \geq 1, \tag{2}$$

and the connection between y_j and z_k via the partial Bell polynomials [7, 8, 11, 14, 21, 28, 29]:

$$y_n = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k k! B_{n,k}(1!z_1, 2!z_2, \cdots, (n-k+1)!z_{n-k+1}), \ n \geq 0. \tag{3}$$

For example, for the Bernoulli polynomials [5, 20, 25] we can construct the differences between them and the corresponding Bernoulli numbers:

$$Q_m(y) := B_m(y) - B_m, \quad B_m = B_m(0), \quad m \geq 0, \ Q_0(y) = 0, \tag{4}$$

that is:

$$Q_1(y) = y, \ Q_2(y) = y^2 - y, \ Q_3(y) = y^3 - \frac{3}{2}y^2 + \frac{1}{2}y, \ Q_4(y) = y^4 - 2y^3 + y^2, \tag{5}$$

then it is possible to obtain the properties [9, 19]:

$$\sum_{n=0}^{\infty} \frac{x^{n+1}t^n}{(n+1)!} Q_{n+1}\left(\frac{1}{x}\right) = \frac{1}{\displaystyle\sum_{m=0}^{\infty} \frac{t^m}{(m+1)!x} Q_{m+1}(x)}, \tag{6}$$

$$\sum_{k=0}^{n} \frac{x^k}{(k+1)!(n-k+1)!} Q_{k+1}\left(\frac{1}{x}\right) Q_{n-k+1}(x) = 0, \quad n \geq 1, \tag{7}$$

$$Q_{n+1}\left(\frac{1}{x}\right) = (n+1) \sum_{k=0}^{n} \frac{(-1)^k k!}{x^{n+k+1}} B_{n,k}\left(\frac{1}{2}Q_2(x), \frac{1}{3}Q_3(x), \cdots, \frac{1}{n-k+2}Q_{n-k+2}(x)\right), \quad n \geq 0, \tag{8}$$

in harmony with (1), (2) and (3).

The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ of (real or complex) order α, are usually defined by means of the following generating functions (See [18, 23]):

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!} \quad (|z|<2\pi;\ 1^{\alpha}:=1) \tag{9}$$

and

$$\left(\frac{2}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!} \quad (|z|<\pi;\ 1^{\alpha}:=1), \tag{10}$$

so that, obviously,

$$B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x) \ \ (n\in\mathbb{N}_0), \tag{11}$$

where

$$\mathbb{N}_0 = \mathbb{N}\cup\{0\} \ \ (\mathbb{N}:=\{1,2,3,\cdots\}).$$

For the classical Bernoulli numbers B_n and the classical Euler numbers E_n, we readily find from (11) that

$$B_n := B_n(0) = B_n^{(1)}(0) \quad \text{and} \quad E_n := E_n(0) = E_n^{(1)}(0) \ \ (n\in\mathbb{N}_0). \tag{12}$$

Some interesting analogues of the classical Bernoulli polynomials and numbers were investigated by Apostol [4] and by Srivastava [24].

Motivated by the generalizations in (9) and (10) of the classical Bernoulli polynomials and the classical Euler polynomials involving a real or complex parameter α, the authors [15] introduced and investigated the so-called Apostol–Bernoulli polynomials of order α and the Apostol–Euler polynomials of order α. These polynomials and numbers have numerous important applications in Combinatorics, Number theory and Numerical analysis. They have therefore been studied extensively over the last two decades.

The difference between Apostol-Bernoulli polynomials, Apostol-Euler polynomials and other polynomials are elaborated in [27] (also see in [16, 17]).

In [26], the authors have established some generating functions for the generalized Gauss type hypergeometric type function which was introduced by them. Further, in [1], the authors have established certain generating functions for the generalized Gauss hypergeometric functions.

In [10], the authors discussed a comparative study on generating function relations for generalized hypergeometric functions via generalized fractional operators.

By the motivation of the works reported in [1, 26], in [2], the authors have established certain new integrals involving the generalized Gauss hypergeometric function, generalized confluent hypergeometric function, and the Laguerre Polynomials.

In [3], the authors have introduced a trustworthy method for solving a mathematical physics model of fractional-order, advection–dispersion equation, numerically. This method was based on a class of orthogonal polynomials which is called the shifted Vieta–Lucas polynomials. In [12], the authors considered the shifted Legendre polynomials for constructing the numerical solution for a class of multi-term variable-order fractional differential equations.

This paper is organized as follows. In the Section 2, we consider certain type of generating functions which allow to deduce relations like to (1), which have immediate application to Apostol-Bernoulli and Apostol-Euler polynomials [6, 16, 17].

2. Generating Functions

We put attention to functions with the following structure:

$$R(x,t) = \gamma(t)\frac{f(xt)}{f(t)}, \tag{13}$$

where we can make the changes:

$$x \to \frac{1}{x} \quad \text{and} \quad t \to xt, \tag{14}$$

to obtain that:

$$R(\frac{1}{x}, xt) = \gamma(xt)\frac{f(t)}{f(xt)} = \gamma(xt)\gamma(t)\frac{1}{\gamma(t)\dfrac{f(xt)}{f(t)}} = \frac{\gamma(xt)\gamma(t)}{R(x,t)}. \tag{15}$$

If now we accept that $R(x,t)$ is a generating function:

$$\sum_{n=0}^{\infty} N_n(x)\frac{t^n}{n!} = R(x,t), \tag{16}$$

then (15) gives the opportunity to construct relations type (1) if in (16) we apply the changes (14), in fact:

$$\sum_{n=0}^{\infty} N_n\left(\frac{1}{x}\right)\frac{x^n t^n}{n!} = \frac{\gamma(xt)\gamma(t)}{\sum_{m=0}^{\infty} N_m(x)\frac{t^m}{m!}}. \tag{17}$$

We can consider two special cases:
Case 1. $\gamma(t) = \text{Constant} = a$ with $f(0) \neq 0$
Thus, from (13) and (16):

$$\sum_{n=0}^{\infty} N_n(x)\frac{t^n}{n!} = a\frac{f(xt)}{f(t)} \quad \therefore N_0(x) = a \neq 0, \tag{18}$$

hence (17) implies the expression:

$$\sum_{n=0}^{\infty} \frac{1}{a}N_n\left(\frac{1}{x}\right)\frac{x^n t^n}{n!} = \frac{1}{\sum_{m=0}^{\infty} \frac{1}{a}N_m(x)\frac{t^m}{m!}}, \tag{19}$$

with the structure (1) such that:

$$y_n = \frac{1}{a}N_n\left(\frac{1}{x}\right)\frac{x^n}{n!}, \quad z_m = \frac{1}{a}N_m(x)\frac{1}{m!}, \quad y_0 = z_0 = 1, \tag{20}$$

thus (2) and (3) generate the interesting identities:

$$\sum_{k=0}^{n} \binom{n}{k} N_k(x) N_{n-k}\left(\frac{1}{x}\right) x^{n-k} = 0, \quad n \geq 1, \tag{21}$$

$$N_n\left(\frac{1}{x}\right) = \frac{1}{x^n}\sum_{k=0}^{n} \frac{(-1)^k k!}{a^{k-1}} B_{n,k}\left(N_1(x), N_2(x), \cdots, N_{n-k+1}(x)\right), \ n \geq 0. \tag{22}$$

For the Apostol-Euler polynomials[6, 16, 17] we have the relation:

$$\sum_{n=0}^{\infty} H_n\left(x;\lambda\right)\frac{t^n}{n!} = \frac{2(\lambda e^{xt}+1)}{\lambda e^t + 1}, \tag{23}$$

that is:

$$N_n(x) = H_n(x;\lambda), \quad f(t) = \lambda e^t + 1, \quad a = 2, \tag{24}$$

therefore, from (21) and (22):

$$\sum_{k=0}^{n} \binom{n}{k} H_k(x;\lambda) H_{n-k}\left(\frac{1}{x};\lambda\right) x^{n-k} = 0, \quad n \geq 1, \tag{25}$$

$$H_n\left(\frac{1}{x};\lambda\right) = \frac{1}{x^n} \sum_{k=0}^{n} \frac{(-1)^k k!}{2^{k-1}} B_{n,k}\left(H_1(x;\lambda), H_2(x;\lambda), \cdots, H_{n-k+1}(x;\lambda)\right), \quad n \geq 0. \tag{26}$$

Case 2. $\gamma(t) = t$ with $f(0) \neq 0$
Then, from (13) and (16):

$$\sum_{n=0}^{\infty} N_n(x) \frac{t^n}{n!} = t \frac{f(xt)}{f(t)}, \quad \therefore N_0(x) = 0, N_1(x) = 1, \tag{27}$$

and (17) gives the expression:

$$\sum_{j=0}^{\infty} N_{j+1}\left(\frac{1}{x}\right) \frac{x^j}{(t+1)!} t^j = \frac{1}{\sum_{l=0}^{\infty} N_{l+1}(x) \frac{t^l}{(l+1)!}}, \tag{28}$$

in accordance with (1) such that:

$$y_j = N_{j+1}\left(\frac{1}{x}\right) \frac{x^j}{j+1}, \quad z_l = N_{l+1}(x) \frac{1}{(l+1)!}, \quad y_0 = z_0 = 1, \tag{29}$$

hence from (2) and (3):

$$\sum_{k=0}^{n} \binom{n+2}{k+1} N_{k+1}(x) N_{n-k+1}\left(\frac{1}{x}\right) x^{n-k} = 0, n \geq 1, \tag{30}$$

$$N_{n+1}\left(\frac{1}{x}\right) = \frac{n+1}{x^n} \sum_{k=0}^{n} (-1)^k k! B_{n,k}\left(\frac{1}{2} N_2(x), \frac{1}{3} N_3(x), \cdots, \frac{1}{n-k+2} N_{n-k+2}(x)\right), \quad n \geq 0. \tag{31}$$

For the Apostol-Bernoulli polynomials [6, 16, 17] we have the expression:

$$\sum_{n=0}^{\infty} D_n\,(x;\lambda)\,\frac{t^n}{n!} = \frac{t(\lambda e^{xt}-1)}{\lambda e^t - 1}, \tag{32}$$

that is:

$$N_n(x) = D_n(x;\lambda), \quad f(t) = \lambda e^t - 1, \quad \lambda \neq 1, \tag{33}$$

then (30) and (31) imply the following identities:

$$\sum_{k=0}^{n} \binom{n+2}{k+1} D_{k+1}\,(x;\lambda)\, D_{n-k+1}\left(\frac{1}{x};\lambda\right) x^{n-k} = 0, n \geq 1, \tag{34}$$

$$D_{n+1}\left(\frac{1}{x};\lambda\right) = \\ \frac{n+1}{x^n}\sum_{k=0}^{n}(-1)^k k! B_{n,k}\left(\frac{D_2(x;\lambda)}{2}, \frac{D_3(x;\lambda)}{3}, \cdots, \frac{D_{n-k+2}(x;\lambda)}{n-k+2}\right), \\ n \geq 0. \tag{35}$$

3. Conclusion

In this paper, it is shown that the generating functions of the type $R(x,t) = \gamma(t)\dfrac{f(xt)}{f(t)}$ allow us to construct Cauchy convolutions for the Apostol-Euler and Apostol-Bernoulli polynomials and their connection with partial Bell Polynomials. Furthermore, in the generating functions the series are formal regardless of their convergence; by the way, it is evident that the Bell polynomials are convergent. These works further can be extended to generalizations of the Apostol–Bernoulli and Apostol–Euler polynomials.

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