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# FIXED POINT THEOREMS IN GENERALISED CONE METRIC SPACES

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**Abstract:** Within a new class of recently proposed generalised cone metric space, we present generalised cone metric space and establish various generalisations of fixed point findings for cone version contraction types of Kannan and Chatterjea.

**Keywords and Phrases:** Cone metric space, generalised cone metric space, Cauchy sequence, contraction.

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## 1. Introduction and Definitions

Huang and Zhang [7] proposed the idea of cone metric space. They define cone metric space and order Banach space in lieu of real numbers in this study. The fixed point theorem in cone metric spaces ensures that this map must have a unique fixed point. They also provided an example of a function that is contraction in the cone metric category but not contraction when evaluated across metric spaces.

Later, in cone metric space, Rezapour and Halbarani [13] did not include the normalcy assumption. Subsequently, a number of publications in cone metric space began to surface (see to [3, 4, 8, 12, 14, 15] and their references).

Many authors improved the classical concept of metric space by changing the metric criteria. The Banach contraction principle is an important technique for determining the existence of solutions to mathematical models of real-world situations including functional, differential, integral, matrix, and other forms of equations.

Theorists have long sought to better the contractive condition and the underlying space. The process of developing new fixed point theorems in the complete metric spaces is in progress under various new restrictions. In this regard, we can find very nice results that appeared in [1, 2, 6, 10, 11] and their references.

Let $E$ be a real Banach space. A subset $P$ of $E$ is called a cone if

1. $P$ is closed, non-empty and $P \neq 0$
2. $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$
3. $P \cap (-P) = 0$.

Given a cone $P \subset E$ we define the partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to denote that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in int.P$(interior of $P$).

There are two kinds of cone. They are normal cone and non-normal cones. A cone $P \subset E$ is normal if there is a number $K > 0$ such that for all $x, y \in P$, $0 \leq x \leq y \Rightarrow \| x \| \leq K \| y \|$. In other words if $x_n \leq y_n \leq z_n$ and $lim_{n\to\infty} x_n = lim_{n\to\infty} z_n = x$ imply $lim_{n\to\infty} y_n = x$. Also, a cone $P \subset E$ is regular if every increasing sequence which is bounded above is convergent.

**Definition 1.1.** [7] *Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies the following conditions:*

1. *$0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$.*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*
3. *$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

*Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.*

**Definition 1.2.** [7] *Let $(X, d)$ be a cone metric space(CMS), $x \in X$ and $\{x_n\}_{n\geq1}$ be a sequence in $X$. Then,*

1. *$\{x_n\}_{n\geq1}$ converges to $x$ whenever for every $c \in E$ with $0 << c$ there is a natural number $N$ such that $d(x_n, x) << c$ for all $n \geq N$. It is denoted by $lim_{n\to\infty} x_n = x$ or $x_n \to x$.*
2. *$\{x_n\}_{n\geq1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 << c$ there is a natural number $N$ such that $d(x_n, x_m) << c$ for all $n, m \geq N$*
3. *$(X, d)$ is a complete cone metric space if every Cauchy sequence is convergent.*

**Definition 1.3.** [7] *Let $(X, d)$ be a cone metric space(CMS), $P$ be a normal cone with normal constant $K$ and $\{x_n\}$ be a sequence in $X$. Then, the sequence $\{x_n\}$ converges to $x$ if and only if $d(x_n, x) \to 0$ or $\| d(x_n, x) \| \to 0$ as $n \to \infty$.*

Let us recall that a mapping $T$ on metric space $(X, d)$ is called a Kannan [9] contraction if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty)$$

for all $x, y \in X$.

In the year 2006, Huang and Zhang [7] proved cone version of Kannan [9] contraction as:

**Theorem 1.4.** [7] *Let $(X, d)$ be a complete cone metric space and $P$ a normal cone with normal constant $K$. Suppose a mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)], \quad for \quad all \quad x, y \in X, \tag{1}$$

*where $k \in [0, \frac{1}{2})$ is a constant. Then $T$ has a unique fixed point in $X$. And for any $x \in X$, iterative sequence, $\{T^n x\}$ converges to the fixed point.*

In 1972, Chatterjea [5] obtained a similar result by considering a constant $\lambda \in [0, \frac{1}{2})$ and a mapping $T : X \to X$ such that

$$d(Tx, Ty) \leq \lambda[d(x, Ty) + d(y, Tx)].$$

Also, in the year 2006, Huang and Zhang [7] proved cone version of Chatterjea contraction as:

**Theorem 1.5.** [7] *Let $(X, d)$ be a complete cone metric space and $P$ a normal cone with normal constant $K$. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq k[d(Tx, y) + d(Ty, x)], \quad for \quad all \quad x, y \in X, \tag{2}$$

*where $k \in [0, \frac{1}{2})$ is a constant. Then $T$ has a unique fixed point in $X$. And for any $x \in X$, iterative sequence, $\{T^n x\}$ converges to the fixed point.*

## 2. Preliminary Theorems

Let $X$ be a nonempty set and let $D : X \times X \to E$ be a given mapping. For every $x \in X$, let us define the set

$$C(D, X, x) = \Big\{\{x_n\} \subset X : \lim_{n \to \infty} D(x_n, x) = 0\Big\}$$

**Definition 2.1.** *$D$ is called a generalised cone metric on $X$ if it satisfies the following conditions:*

1. *For every* $(x, y) \in X \times X$, *we have* $D(x, y) = 0 \Rightarrow x = y$.
2. *For all* $(x, y) \in X \times X$, *we have* $D(x, y) = D(y, x)$.
3. *There exixts a real constant* $C > 0$ *such that, for all* $(x, y) in X \times X$ *and* $\{x_n\} \in C(D, X, x)$, *we have* $D(x, y) \leq C \lim_{n\to\infty} sup D(x_n, y)$.

*The pair* $(X, D)$ *is called a generalised cone metric space.*

**Remark 2.2.** *If the set* $C(D, X, x)$ *is empty for every* $x \in X$, *then* $(X, D)$ *is a generalised cone metric space if and only if 1 and 2 are satisfied.*

**Definition 2.3.** *Let* $(X, D)$ *be a generalised cone metric space, let* $\{x_n\}$ *be a sequence in* $X$ *and let* $x \in X$. *We say that* $\{x_n\}$ *is* $D-$*converges to* $x$ *in* $X$ *if* $\{x_n\} \in C(D, X, x)$.

**Remark 2.4.** *Let* $\{x_n\}$ *be the sequence defined by* $x_n = x$ *for all* $n \in N$. *If it* $D$*-converges to* $x$, *then* $D(x, x) = 0$.

**Definition 2.5.** *Let* $(X, D)$ *be a generalised cone metric space. A sequence* $\{x_n\}$ *in* $X$ *is called a* $D$*-Cauchy sequence if*

$$\lim_{m,n\to\infty} D(x_n, x_m, x) = 0.$$

*The space* $(X, D)$ *is said to be* $D-$*complete if every* $D$*-Cauchy sequence in* $X$ *is* $D-$*convergent to some element in* $X$.

**Definition 2.6.** *Let* $(X, D)$ *be a generalised cone metric space and let* $\{x_n\}$ *be a sequence in* $X$. *We say that* $\{x_n\}$ *is a* $D-$*Cauchy sequence in if*

$$\lim_{m,n\to\infty} D(x_n, x_m, x) = 0.$$

**Proposition 2.7.** $C(D, X, x)$ *is a non-empty set if and only if* $D(x, x) = 0$.

**Proof.** If $C(D, X, x) \neq \phi$, then there exists a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} D(x_n, x) = 0$. Using property 3, we obtain

$$D(x, x) \leq C \lim_{n\to\infty} sup D(x_n, x),$$

and thus $D(x, x) = 0$. Assume that $D(x, x) = 0$. The sequence $\{x_n\} \subset X$ defined by $x_n = x$ for all $n \in N$ $D$-converges to $x$, which ends the proof.

The focus of this work is on the cone version contraction types of Kannan and Chatterjea. We prove certain fixed point findings in the recently released generalised cone metric spaces. In order to demonstrate the usefulness of the outcomes, we also provide a few instances.

## 3. Main Theorems

**Proposition 3.1.** *Let $(X, D)$ be a generalised cone metric space and let $T : X \to X$ be a mapping satisfying inequality (1) for some $\lambda \in [0, \frac{1}{2})$. Then any fixed point $u \in X$ of $T$* satisfies

$$D(u, u) < \infty \Rightarrow D(u, u) = 0.$$

**Proof.** Let $u \in X$ be a fixed point of $T$ such that $D(u, u) < \infty$. Using (1), we obtain

$$\begin{aligned} D(u, u) &= D(Tu, Tu) \\ &= \lambda(D(u, Tu) + D(u, Tu)) \\ &= 2\lambda D(u, u). \end{aligned}$$

Since $2\lambda \in [0, 1)$, we obtain $D(u, u) = 0$. For every $x \in X$, we define

$$\delta(D, T, x) = sup\{D(T^i x, T^j x) : i, j \in N\}.$$

**Theorem 3.2.** *Let $(X, D)$ be a $D-$complete generalised cone metric space and let $T$ be a self-mapping on $X$ satisfying (1) for some constant $\lambda \in [0, \frac{1}{2})$ such that $C\lambda < 1$.*

*If there exists an element $x_0 \in X$ such that $\delta(D, T, x_0) < \infty$, then the sequence $\{T^n\}$ D-converges to some $u \in X$. Moreover, if $D(u, Tu) < \infty$, then $u$ is a fixed point of $T$. Moreover, for each fixed point $u'$ of $T$ in $X$ such that $D(u', u') < \infty$, we have $u = u'$.*

**Proof.** Let $n \in N (n \geq 1)$. For all $i, j \in N$, we have

$$D(T^{n+i}x_0, T^{n+j}x_0) \leq \lambda[D(T^{n+i-1})x_0, T^{n+i}x_0 + D(T^{n+j-1})x_0, T^{n+j}x_0]$$

and then

$$D(T^{n+i}x_0, T^{n+j}x_0) \leq 2\lambda\delta(D, T, T^{n-1}x_0),$$

which gives

$$\delta(D, T, T^n x_0) \leq 2\lambda\delta(D, T, T^{n-1}x_0).$$

Consequently, we obtain

$$\delta(D, T, T^n x_0) \leq (2\lambda)^n \delta(D, T, x_0)$$

and

$$D(T^n x_0, T^m x_0) \leq \delta(D, T, T^n x_0) \leq (2\lambda)^n \delta(D, T, x_0) \tag{3}$$

for all integers $m$ such that $m > n$. Since $\delta(D, T, x_0) < \infty$ and $2\lambda \in [0, 1)$, we obtain

$$\lim_{m,n\to\infty} D(T^n x_0, T^m x_0) = 0.$$

It follows that $\{T^n x_0\}$ is a D-Cauchy sequence and thus there exists $u \in X$ such that

$$\lim_{n\to\infty} D(T^n x_0, u) = 0$$

and

$$D(Tu, u) \leq C \lim_{n\to\infty} sup D(Tu, T^{n+1} x_0. \tag{4}$$

By (1), we have

$$D(T^{n+1}x_0, Tu) \leq \lambda(D(T^{n+1}x_0, T^n x_0) + D(u, fu)). \tag{5}$$

By (3) and (5) we obtain

$$\lim_{n\to\infty} D(Tw, T^{n+1}x_0) \leq \lambda D(u, Tu).$$

Using (4), we obtain

$$D(u, Tu) \leq C\lambda D(u, Tu).$$

Since $C\lambda < 1$ and $D(u, Tu) < \infty$, we deduce that $D(u, Tu) = 0$, which implies that $Tu = u$. If $u^{'}$ is any fixed point of $T$ such that $D(u^{'}, u^{'}) < \infty$, we obtain

$$\begin{aligned} D(u, u^{'}) &= D(Tu, Tu^{'}) \\ &\leq \lambda(D(Tu, u) + D(Tu^{'}, u^{'})) \\ &\leq \lambda(D(u, u) + D(u^{'}, u^{'})) \\ &\leq 0 \end{aligned}$$

which implies $u^{'} = u$.

**Example 3.3.** Let $X = [0, 1], E = [0, \infty)$ and let $D : X \times X \to E$ be the mapping defined by

$$\begin{aligned} D(x, y) &= \{x + y \qquad if \quad x \neq 0 \quad and \quad y \neq 0 \\ D(0, x) = D(x, 0) &= \frac{x}{2} \qquad for \;\; all \quad x \in X. \end{aligned}$$

conditions 1 and 2 are trivially satisfied. By Proposition 2.7 we need to verify condition 3 only for elements $x$ of $X$ such that $D(x, x) = 0$, which implies that $x = 0$.

Let $\{x_n\} \subset X$ be a sequence such that $\lim_{n\to\infty} D(x_n, 0) = 0$. For all $n \in N$ and $y \in X$, we have:

$$\begin{aligned} D(x_n, y) &= x_n + y \quad if \quad x_n \neq 0 \\ D(x_n, y) &= \frac{y}{2} \quad if \quad x_n = 0. \end{aligned}$$

Then

$$\frac{y}{2} \leq D(x_n, y),$$

which implies that

$$D(0, y) = \frac{y}{2} \leq \lim_{n\to\infty} sup D(x_n, y).$$

It follows that $(X, D)$ is a generalised cone metric space that is not a standard metric space since the triangular inequality does not hold: If $x, y \in X - \{0\}$, then we have $D(x, y) = x + y$ and $D(x, 0) + D(0, y) = \frac{x+y}{2}$, and thus

$$D(x, y) > D(x, 0) + D(0, y).$$

Note that $(X, D)$ is D-complete. Define the mapping $f$ on $X$ by

$$f(x) = \frac{x}{x+2} \quad for \quad all \quad x \in X.$$

For any $x \in X$, we have:

$$D(f(x), f(0)) = D(\frac{x}{x+2}, 0) = \frac{x}{2(x+2)}$$

and

$$D(f(x), x) + D(0, T(0)) = D(\frac{x}{x+2}, x) + D(0, 0) = \frac{x}{x+2} + x.$$

Then

$$D(f(x), f(0)) \leq \frac{1}{3}(D(f(x), x) + D(0, f(0))).$$

For $x, y \in X - \{0\}$, we have

$$D(f(x), f(y)) = D(\frac{x}{x+2}, \frac{y}{y+2}) = \frac{x}{2(x+2)} + \frac{y}{2(y+2)}$$

and

$$D(f(x), x) + D(y, f(y)) = D(\frac{x}{x+2}, x) + D(y, \frac{y}{y+2}) = \frac{x}{x+2} + \frac{y}{y+2} + x + y.$$

Then

$$D(f(x), f(y)) \leq \frac{1}{3}[D(f(x), x) + D(y, f(y))].$$

The hypothesis of Theorem 3.2 are satisfied. Therefore $f$ has a unique fixed point since $D$ is bounded and $f(0) = 0$.

**Lemma 3.4.** *Let $\lambda$ be a real number such that $0 \leq \lambda < 1$ and let $\{b_n\}$ be a sequence of positives real numbers such that $\lim_{n\to\infty} b_n = 0$. Then, for any sequence of positives numbers $\{a_n\}$ satisfying*

$$a_{n+1} \leq \lambda a_n + b_n \quad for \;\; all \;\; n \in N,$$

we have

$$\lim_{n\to\infty} a_n = 0$$

**Theorem 3.5.** *Let $(X, D)$ be a $D-$complete generalised cone metric space, $\lambda \in [0, \frac{1}{2})$ and let $T$ a self-mapping on $X$ such that*

$$D(Tx, Ty) \leq \lambda(D(y, Tx) + D(x, Ty)) \tag{6}$$

*for all $x, y \in X$. If there exists a point $x_0 \in X$ such that $\delta(D, T, x_0) < \infty$, then the sequence $\{T^n x_0\}$ $D$-converges to $u \in X$. Moreover, if $D(x_0, Tu) < \infty$, then $u$ is a fixed point of $T$, and for any fixed point $u'$ of $T$ such that $D(u, u') < \infty$, we have $u = u'$.*

**Proof.** Let $n \in N(n \geq 1)$. For all integers $i, j$, we have

$$D(T^{n+i}x_0, T^{n+j}x_0) \leq \lambda[D(T^{n+i}x_0, T^{n+j-1}x_0) + D(T^{n+i-1}x_0, T^{n+j}x_0)]$$

which implies that

$$D(T^{n+i}x_0, T^{n+j}x_0) \leq 2\lambda\delta(D, T, T^{n-1}x_0).$$

Hence

$$\delta(D, T, T^n x_0) \leq 2\lambda\delta(D, T, T^{n-1}x_0)$$

and consequently

$$\delta(D, T, T^n x_0) \leq (2\lambda)^n \delta(D, T, x_0).$$

This inequality implies that

$$D(T^n x_0, T^m x_0) \leq \delta(D, T, T^n x_0) \leq (2\lambda)^n \delta(D, T, x_0)$$

for all integers $n, m$ such that $m > n$. Since $\delta(D, T, x_0) < \infty$ and $2\lambda \in [0, 1)$, we obtain

$$\lim_{m,n\to\infty} D(T^n x_0, T^m x_0) = 0.$$

It follows that $\{T^n x_0\}$ is a D-Cauchy sequence and thus there exists $u \in X$ such that

$$\lim_{n\to\infty} D(T^n x_0, u) = 0.$$

By (3) we have

$$D(T^n x_0, u) \leq C \lim_{n\to\infty} sup D(T^n x_0, T^m x_0) \leq (2\lambda)^n C\delta(D, T, x_0) \leq C\delta(D, T, x_0).$$

Then

$$D(T^n x_0, u) < \infty \quad for \ \ all \ \ n \in \mathbb{N}.$$

By (6) we have

$$D(T^{n+1} x_0, Tu) \leq \lambda[D(T^{n+1} x_0, u) + D(T^n x_0, Tu)].$$

Since $D(x_0, Tu) < \infty$, we have $D(T^n x_0, Tu) < \infty$ for all $n \in N$. By Lemma 3.4, we obtain

$$\lim_{n\to\infty} D(T^n x_0, Tu) = 0.$$

It follows that $Tu = u$.

Let $u^{'}$ be any fixed point of $X$. Then we have

$$\begin{aligned} D(u, u^{'}) &= D(Tu, Tu^{'}) \\ &\leq \lambda(D(Tu, u^{'}) + D(Tu^{'}, u)) \\ &\leq \lambda(D(u, u^{'}) + D(u^{'}, u)) \\ &\leq 2\lambda D(u, u^{'}). \end{aligned}$$

Since $D(u, u^{'}) < \infty$, we obtain $D(u, u^{'}) = 0$ which ends the proof.

**Example 3.6.** Let $X = [0, 1], E = [0, \infty)$ be defined by

$$D(x, 1) = D(1, x) = \infty \quad for \ \ all \ \ x \in [0, 1]$$

$$D(x, y) = x + y \quad if \ \ x \neq 1 \ \ and \ \ y \neq 1.$$

It is easy to see that $(X, D)$ is a D-complete generalised come metric space with $C = 1$.

Consider the function $f : [0,1] \to [0,1]$ given by

$$f(x) = \frac{1}{2}x \quad if \quad x \in [0,1[,$$

$$f(1) = 1.$$

The function $f$ satisfies (6) with $\lambda = \frac{1}{3}$ in $(X, D)$. By theorem 3.5 $f$ has a fixed point.

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