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FRACTIONAL CALCULUS AND FRACTIONAL KINETIC EQUATION INVOLVING EXTENDED HYPERGEOMETRIC FUNCTION

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Abstract: In this article, generalized pathway integral operator with the classical Gauss hypergeometric function kernel and the fractional differential operators are used to studied new extended hypergeometric function. Furthermore, extended fractional kinetic equation involving new extended hypergeometric function that contained two Fox-Wright function as regularizer is investigated using Sumudu transform and Wiman's function.

Keywords and Phrases: Beta function, Fox-Wright function, Wiman function, Sumudu transform.

2020 Mathematics Subject Classification: 33B15, 33B20, 33C15.

1. Introduction

Special functions of fractional calculus such as gamma, beta, zeta, hypergeometric and Mittag-Leffler functions are plays a very important role in various fields such as physics, statistics, black hole theory, string theory and so on. Many researchers have studied these functions, the interested readers may refer to some recent articles on topics Abubakar and Patel [1], Abubakar [2], Ata and Kıymaz [3], [4] and Mobeen et al. [11]. The following new extended beta function with bi-Fox-Wright function kernel was studied by Kaurangini et al. [7]:

$$\begin{aligned} {}^{\Psi}B_{p,q}^{\alpha,\beta}(x,y) &= {}^{\Psi}B_{p,q}^{\alpha,\beta}\left[\begin{array}{c|c|c} (a_i,A_i)_{1,\xi} & (c_k,C_k)_{1,\kappa} & \\ & & x,y \\ (b_j,B_j)_{1,\eta} & (d_l,D_l)_{1,\upsilon} & \end{array}\right] \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t^{\alpha}}\right){}_{\kappa}\Psi_{\upsilon}\left(-\frac{q}{(1-t)^{\beta}}\right)dt, \end{aligned}$$

where $\min\{\Re(\alpha),\Re(\beta)\}>0$, $\min\{\Re(p),\Re(q)\}>0$, $\min\{\Re(x),\Re(y)\}>0$. Also here the ${}_{\xi}\Psi_{\eta}(\cdot)$ is the generalized Wright function defined for $z,a_i,b_j\in\mathbb{C}$ and $A_i,B_j\in\mathbb{R}$ for all $i=1,2,\ldots,\xi;\ j=1,2,\ldots,\eta$ by the series in [8]

$${}_{\xi}\Psi_{\eta}(z) = {}_{\xi}\Psi_{\eta}\left[\begin{array}{c|c} (a_i,A_i)_{1,\xi} & \\ & z \\ (b_j,B_j)_{1,\eta} & \end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{\xi}\Gamma(a_i+nA_i)}{\prod_{j=1}^{\eta}\Gamma(b_j+nB_j)}\frac{z^n}{n!}. \tag{1}$$

They also (Kaurangini et al. [7]) introduced the following new extended Gauss and confluent hypergeometric functions for $\Re(\nu)>\Re(\mu)>0$ respectively:

$$\begin{aligned} {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;z) &= {}^{\Psi}F_{p,q}^{\alpha,\beta}\left[\begin{array}{c|c|c} (a_i,A_i)_{1,\xi} & (c_k,C_k)_{1,\kappa} & \\ & & \lambda,\mu;\nu;z \\ (b_j,B_j)_{1,\eta} & (d_l,D_l)_{1,\upsilon} & \end{array}\right] \\ &= \sum_{n=0}^{\infty}(\lambda)_n\frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^n}{n!} \qquad (|z|<1) \end{aligned} \tag{2}$$

and

$$\begin{aligned} {}^{\Psi}\Phi_{p,q}^{\alpha,\beta}(\mu;\nu;z) &= {}^{\Psi}\Phi_{p,q}^{\alpha,\beta}\left[\begin{array}{c|c|c} (a_i,A_i)_{1,\xi} & (c_k,C_k)_{1,\kappa} & \\ & & \mu;\nu;z \\ (b_j,B_j)_{1,\eta} & (d_l,D_l)_{1,\upsilon} & \end{array}\right] \\ &= \sum_{n=0}^{\infty}\frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^n}{n!}. \end{aligned} \tag{3}$$

The Merichev-Saigo-Maeda fractional integral operators are a type of generalized fractional integral operator that use Horn's function as their kernel. They were first introduced by Merichev in 1974 [10] and later studied in 1998 by Saigo and Maeda [16]. The operators are defined as follows:

$$\left(\mathcal{I}_{0,x}^{\eta,\eta',\rho,\rho',\tau} f\right)(x) = \frac{x^{-\eta}}{\Gamma(\tau)} \int_0^x t^{-\eta'}(x-t)^{\tau-1} F_3\left(\eta,\eta',\rho,\rho',\tau; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt \quad (4)$$

and

$$\left(\mathcal{I}_{x,\infty}^{\eta,\eta',\rho,\rho',\tau} f\right)(x) = \frac{x^{-\eta'}}{\Gamma(\tau)} \int_x^\infty t^{-\eta}(t-x)^{\tau-1} F_3\left(\eta,\eta',\rho,\rho',\tau; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt, \quad (5)$$

where $\eta, \eta', \rho, \rho', \tau \in \mathbb{C}$, $x \in \mathbb{R}^+$ and $\Re(\tau) > 0$. Also here the $F_3(\cdot)$ is the Horn's function defined in Rainville [15] as

$$F_3(\eta,\eta',\rho,\rho',\tau; y, z) = \sum_{n,r=0}^{\infty} \frac{(\eta)_n(\eta')_r(\rho)_n(\rho')_r}{(\tau)_{n+r}} \frac{y^n}{n!} \frac{z^r}{r!} \qquad (\max\{|y|,|z|\} < 1).$$

The Merichev-Saigo-Maeda fractional differential operators are defined by using integral formulas in (4) and (5) as follows:

$$\left(\mathcal{D}_{0,x}^{\eta,\eta',\rho,\rho',\tau} f\right)(x) = \left(\frac{d}{dx}\right)^{\Delta} \left(\mathcal{I}_{0,x}^{-\eta',-\eta,-\rho'+\Delta,-\rho,-\tau+\Delta} f\right)(x)$$

and

$$\left(\mathcal{D}_{x,\infty}^{\eta,\eta',\rho,\rho',\tau} f\right)(x) = \left(-\frac{d}{dx}\right)^{\Delta} \left(\mathcal{I}_{x,\infty}^{-\eta',-\eta,-\rho',-\rho+\Delta,-\tau+\Delta} f\right)(x),$$

where $\Delta = \Re(\tau) + 1$.

In this article, we will explore other properties and applications of the generalized Gauss and confluent hypergeometric functions in (2) and (3) by using the generalized pathway fractional integral operator, fractional differentiation and fractional kinetic equation. This article is organized as follows: In Section 2, the basic definitions are given. In Section 3, the generalized pathway fractional integral operator involving new generalized hypergeometric function is investigated. In Section 4, fractional differential operators of the new generalized hypergeometric function is studied. In Section 5, covers the applications of the new generalized fractional kinetic equation. In Section 6, conclusion is discussed.

2. Preliminaries

In this section, we give the basic definitions needed throughout this article.

Definition 2.1. ([14]) *Pohlen defined the Hadamard convolution (product) for the two power series* $h(z) = a_n z^n$ $(|z| < R_h)$ *and* $k(z) = b_n z^n$ $(|z| < R_k)$, *where* R_h *and* R_k *are the radii of convergence is given by*

$$(h * k)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (k * h)(z) \quad (R_h R_k \leq R). \tag{6}$$

Definition 2.2. ([20]) *The Wiman's function is defined by*

$$E_{\vartheta,\varsigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\vartheta + \varsigma)} \qquad (\vartheta, \varsigma \in \mathbb{C};\ \Re(\vartheta) > 0, \Re(\varsigma) > 0). \tag{7}$$

Definition 2.3. ([19]) *The Sumudu transform is defined by*

$$\mathcal{S}\{f(t); s\} = \int_0^{\infty} f(st)\exp(-t)dt = \mathcal{G}(s) \tag{8}$$

and

$$\mathcal{S}^{-1}\{s^{\tau}; t\} = \frac{t^{\tau-1}}{\Gamma(\tau)} \quad (\Re(\tau) > 0). \tag{9}$$

The Sumudu transform of the Riemann-Liouville fractional derivative operator is also given in [18] *as*

$$\mathcal{S}\left\{{}_0D_t^{-\tau} f(t); s\right\} = s^{\tau}\mathcal{G}(s). \tag{10}$$

3. Generalized Pathway Fractional Integral

Definition 3.1. ([12], [13]) *If* $f(x) \in \mathbb{R}^+$ *and let* $\gamma, \delta, \epsilon \in \mathbb{C}$, $x \in \mathbb{R}^+$, *such that* $a > 0$, $\Re\left(1 + \frac{\gamma-1}{1-\zeta}\right) > 0$, $\Re(\epsilon - \delta) > 0$, *then for the pathway parameter* $\zeta < 1$, *the generalized pathway fractional integral operator is defined by*

$$\left(P_{0^+}^{\gamma,\delta,\epsilon,\zeta;a} f\right)(x) = x^{-\delta} \int_0^b \left(1 - \frac{a(1-\zeta)t}{x}\right)^{\frac{\gamma-1}{1-\zeta}} \times {}_2F_1\left(\frac{\gamma-1}{1-\zeta} + \delta + 1, -\epsilon; \frac{\gamma-1}{1-\zeta} + 1; 1 - \frac{a(1-\zeta)t}{x}\right) f(t)dt, \tag{11}$$

where $b = \frac{x}{a(1-\zeta)}$.

Remark 3.1. ([13]) *The fractional integral operator in the equation* (11) *also allows for the following deductions.*

1. *If* $\Re(\gamma) > 0$ *when* $a = 1$ *and* $\zeta = 0$, *the equation* (11) *reduces to the Saigo fractional integral operator in the following:*

$$\left(P_{0^+}^{\gamma,\delta,\epsilon,0;a} f\right)(x) = x\Gamma(\gamma)\left(I_{0^+}^{\gamma,\delta,\epsilon} f\right)(x).$$

2. *If* $\Re(\gamma) > 0$ *when* $a = 1$, $\zeta = 0$, $\delta = -\gamma$, *then the equation* (11) *reduces to the Riemann-Liouville fractional integral operator below:*

$$\left(P_{0^+}^{\gamma,-\gamma,\epsilon,0;1} f\right)(x) = x\Gamma(\gamma)\left(I_{0^+}^{\gamma} f\right)(x).$$

3. *If* $\Re(\gamma) > 0$ *when* $a = 1$, $\zeta = 0$, $\delta = 0$, *then the equation* (11) *reduces to the Erdelyi-Kober fractional integral operator given by:*

$$\left(P_{0^+}^{\gamma,0,\epsilon,0;1} f\right)(x) = x\Gamma(\gamma)\left(I_{\gamma,\epsilon}^{+} f\right)(x).$$

4. *If* $\Re(\gamma - 1) > 0$ *and putting* $\epsilon = 0$, *then the equation* (11) *reduces to the classical fractional integral operator given as:*

$$\left(P_{0^+}^{\gamma,\delta,0,\zeta;a} f\right)(x) = x^{1-\gamma-\delta}\left(P_{0^+}^{\gamma-1,\zeta;a} f\right)(x).$$

5. *If* $\Re(\gamma - 1) > 0$ *and substituting* $\epsilon = 0$ *and* $\zeta \to 1_-$, *then the equation* (11) *reduces to the classical pathway fractional integral operator given as:*

$$\lim_{\zeta \to 1_-}\left(P_{0^+}^{\gamma,\delta,0,\zeta;a} f\right)(x) = x^{-\delta}\mathcal{L}\left\{f(t); \frac{a(\gamma-1)}{x}\right\}.$$

Lemma 3.1. ([13]) *The pathway factional integral operator of power function is*

$$\left(P_{0^+}^{\gamma,\delta,\epsilon,\zeta;a} t^{\aleph-1}\right)(x) = \frac{x^{\aleph-\delta}}{[a(1-\zeta)]^{\aleph}} \frac{\Gamma(\aleph)\Gamma(\aleph-\delta+\epsilon)\Gamma\left(1+\frac{\gamma-1}{1-\zeta}\right)}{\Gamma(\aleph-\delta)\Gamma\left(1+\aleph+\epsilon+\frac{\gamma-1}{1-\zeta}\right)}, \tag{12}$$

where $x \in \mathbb{R}^+$, $\gamma, \delta, \epsilon, \aleph \in \mathbb{C}$, $a > 0$, $\Re\left(1+\frac{\gamma-1}{1-\zeta}\right) > 0$, $\Re(\epsilon-\delta) > 0$ *and the pathway parameter* $\zeta < 1$.

Theorem 3.1. *Let* $x \in \mathbb{R}^+$ *and* $z, \gamma, \delta, \epsilon, \aleph \in \mathbb{C}$, *such that* $a > 0$, $\Re\left(1+\frac{\gamma-1}{1-\zeta}\right) > 0$,

$\Re(\epsilon-\delta)>0$, $\Re(\omega)>0$, $\Re(\lambda)>0$, $\Re(\nu)>\Re(\mu)>0$, $\min\{\Re(p),\Re(q)\}>0$, $\min\{\Re(\alpha),\Re(\beta)\}>0$ and the pathway parameter $\zeta<1$. Then, we have

$$\begin{aligned}&\left(P_{0^+}^{\gamma,\delta,\epsilon,\zeta;a}t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;zt^{\omega})\right)(x)\\&=\frac{x^{\aleph-\delta}\Gamma\left(1+\frac{\gamma-1}{1-\zeta}\right)}{[a(1-\zeta)]^{\aleph}}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}\left(\lambda,\mu;\nu;\frac{zx^{\omega}}{[a(1-\zeta)]^{\omega}}\right)\\&*{}_3\Psi_2\left[\begin{array}{c|c}(\aleph,\omega),(\aleph-\delta+\epsilon,\omega),(1,1)&\\&\frac{zx^{\omega}}{[a(1-\zeta)]^{\omega}}\\(\aleph-\delta,\omega),\left(1+\aleph+\epsilon+\frac{\gamma-1}{1-\zeta},\omega\right)&\end{array}\right].\end{aligned}\tag{13}$$

Proof. Let P be the left-hand side of (13), then using (2), (11) and changing the order of summation and pathway fractional integral operator, gives

$$P=\sum_{n=0}^{\infty}(\lambda)_n\frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^n}{n!}\left(P_{0^+}^{\gamma,\delta,\epsilon,\zeta;a}t^{\aleph+\omega n-1}\right)(x).\tag{14}$$

Applying equation (12) to (14), lead us to

$$\begin{aligned}P=&\frac{x^{\aleph-\delta}\Gamma\left(1+\frac{\gamma-1}{1-\zeta}\right)}{[a(1-\zeta)]^{\aleph}}\sum_{n=0}^{\infty}(\lambda)_n\frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)n!}\\&\times\frac{\Gamma(\aleph+n\omega)\Gamma(\aleph-\delta+\epsilon+n\omega)\Gamma(1+n)}{\Gamma(\aleph-\delta+n\omega)\Gamma\left(1+\aleph+\epsilon+\frac{\gamma-1}{1-\zeta}+n\omega\right)}\left(\frac{zx^{\omega}}{[a(1-\zeta)]^{\omega}}\right)^n.\end{aligned}$$

Using the Hadamard convolution (product) in (6) and Fox-Wright function in (1), the desired result in (13) is obtained.

Corollary 3.1. *Let $x\in\mathbb{R}^+$, $z,\gamma,\delta,\epsilon,\aleph\in\mathbb{C}$, $\Re(\gamma)>0$, $\Re(\omega)>0$, $\Re(\lambda)>0$, $\Re(\epsilon-\delta)>0$, $\Re(\nu)>\Re(\mu)>0$, $\min\{\Re(p),\Re(q)\}>0$, $\min\{\Re(\alpha),\Re(\beta)\}>0$. Then, we get*

$$\begin{aligned}\left(P_{0^+}^{\gamma,\delta,\epsilon,0;1}t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;zt^{\omega})\right)(x)&=x^{\aleph-\delta}\Gamma(\gamma)\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;zx^{\omega})\\&*{}_3\Psi_2\left[\begin{array}{c|c}(\aleph,\omega),(\aleph-\delta+\epsilon,\omega),(1,1)&\\&zx^{\omega}\\(\aleph-\delta,\omega),(\gamma+\aleph+\epsilon,\omega)&\end{array}\right].\end{aligned}$$

Corollary 3.2. *Let $x \in \mathbb{R}^+$, $z, \gamma, \epsilon, \aleph \in \mathbb{C}$, $\Re(\gamma) > 0$, $\Re(\omega) > 0$, $\Re(\lambda) > 0$, $\Re(\nu) > \Re(\mu) > 0$, $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$. Then, we obtain*

$$\left(P_{0^+}^{\gamma,-\gamma,\epsilon,0;1} t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;zt^{\omega})\right)(x) = x^{\aleph+\gamma}\Gamma(\gamma)\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;zx^{\omega}) * {}_2\Psi_1\left[\begin{array}{c|c} (\aleph,\omega),(1,1) & \\ & zx^{\omega} \\ (\aleph+\gamma,\omega) & \end{array}\right].$$

Corollary 3.3. *Let $x \in \mathbb{R}^+$, $z, \gamma, \epsilon, \aleph \in \mathbb{C}$, $\Re(\gamma) > 0$, $\Re(\epsilon) > 0$, $\Re(\omega) > 0$, $\Re(\lambda) > 0$, $\Re(\nu) > \Re(\mu) > 0$, $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$. Then, we have*

$$\left(P_{0^+}^{\gamma,0,\epsilon,0;1} t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;zt^{\omega})\right)(x) = x^{\aleph+\gamma}\Gamma(\gamma)\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;zx^{\omega}) * {}_2\Psi_1\left[\begin{array}{c|c} (\aleph+\epsilon,\omega),(1,1) & \\ & zx^{\omega} \\ (\aleph+\gamma+\epsilon,\omega) & \end{array}\right].$$

Corollary 3.4. *Let $x \in \mathbb{R}^+$, $z, \gamma, \delta, \aleph \in \mathbb{C}$, $a > 0$, $\Re(\delta) > 0$, $\Re(\omega) > 0$, $\Re(\lambda) > 0$, $\Re(\nu) > \Re(\mu) > 0$, $\Re\left(1 + \frac{\gamma-1}{1-\zeta}\right) > 0$, $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$ and the pathway parameter $\zeta < 1$. Then, we get*

$$\left(P_{0^+}^{\gamma,\delta,0,\zeta;a} t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;zt^{\omega})\right)(x) = \frac{x^{\aleph-\delta}\Gamma\left(1+\frac{\gamma-1}{1-\zeta}\right)}{[a(1-\zeta)]^{\aleph}} * {}^{\Psi}F_{p,q}^{\alpha,\beta}\left(\lambda,\mu;\nu;\frac{zx^{\omega}}{[a(1-\zeta)]^{\omega}}\right) {}_2\Psi_1\left[\begin{array}{c|c} (\aleph,\omega),(1,1) & \\ & \frac{zx^{\omega}}{[a(1-\zeta)]^{\omega}} \\ \left(1+\aleph+\frac{\gamma-1}{1-\zeta},\omega\right) & \end{array}\right].$$

Corollary 3.5. *Let $x \in \mathbb{R}^+$, $z, \gamma, \aleph \in \mathbb{C}$, $a > 0$, $\Re(\delta) > 0$, $\Re(\omega) > 0$, $\Re(\lambda) > 0$, $\Re(\nu) > \Re(\mu) > 0$, $\Re(\gamma - 1) > 0$, $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$. Then, we obtain*

$$\left(P_{0^+}^{\gamma,\delta,0,1;a} t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;zt^{\omega})\right)(x) = \frac{x^{\aleph-\delta}}{[a(\gamma-1)]^{\aleph}}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}\left(\lambda,\mu;\nu;\frac{zx^{\omega}}{[a(\gamma-1)]^{\omega}}\right) * {}_2\Psi_0\left[\begin{array}{c|c} (\aleph,\omega),(1,1) & \\ & \frac{zx^{\omega}}{[a(\gamma-1)]^{\omega}} \\ - & \end{array}\right].$$

4. Fractional Differentiations

Lemma 4.1. ([9]) *Let $\eta, \eta', \rho, \rho', \tau \in \mathbb{C}$ and $x > 0$, such that $\Re(\tau) > 0$ and $\Re(\aleph) > \max\{0, \Re(\tau - \eta - \eta' - \rho'), \Re(\rho - \eta)\}$. Then, we have*

$$\left(\mathcal{D}_{0,x}^{\eta,\eta',\rho,\rho',\tau} t^{\aleph-1}\right)(x) = x^{\aleph-\tau+\eta+\eta'-1} \frac{\Gamma(\aleph)\Gamma(\aleph-\tau+\eta+\eta'+\rho')\Gamma(\aleph-\rho+\eta)}{\Gamma(\aleph-\rho)\Gamma(\aleph-\tau+\eta+\eta')\Gamma(\aleph-\tau+\eta+\rho')}.$$

The following equations are true for $x > 0$:

1. If $\Re(\tau) > 0$ and $\Re(\eta) > -\max\{0, \Re(\eta + \rho + \tau)\}$, then
$$\left(\mathcal{D}_{0,x}^{\eta,\rho,\tau} t^{\aleph-1}\right)(x) = x^{\aleph+\rho-1} \frac{\Gamma(\aleph)\Gamma(\aleph+\eta+\rho+\tau)}{\Gamma(\aleph+\rho)\Gamma(\aleph+\tau)}.$$

2. If $\Re(\eta) > \Re(\aleph)$, then
$$\left(\mathcal{D}_{0,x}^{\eta} t^{\aleph-1}\right)(x) = x^{\aleph-\eta-1} \frac{\Gamma(\aleph)}{\Gamma(\aleph-\rho)}.$$

3. If $\Re(\eta) > 0$, and $\Re(\aleph) > -\Re(\eta + \tau)$, then
$$\left(\mathcal{D}_{0,x}^{\eta,\tau} t^{\aleph-1}\right)(x) = x^{\aleph-1} \frac{\Gamma(\aleph+\eta+\tau)}{\Gamma(\aleph+\tau)}.$$

Lemma 4.2. ([9]) *Let $\eta, \eta', \rho, \rho', \tau \in \mathbb{C}$ and $x > 0$, such that $\Re(\tau) > 0$ and $\Re(\aleph) < 1 + \min\{\Re(\rho'), \Re(\tau - \eta - \eta'), \Re(\rho - \eta)\}$. Then, we get*

$$\left(\mathcal{D}_{x,\infty}^{\eta,\eta',\rho,\rho',\tau} t^{\aleph-1}\right)(x) = x^{\aleph-\tau+\eta+\eta'-1} \times \frac{\Gamma(1-\aleph+\eta')\Gamma(1-\aleph+\tau+-\eta-\eta')\Gamma(1-\aleph+\tau-\eta'-\rho)}{\Gamma(1-\aleph)\Gamma(1-\aleph+\tau-\eta-\eta'-\rho)\Gamma(1-\aleph-\eta'-\rho)}.$$

The following equations are true for $x > 0$:

1. If $\Re(\eta) > 0$ and $\Re(\aleph) < 1 + \min\{\Re(-\rho - \Delta), \Re(\eta + \tau)\}$, then
$$\left(\mathcal{D}_{x,\infty}^{\eta,\rho,\tau} t^{\aleph-1}\right)(x) = x^{\aleph+\rho-1} \frac{\Gamma(1-\aleph-\rho)\Gamma(1-\aleph+\eta+\tau)}{\Gamma(1-\aleph)\Gamma(1-\aleph+\tau-\rho)}.$$

2. If $\Re(\eta) > 0$, and $\Re(\aleph) < 1 + \Re(\eta) - \Delta$, then
$$\left(\mathcal{D}_{x,\infty}^{\eta} t^{\aleph-1}\right)(x) = x^{\aleph-\eta-1} \frac{\Gamma(1-\aleph+\eta)}{\Gamma(1-\aleph)}.$$

3. If $\Re(\eta) > 0$, and $\Re(\aleph) < 1 + \Re(\eta + \tau) - \Delta$, then

$$\left(\mathcal{D}_{x,\infty}^{\eta,\tau} t^{\aleph-1}\right)(x) = x^{\aleph-1} \frac{\Gamma(1-\aleph+\eta+\tau)}{\Gamma(1-\aleph-\tau)}.$$

Theorem 4.1. *Let* $\eta, \eta', \rho, \rho', \tau \in \mathbb{C}$ *and* $x > 0$, *such that* $\Re(\omega) > 0$, $\Re(\lambda) > 0$, $\Re(\tau) > 0$, $\Re(\nu) > \Re(\mu) > 0$ *and* $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$, $\Re(\aleph) > \max\{0, \Re(\tau - \eta - \eta' - \rho'), \Re(\rho - \eta)\}$. *Then, we obtain*

$$\left(\mathcal{D}_{0,x}^{\eta,\eta',\rho,\rho',\tau} t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda, \mu; zt^{\omega})\right)(x) = x^{\aleph-\tau+\eta+\eta'-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda, \mu; zx^{\omega})$$
$$* {}_3\Psi_2 \left[\begin{array}{c|c} (\aleph, \omega), (\aleph-\tau+\eta+\eta'+\rho', \omega), (\aleph-\rho+\eta, \omega), (1,1) & \\ & zx^{\omega} \\ (\aleph+\rho, \omega), (\aleph-\tau+\eta+\eta', \omega), (\aleph-\tau+\eta+\rho', \omega) & \end{array} \right].$$

Proof. Using (2) and changing the order of summation and integral operator gives

$$D_1 = \sum_{n=0}^{\infty} (\lambda)_n \frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu+n, \nu-\mu)}{B(\mu, \nu-\mu)} \frac{z^n}{n!} \left(\mathcal{D}_{0,x}^{\eta,\eta',\rho,\rho',\tau} t^{\tau+n\ell-1}\right)(x).$$

Then, we have

$$D_1 = \sum_{n=0}^{\infty} (\lambda)_n \frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu+n, \nu-\mu)}{B(\mu, \nu-\mu) n!}$$
$$\times \frac{\Gamma(\aleph+n\omega)\Gamma(\aleph-\tau+\eta+\eta'+\rho'+n\omega)\Gamma(\aleph-\rho+\eta+n\omega)\Gamma(1+n)}{\Gamma(\aleph+\rho+n\omega)\Gamma(\aleph-\tau+\eta+\eta'+n\omega)\Gamma(\aleph-\tau+\eta+\rho'+n\omega)n!}(x^{\omega}z)^n.$$

Considering the Hadamard convolution (product), the desired result is achieved.

Corollary 4.1. *Let* $\eta, \eta', \rho, \rho', \tau \in \mathbb{C}$, $x > 0$, $\Re(\omega) > 0$, $\Re(\nu) > \Re(\mu) > 0$, $\Re(\tau) > 0$, $\Re(\aleph) > \max\{0, \Re(\tau - \eta - \eta' - \rho'), \Re(\rho - \eta)\}$, $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$. *Then, we have*

$$\left(\mathcal{D}_{0,x}^{\eta,\eta',\rho,\rho',\tau} t^{\aleph-1}\ {}^{\Psi}\Phi_{p,q}^{\alpha,\beta}(\mu; \nu; zt^{\omega})\right)(x) = x^{\aleph-\tau+\eta+\eta'-1}\ {}^{\Psi}\Phi_{p,q}^{\alpha,\beta}(\mu; zx^{\omega})$$
$$* {}_3\Psi_2 \left[\begin{array}{c|c} (\aleph, \omega), (\aleph-\tau+\eta+\eta'+\rho', \omega), (\aleph-\rho+\eta, \omega), (1,1) & \\ & zx^{\omega} \\ (\aleph+\rho, \omega), (\aleph-\tau+\eta+\eta', \omega), (\aleph-\tau+\eta+\rho', \omega) & \end{array} \right].$$

Corollary 4.2. *Let* $\eta, \eta', \rho, \rho', \tau \in \mathbb{C}$ *and* $x > 0$, *such that* $\Re(\omega) > 0$, $\Re(\lambda) > 0$, $\Re(\nu) > \Re(\mu) > 0$ *and* $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$, $\Re(\tau) > 0$,

$\Re(\eta) > -\max\{0, \Re(\eta+\rho+\tau)\}$. *Then, we get*

$$\left(\mathcal{D}_{0,x}^{\eta,\rho,\tau} t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;;zt^{\omega})\right)(x)$$
$$= x^{\tau+\lambda-1\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zx^{\omega}) * {}_3\Psi_2\left[\begin{array}{c|c} (\aleph,\omega),(\aleph+\tau+\rho,\omega),(1,1) & \\ & zx^{\omega} \\ (\aleph+\rho,\omega),(\aleph+\rho,\omega) & \end{array}\right].$$

Corollary 4.3. *Let* $\eta,\eta',\rho,\rho',\tau\in\mathbb{C}$ *and* $x>0$, *such that* $\Re(\omega)>0$, $\Re(\lambda)>0$, $\Re(\nu)>\Re(\mu)>0$ *and* $\min\{\Re(p),\Re(q)\}>0$, $\min\{\Re(\alpha),\Re(\beta)\}>0$, $\Re(\eta)>\Re(\aleph)$. *Then, we obtain*

$$\left(\mathcal{D}_{0,x}^{\eta} t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;;zt^{\omega})\right)(x)$$
$$= x^{\aleph-\eta-1\Psi}F_{,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zx^{\ell}) * {}_2\Psi_1\left[\begin{array}{c|c} (\tau,\ell),(1,1) & \\ & zx^{\ell} \\ (\tau-\kappa,\ell) & \end{array}\right].$$

Corollary 4.4. *Let* $\eta,\eta',\rho,\rho',\tau\in\mathbb{C}$ *and* $x>0$, *such that* $\Re(\omega)>0$, $\Re(\lambda)>0$, $\Re(\nu)>\Re(\mu)>0$ *and* $\min\{\Re(p),\Re(q)\}>0$, $\min\{\Re(\alpha),\Re(\beta)\}>0$, $\Re(\eta)>0$ *and* $\Re(\aleph)>-\Re(\eta+\tau)$. *Then, we have*

$$\left(\mathcal{D}_{0,x}^{\eta,\tau} t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;zt^{\omega})\right)(x)$$
$$= x^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu;zx^{\omega}) * {}_2\Psi_1\left[\begin{array}{c|c} (\aleph+\tau+\eta,\omega),(1,1) & \\ & zx^{\omega} \\ (\aleph+\tau,\omega) & \end{array}\right].$$

Theorem 4.2. *Let* $\eta,\eta',\rho,\rho',\tau\in\mathbb{C}$ *and* $x>0$, *such that* $\Re(\omega)>0$, $\Re(\lambda)>0$, $\Re(\tau)>0$, $\Re(\nu)>\Re(\mu)>0$ *and* $\min\{\Re(p),\Re(q)\}>0$, $\min\{\Re(\alpha),\Re(\beta)\}>0$, $\Re(\aleph)<1+\min\{\Re(\rho'),\Re(\tau-\eta-\eta'),\Re(\rho-\eta)\}$. *Then, we get*

$$\left(\mathcal{D}_{x,\infty}^{\eta,\eta',\rho,\rho',\tau} t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;zt^{-\omega})\right)(x) = x^{\aleph-\tau+\eta+\eta'-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;zx^{-\omega})$$
$$* {}_3\Psi_2\left[\begin{array}{c|c} (1-\aleph+\rho',\omega),(1-\aleph+\tau-\rho-\rho',\omega),(1-\aleph+\tau-\eta-\rho',\omega),(1,1) & \\ & zx^{-\omega} \\ (1-\aleph,\omega),(1-\aleph+\tau-\eta-\eta'+\rho,\omega),(1-\aleph-\eta'+\rho',\omega) & \end{array}\right].$$

Proof. Using (2) and changing the order of summation and integral operator gives

$$D_2 = \sum_{n=0}^{\infty}(\lambda)_n \frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^n}{n!}\left(\mathcal{D}_{x,\infty}^{\eta,\eta',\rho,\rho',\tau} t^{\tau-n\ell-1}\right)(x).$$

Then, we get

$$D_2 = \sum_{n=0}^{\infty} (\lambda)_n \frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)n!} \frac{\Gamma(1-\aleph+\rho',\omega)\Gamma(1-\aleph+\tau-\rho-\rho'+n\omega)}{\Gamma(1-\aleph+n\omega)\Gamma(1-\aleph+\tau-\eta-\eta'+\rho+n\omega)}$$
$$\times \frac{\Gamma(1-\aleph+\tau-\eta-\rho'+n\omega)\Gamma(1+n)}{\Gamma(1-\aleph-\eta'+\rho'+n\omega)n!}(x^{\omega}z)^n$$

Considering the Hadamard convolution (product), the desired result is achieved.

Corollary 4.5. *Let* $\eta, \eta', \rho, \rho', \tau \in \mathbb{C}$ *and* $x > 0$*, such that* $\Re(\omega) > 0$*,* $\Re(\tau) > 0$*,* $\Re(\nu) > \Re(\mu) > 0$ *and* $\min\{\Re(p), \Re(q)\} > 0$*,* $\min\{\Re(\alpha), \Re(\beta)\} > 0$ *and also let* $\Re(\aleph) < 1 + \min\{\Re(\rho'), \Re(\tau-\eta-\eta'), \Re(\rho-\eta)\}$*. Then, we obtain*

$$\left(\mathcal{D}_{x,\infty}^{\eta,\eta',\rho,\rho',\tau} t^{\aleph-1}\ {}^{\Psi}\Phi_{p,q}^{\alpha,\beta}(\mu; zt^{-\omega})\right)(x) = x^{\aleph-\tau+\eta+\eta'-1}\ {}^{\Psi}\Phi_{p,q}^{\alpha,\beta}(\mu; zx^{-\omega})$$
$$* {}_3\Psi_2 \left[\begin{array}{c|c} (1-\aleph+\rho',\omega), (1-\aleph+\tau-\rho-\rho',\omega), (1-\aleph+\tau-\eta-\rho',\omega), (1,1) & \\ & zx^{-\omega} \\ (1-\aleph,\omega), (1-\aleph+\tau-\eta-\eta'+\rho,\omega), (1-\aleph-\eta'+\rho',\omega) & \end{array} \right].$$

Corollary 4.6. *Let* $\eta, \eta', \rho, \rho', \tau \in \mathbb{C}$ *and* $x > 0$*, such that* $\Re(\omega) > 0$*,* $\Re(\lambda) > 0$*,* $\Re(\nu) > \Re(\mu) > 0$*,* $\min\{\Re(p), \Re(q)\} > 0$*,* $\min\{\Re(\alpha), \Re(\beta)\} > 0$*,* $\Re(\eta) > 0$ *and* $\Re(\aleph) < 1 + \min\{\Re(-\rho-\Re(\eta)-1), \Re(\eta+\tau)\}$*. Then, we have*

$$\left(\mathcal{D}_{x,\infty}^{\eta,\rho,\tau} t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu; zt^{-\omega})\right)(x) = x^{\aleph+\rho-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu; zx^{-\omega})$$
$$* {}_3\Psi_2 \left[\begin{array}{c|c} (1-\rho-\aleph,\omega), (\tau.\eta-\aleph+1,\omega), (1,1) & \\ & zx^{-\omega} \\ (1-\aleph,\omega), (\tau-\rho-\aleph+1,\omega) & \end{array} \right].$$

Corollary 4.7. *Let* $\eta, \eta', \rho, \rho', \tau \in \mathbb{C}$ *and* $x > 0$*, such that* $\Re(\omega) > 0$*,* $\Re(\lambda) > 0$*,* $\Re(\nu) > \Re(\mu) > 0$*,* $\min\{\Re(p), \Re(q)\} > 0$*,* $\min\{\Re(\alpha), \Re(\beta)\} > 0$*,* $\Re(\eta) > 0$ *and* $\Re(\aleph) < 1 + \Re(\eta) - \Re(\eta) - 1$*. Then, we get*

$$\left(\mathcal{D}_{x,\infty}^{\eta} t^{\aleph-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu; zt^{-\omega})\right)(x)$$
$$= x^{\aleph-\eta-1}\ {}^{\Psi}F_{p,q}^{\alpha,\beta}(\lambda,\mu;\nu; zx^{-\omega}) * {}_2\Psi_1 \left[\begin{array}{c|c} (1+\eta-\aleph,\omega), (1,1) & \\ & zx^{-\omega} \\ (1-\aleph,\omega) & \end{array} \right].$$

Corollary 4.8. *Let* $\eta, \eta', \rho, \rho', \tau \in \mathbb{C}$ *and* $x > 0$*, such that* $\Re(\omega) > 0$*,* $\Re(\lambda) > 0$*,* $\Re(\nu) > \Re(\mu) > 0$*,* $\min\{\Re(p), \Re(q)\} > 0$*,* $\min\{\Re(\alpha), \Re(\beta)\} > 0$*,* $\Re(\eta) > 0$ *and*

$\Re(\aleph) < 1 + \Re(\eta + \tau) - \Re(\eta) - 1$. *Then, we obtain*

$$\left(\mathcal{D}^{\kappa,\mu}_{x,\infty} t^{\aleph-1}\ {}^{\Psi}F^{\alpha,\beta}_{p,q}(\lambda,\mu;\nu;zt^{-\omega})\right)(x)$$

$$= x^{\aleph-1}\ {}^{\Psi}F^{\alpha,\beta}_{p,q}(\lambda,\mu;\nu;zx^{-\ell}) * {}_2\Psi_1\left[\begin{array}{c|c} (\eta+\tau-\aleph+1,\omega),(1,1) & \\ & zx^{-\omega} \\ (1-\tau-\aleph,\omega) & \end{array}\right].$$

5. Applications to Fractional Kinetic Equation

The solution of fractional kinetic equations has been observed by many mathematicians using various generalizations of the Mittag-Leffler function. In this section, we use the two-parameter Mittag-Leffler (Wiman's) function and Sumudu transform to study the solution of the fractional kinetic equation.

Theorem 5.1. *Let* $\alpha, d, \omega > 0$, $\lambda, \mu, \nu \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\lambda) > 0$, $\Re(\nu) > \Re(\mu) > 0$, $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$ *when* $p = q = 0$. *The the solution of fractional kinetic equation*

$$\Omega(t) - \Omega_0\ {}^{\Psi}F^{\alpha,\beta}_{p,q}\left(\lambda,\mu;\nu;d^{\omega}t^{\omega}\right) = -\alpha^{\omega}\,{}_0D_t^{-\omega}\Omega(t), \tag{15}$$

is given by

$$\Omega(t) = \Omega_0 \sum_{n=0}^{\infty} (\lambda)_n \frac{{}^{\Psi}B^{\alpha,\beta}_{p,q}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{(d^{\omega}t^{\omega})^n}{n!} \Gamma(n\omega+1) E_{\omega,n\omega}\left(-\alpha^{\omega}t^{\omega}\right). \tag{16}$$

Proof. Using the Sumudu transform in (8) to (15), gives

$$\mathcal{S}\{\Omega(t);s\} = \Omega_0 \sum_{n=0}^{\infty} (\lambda)_n \frac{{}^{\Psi}B^{\alpha,\beta}_{p,q}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{d^{n\omega}}{n!} \mathcal{S}\left\{t^{n\omega};s\right\} - \alpha^{\omega}\mathcal{S}\left\{{}_0D_t^{-\omega};s\right\}. \tag{17}$$

Using the result from (10) to (17) and simplifying, we have

$$\Omega(s) = \Omega_0 \sum_{n=0}^{\infty} (\lambda)_n \frac{{}^{\Psi}B^{\alpha,\beta}_{p,q}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{d^{n\omega}}{n!} \Gamma(n\omega+1) \left\{\sum_{u=0}^{\infty} (-1)^u \alpha^{u\omega} s^{\omega(n+u)}\right\}. \tag{18}$$

Applying the inverse Sumudu transform in (9) to (18) and simplifying, we get

$$\Omega(t) = \Omega_0 \sum_{n=0}^{\infty} (\lambda)_n \frac{{}^{\Psi}B^{\alpha,\beta}_{p,q}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{(d^{\omega}t^{\omega})^n}{n!} \Gamma(n\omega+1) \sum_{u=0}^{\infty} \frac{(-\alpha^{\omega}t^{\omega})}{\Gamma(n\omega+u\omega+1)}. \tag{19}$$

Applying (7) to (19) gives the desired result in (16).

Corollary 5.1. *Let $\alpha, d, \omega > 0$, $\mu, \nu \in \mathbb{C}$, $\Re(\nu) > \Re(\mu) > 0$, $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$ when $p = q = 0$. The fractional kinetic equation*

$$\Omega(t) - \Omega_0 \ {}^{\Psi}\Phi_{p,q}^{\alpha,\beta}\left(\mu; \nu; d^\omega t^\omega\right) = -\alpha^\omega {}_0D_t^{-\omega}\aleph(t),$$

has a solution

$$\Omega(t) = \Omega_0 \sum_{n=0}^{\infty} \frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu + n, \nu - \mu)}{B(\mu, \nu - \mu)} \frac{(d^\omega t^\omega)^n}{n!} \Gamma(n\omega + 1) E_{\omega, n\omega}\left(-\alpha^\omega t^\omega\right).$$

Theorem 5.2. *Let $d, \omega > 0$, $\lambda, \mu, \nu \in \mathbb{C}$, $\Re(\lambda) > 0$, $\Re(\nu) > \Re(\mu) > 0$, $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$ when $p = q = 0$. Then the fractional kinetic equation*

$$\Omega(t) - \Omega_0 \ {}^{\Psi}F_{p,q}^{\alpha,\beta}\left(\lambda, \mu; \nu; d^\omega t^\omega\right) = -d^\omega {}_0D_t^{-\omega}\Omega(t),$$

is given by

$$\Omega(t) = \Omega_0 \sum_{n=0}^{\infty} (\lambda)_n \frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu + n, \nu - \mu)}{B(\mu, \nu - \mu)} \frac{(d^\omega t^\omega)^n}{n!} \Gamma(n\omega + 1) E_{\omega, n\omega}\left(-d^\omega t^\omega\right).$$

Proof. The proof of this theorem follow directly from Theorem 5.1.

Corollary 5.2. *Let $d, \omega > 0$, $\mu, \nu \in \mathbb{C}$, $\Re(\nu) > \Re(\mu) > 0$, $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$ when $p = q = 0$. Then the solution of fractional kinetic equation*

$$\Omega(t) - \Omega_0 \ {}^{\Psi}\Phi_{p,q}^{\alpha,\beta}\left(\mu; \nu; d^\omega t^\omega\right) = -d^\omega {}_0D_t^{-\omega}\Omega(t),$$

is given by

$$\Omega(t) = \Omega_0 \sum_{n=0}^{\infty} \frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu + n, \nu - \mu)}{B(\mu, \nu - \mu)} \frac{(d^\omega t^\omega)^n}{n!} \Gamma(n\omega + 1) E_{\omega, n\omega}\left(-d^\omega t^\omega\right).$$

Theorem 5.3. *Let $d, \omega > 0$, $\lambda, \mu, \nu \in \mathbb{C}$, $\Re(\lambda) > 0$, $\Re(\nu) > \Re(\mu) > 0$, $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$ when $p = q = 0$. Then the solution of fractional kinetic equation*

$$\Omega(t) - \Omega_0 \ {}^{\Psi}F_{p,q}^{\alpha,\beta}\left(\lambda, \mu; \nu; t\right) = -d^\omega {}_0D_t^{-\omega}\aleph(t),$$

is given as

$$\Omega(t) = \Omega_0 \sum_{n=0}^{\infty} (\lambda)_n \frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu + n, \nu - \mu)}{B(\mu, \nu - \mu)} t^n E_{\omega, n}\left(-d^\omega t^\omega\right).$$

Proof. The proof of this theorem follow directly from Theorem 5.1.

Corollary 5.3. *Let* $d, \omega > 0$, $\mu, \nu \in \mathbb{C}$, $\Re(\nu) > \Re(\mu) > 0$, $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$ *when* $p = q = 0$. *Then the fractional kinetic equation*

$$\Omega(t) - \Omega_0 \ {}^{\Psi}\Phi_{p,q}^{\alpha,\beta}(\mu;\nu;t) = -d^{\omega} {}_0D_t^{-\omega} \aleph(t),$$

is given as

$$\Omega(t) = \Omega_0 \sum_{n=0}^{\infty} \frac{{}^{\Psi}B_{p,q}^{\alpha,\beta}(\mu+n, \nu-\mu)}{B(\mu, \nu-\mu)} t^n E_{\omega,n}\left(-d^{\omega} t^{\omega}\right).$$

6. Conclusion

The generalized Gauss and confluent hypergeometric functions given in (2) and (3) are generalization of many existing Gauss and confluent hypergeometric functions, see for example Chaudhry et al. [5], Şahin et al. [17] and Choi et al. [6], therefore, a variety of pathway fraction integral formulas, fractional differential composition formulas, and fractional kinetic equations can be obtained as special cases by suitably replacing the parameters.

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