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Mixture of Pathway Weibull Model and Mellin Convolutions

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Abstract: The efficiency of the pathway idea in modeling heterogeneous real-life data is illustrated in this paper. Usually mixture models are used for modeling heterogeneous data. Here we introduce a finite mixture of two pathway Weibull models, denoted by WM_q (or $_qWM$ or W_qM). Properties of this new model are examined and identifiability is proved. Some important special cases of WM_q are given. Stress-strength reliability is found by using the Mellin convolution technique. With the help of a real heterogeneous data set, it is shown that the proposed model fits the data better than all other popular models in the literature.

Keywords: Pathway Weibull model; Fox H-function; mellin convolution; finite mixture distribution.

AMS Subject Classification : 03H05, 33C60, 60E99, 60E05.

1- Introduction :

Mixed failure populations are commonly encountered in life testing, reliability and quality control problems. A mixture model is a compounding of statistical distributions, which arise when sampling is done from heterogeneous populations with a different probability density function in each component. It has been considered for long-time as a flexible and powerful statistical modeling technique, especially to account for unobserved heterogeneity. Applications of mixture models are common in physics, biology, medicine, economics, finance and insurance among others. These models provide a framework not only for heterogeneous population but also a rationale for some thick-tailed distributions.

Mixture models received considerable attention in the area of survival analysis and reliability. In many real-life applications, the use of mixture models becomes inevitable when the data are not available for each component of the mixture rather for the overall mixture as described in Everitt and Hand (1981). In life-testing and reliability estimation problems, the underlying failure time distribution need not be homogeneous but can be a mixture of several distinct lifetime distributions. Each of these distinct lifetime distributions can represent a different type of cause of failure for the population. For instance, we assume that the survival function of treated cancer patients is a mixture of two sub populations; one which die due to their disease with a given proportion and the rest die of other reasons with a complementary proportion. For a recent review on finite mixtures and their applications, see Al-Hussaini and Sultan (2001) and the references therein.

Finite mixture distributions have provided a mathematically based approach to the statistical modeling of a wide variety of random phenomena. They consist of a weighted sum of standard distributions and are useful tools for reliability analysis of a heterogeneous population. They provide the necessary flexibility to model failure distributions of components with multiple failure modes. Mixture distributions are mostly used to model heterogeneous survival data sets (Aalen, (1992) Angelis et al., (1999) Al-Hussaini et al., (2000)) and have been successfully applied in the field of astronomy, biology, genetics, medicine, economics, engineering etc. These distributions provide the basis for models in which a random variable has a distribution of a particular form, but one or more of the parameters therein may be random. Mixture distributions have been extensively considered by many authors (Titterington et al., (1985), Maclachlan and Peel, (2000), Sultan et al., (2007)).

The pathway Weibull model which is a special case of Mathai's pathway model (Mathai, (2005)), has wide range of applications in modeling of survival and life-time data. It smoothly interpolates the Weibull and q-exponential densities which makes the pathway Weibull model more fat-tailed than the Weibull distribution. Costa et al. (2006) show that in the study of dielectric breakdown in oxides of electronic devices, a pathway Weibull model gives a good fit for the data. The various properties and applications of the pathway Weibull model are discussed in Naik (2008). When observations are taken from a possibly heterogeneous population, a simple Weibull model or a pathway Weibull model may not always be appropriate and in that situation we need to consider the mixture models. The motivation for a mixture of pathway Weibull model is that this model can move from one functional form to another through the pathway parameter q. The density function of a pathway Weibull random variable is given by

$$f_1(x;\alpha,\lambda,q) = \begin{cases} \alpha\lambda^{\alpha}(2-q)x^{\alpha-1}[1+(q-1)(\lambda x)^{\alpha}]^{-\frac{1}{q-1}} \\ 0, \text{ otherwise} \end{cases} \tag{1}$$

for $x \geq 0$, $\alpha > 0$, $\lambda > 0$, $1 < q < 2$.
For $q < 1$, $f_1(x;\alpha,\lambda,q)$ reduces to the following density function of the pathway Weibull random variable:

$$f_2(x;\alpha,\lambda,q) = \begin{cases} \alpha\lambda^{\alpha}(2-q)x^{\alpha-1}[1-(1-q)(\lambda x)^{\alpha}]^{\frac{1}{1-q}} \\ 0, \text{ otherwise} \end{cases} \tag{2}$$

for $0 \leq x \leq \frac{1}{\lambda(1-q)^{\frac{1}{\alpha}}}$, $\alpha > 0$, $\lambda > 0$. Clearly when $q \to 1$, $f_1(x;\alpha,\lambda,q)$, $f_2(x;\alpha,\lambda,q)$ tend to the usual Weibull density with two parameters α and λ. In this paper we consider the case $1 < q < 2$.

In the remainder of this paper we introduce a finite mixture of two pathway Weibull models denoted by WM_q and discuss its statistical properties. Some special cases and an application in the area of stress-strength analysis is given in this chapter. Finally, we apply the WM_q to a heterogeneous survival real data set and compare it with the models that are commonly used for survival analysis.

2- Mixture model and its properties :

In this section, we introduce the density function of a finite mixture of two pathway Weibull models (WM_q) and discuss some of its properties:
The mixture of two pathway Weibull models has its pdf given by

$$f(x;\psi) = \begin{cases} \pi f_1(x;\psi_1) + (1-\pi) f_2(x;\psi_2), \; 0<\pi<1, \; x \geq 0 \\ 0, \; \text{otherwise} \end{cases} \tag{3}$$

where $\psi_i = \{\psi_1, \psi_2\} = \{\alpha_i, \; \lambda_i, \; q\}$, i=1,2, and $f_i(x;\psi_i)$ is the density function of the ith component given by:

$$f_i(x;\alpha_i,\lambda_i,q) = \begin{cases} \alpha_i \lambda_i^{\alpha_i}(2-q)x^{\alpha_i-1}[1+(q-1)(\lambda_i x)^{\alpha_i}]^{-\frac{1}{q-1}} \\ 0, \; \text{otherwise} \end{cases} \tag{4}$$

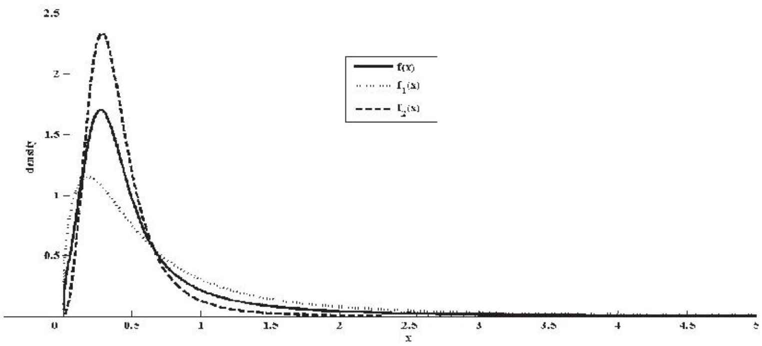
for $x \geq 0$, $\alpha_i > 0$, $\lambda_i > 0$, $1 < q < 2$ and $i = 1, 2$.
For $q < 1$, the mixture model has the density

$$g(x;\psi) = \begin{cases} \pi g_1(x;\psi_1) + (1-\pi) g_2(x;\psi_2), \; 0<\pi<1, \; 0 \leq x \leq (\frac{1}{\lambda_1(1-q)^{\frac{1}{\alpha_1}}} + \frac{1}{\lambda_2(1-q)^{\frac{1}{\alpha_2}}}) \\ \\ 0, \; \text{otherwise} \end{cases}$$

where $\psi_i = \{\psi_1, \psi_2\} = \{\alpha_i, \; \lambda_i, \; q\}$, i=1,2 and $g_i(x;\psi_i)$ is the density function of the ith component given by:

$$g_i(x;\alpha_i,\lambda_i,q) = \begin{cases} \alpha_i \lambda_i^{\alpha_i}(2-q)x^{\alpha_i-1}[1-(1-q)(\lambda_i x)^{\alpha_i}]^{\frac{1}{1-q}} \\ 0, \; \text{otherwise} \end{cases} \tag{5}$$

for $0 \leq x \leq \frac{1}{\lambda_i(1-q)^{\frac{1}{\alpha_i}}}$, $\alpha_i > 0$, $\lambda_i > 0$, $q < 1$ and $i = 1, 2$.
The graphs of $f(x;\psi)$ along with their components for different values of the parameters are given in the following Figure 1:



(i)

Figure 1 - Density functions: components and their mixture with parameters $\pi = 0.5,\ \alpha_1 = 1.5,\ \alpha_2 = 3,\ \lambda_1 = 3,\ \lambda_2 = 3.5,\ q = 1.5$.

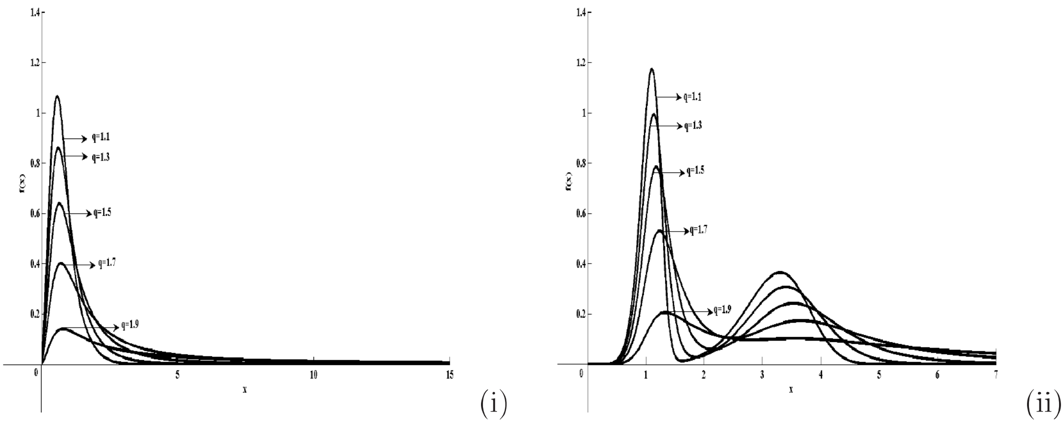


Figure 2 - Graph of $f(x)$ for (i): $\pi = 0.5,\ \alpha_1 = 2,\ \alpha_2 = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 1.5$, (ii): $\pi = 0.5,\ \alpha_1 = 7,\ \alpha_2 = 7.5,\ \lambda_1 = 0.3,\ \lambda_2 = 0.9$.

Figure 2 (i) and (ii) show the density curves of $f(x;\psi)$ for selected values of the parameters $\alpha_i,\ \lambda_i,\ i = 1,2$ and for selected values of the pathway parameter q. From the graphs it is clear that we can model both thicker and thinner tailed curves and bimodal densities by simply varying the pathway parameter q.

The cumulative distribution function of the WM$_q$ is given by

$$F(x;\psi) = \pi F_1(x;\psi_1) + (1-\pi)F_2(x;\psi_1),\ 0 < \pi < 1 \tag{6}$$

where

$$F_i(x;\psi_i) = 1 - [1 + (q-1)(\lambda_i x)^{\alpha_i}]^{\frac{q-2}{q-1}},\ i = 1,2,\ x \geq 0,\ \alpha_i > 0,\ \lambda_i > 0,\ 1 < q < 2. \tag{7}$$

2.1 Moments

The s^{th} moment of the WM$_q$ is given by

$$\mathbf{E}(x^s) = \pi \frac{(2-q)}{(\lambda_1)^s (q-1)^{\frac{s}{\alpha_1}+1}} \frac{\Gamma(\frac{s}{\alpha_1}+1)(\Gamma(\frac{1}{q-1} - \frac{s}{\alpha_1} - 1))}{\Gamma(\frac{1}{q-1})}$$

$$+(1-\pi)\frac{(2-q)}{(\lambda_2)^s (q-1)^{\frac{s}{\alpha_2}+1}} \frac{\Gamma(\frac{s}{\alpha_2}+1)(\Gamma(\frac{1}{q-1} - \frac{s}{\alpha_2} - 1))}{\Gamma(\frac{1}{q-1})}, \tag{8}$$

$0 < \pi < 1,\ \alpha_i > 0,\ \lambda_i > 0,\ q > 1,\ i = 1,2,\ \min\{-\alpha_i\} < \mathbf{R}(s) < \min\{\frac{(2-q)\alpha_i}{q-1}\}$ where $\mathbf{R}(.)$ denotes the real part of (.).

From (8), the explicit expressions for the mean and variance of the WM$_q$ can easily be derived.

2.2 Reliability and failure rate functions

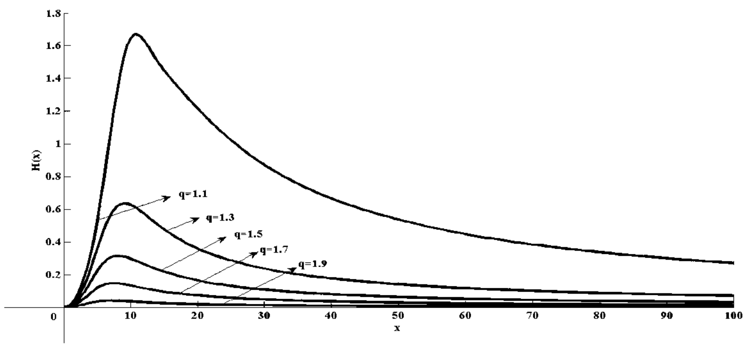
The reliability function (survival function) of the WM_q is given by

$$\begin{aligned} R(x) &= Pr\{\text{system survives beyond a time } x\} \\ &= 1 - F(x) = \pi[(1+(q-1)(\lambda_1 x)^{\alpha_1})]^{\frac{q-2}{q-1}} + (1-\pi) \times \\ &\qquad [(1+(q-1)(\lambda_2 x)^{\alpha_2})]^{\frac{q-2}{q-1}} \end{aligned} \tag{9}$$

for $x \geq 0,\ \alpha_i > 0,\ \lambda_i > 0,\ 1 < q < 2,\ 0 < \pi < 1,\ i = 1, 2.$
By using (3) and (9), the failure rate (hazard rate) function of the WM_q is given by the following relation:

$$H(x) = \frac{f(x)}{R(x)} = \frac{\sum_{i=1}^{2} \pi_i \alpha_i \lambda_i^{\alpha_i} (2-q) x^{\alpha_i - 1} (1+(q-1)(\lambda_i x)^{\alpha_i})^{-\frac{1}{q-1}}}{\sum_{i=1}^{2} \pi_i (1+(q-1)(\lambda_i x)^{\alpha_i})^{\frac{q-2}{q-1}}}, \tag{10}$$

where $\pi_i > 0,\ \pi_1 + \pi_2 = 1,\ \alpha_i, \lambda_i > 0,\ 1 < q < 2,\ i = 1, 2.$
The hazard function, for different values of the parameter is given in the following figures.



(i)

Figure 3 - Hazard function of the WM_q with parameters $\pi_1 = \pi_2 = 0.5, \alpha_1 = 3, \alpha_2 = 4, \lambda_1 = 4.5, \lambda_2 = 5.5$.

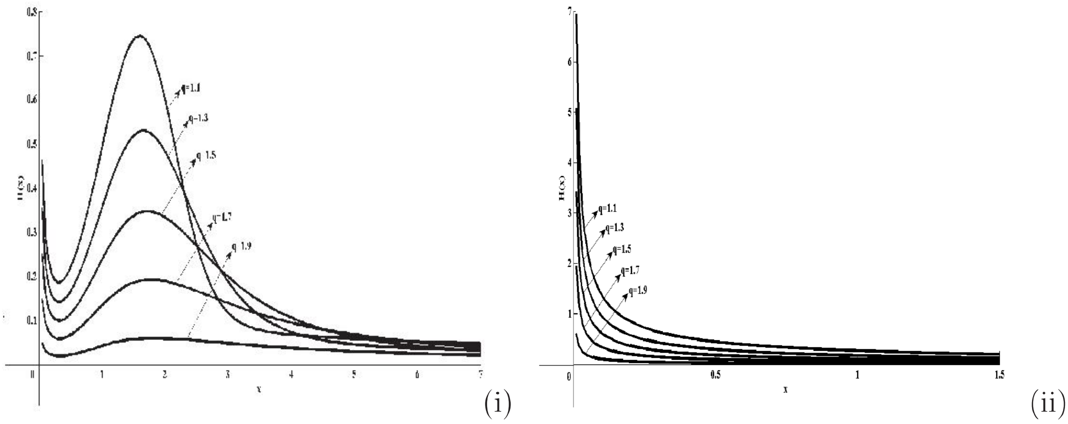


Figure 4 - Hazard function of the WM_q with parameters (i): $\pi_1 = \pi_2 = 0.5, \alpha_1 = 3, \alpha_2 = 0.4, \lambda_1 = 1.5, \lambda_2 = 7$ and (ii): $\pi_1 = \pi_2 = 0.7, \alpha_1 = 0.2, \alpha_2 = 0.5, \lambda_1 = 0.5, \lambda_2 = 0.7$.

From Al-Hussaini and Sultan (2001) we have

$$H(x) = h(x)H_1(x) + (1 - h(x))H_2(x) \tag{11}$$

where $h(x) = \frac{1}{1+\frac{(1-\pi)R_2(x)}{\pi R_1(x)}}$, $H_i(x) = \frac{f_i(x)}{R_i(x)}$, $R_i(x) = [(1 + (q-1)(\lambda_i x)^{\alpha_i})]^{\frac{q-2}{q-1}}$, $i = 1, 2$.
Then failure rate function given in (10) satisfies the following limits.

Lemma 2.1

$$\lim_{x\to 0} H(x) = 0$$

and

$$\lim_{x\to\infty} H(x) = 0.$$

Proof. We have $H_i(x) = \frac{f_i(x)}{R_i(x)}$, i=1,2.
Then

$$H_1(x) = \frac{\alpha_1 {\lambda_1}^{\alpha_1}(2-q)x^{\alpha_1-1}[1+(q-1)(\lambda_1 x)^{\alpha_1}]^{-\frac{1}{q-1}}}{[1+(q-1)(\lambda_1 x)^{\alpha_1}]^{\frac{q-2}{q-1}}} \tag{12}$$

$$= \frac{\alpha_1(2-q)}{(q-1)x[1+\frac{1}{(q-1)(\lambda_1 x)^{\alpha_1}}]} = \frac{\alpha_1(2-q)}{(q-1)x+\frac{1}{(\lambda_1)^{\alpha_1}x^{\alpha_1-1}}}. \tag{13}$$

When $x \to 0$, the denominator in (13) tends to infinity, so

$$\lim_{x\to 0} H_1(x) = 0. \text{ Similarly } \lim_{x\to 0} H_2(x) = 0.$$

Now consider

$$\lim_{x\to 0} h(x) = \frac{1}{1 + \lim\limits_{x\to 0}\frac{(1-\pi)R_2(x)}{\pi R_1(x)}} = \pi.$$

Then from (11), it can be shown that

$$\lim_{x\to 0} H(x) = 0.$$

Again we have $\frac{(1-\pi)R_2(x)}{\pi R_1(x)} \geq 0 \Rightarrow h(x) < \infty$. Also from (13), it can be shown that when $x \to \infty$ the denominator tends to ∞, so

$$\lim_{x\to \infty} H(x) = 0.$$

Hence the proof.

3- Identifiability :

Identifiability gives a unique representation for a class of mixtures. Estimation of parameters and testing hypotheses about a mixture distribution can be meaningfully discussed only if the family of mixing distributions is identifiable. Identifiability of mixtures may be found in several papers, including Teicher (1963), Yakowitz and Spragins (1968), Al-Hussaini and Ahmad (1981), Kent (1983), Ahmad (1988). We prove the identifiability of WM_q by using the following theorem of Chandra (1977).

Theorem 3.1 *(Chandra 1977): Let there be associated with each $F_i \in \Phi$ a transform ϕ_i having the domain of definition D_{ϕ_i}, and suppose that the mapping $M : F_i \to \phi_i$ is linear. Suppose also that there exists a total ordering $(\leq)$ of Φ such that*
(i) $F_1 \leq F_2$ $(F_1, F_2 \in \Phi)$ implies $D_{\phi_1} \subseteq D_{\phi_2}$
(ii) for each $F_i \in \Phi$, there exists some $s_1 \in D_{\phi_1}$, $\phi_1(s) \neq 0$ such that $\lim_{s\to s_1} \phi_2(s)/\phi_1(s) = 0$ for $F_1 \leq F_2$ $(F_1, F_2 \in \Phi)$.
Then the class Λ of all finite mixing distributions is identifiable relative to Φ.

By using the above theorem we prove the following Proposition.

Proposition 3.1 *The class of all finite mixing distributions relative to the pathway Weibull model is identifiable.*

Proof. Let x be a random variable having the density function and distribution function given in (4), (7) respectively. Then the s^{th} moment of the i^{th} pathway Weibull model is given by

$$\phi_i(s) = \mathbf{E}(x^s) = \frac{(2-q)}{{\lambda_i}^s (q-1)^{\frac{s}{\alpha_i}+1}} \frac{\Gamma(\frac{s}{\alpha_i}+1)\Gamma(\frac{1}{q-1}-\frac{s}{\alpha_i}-1)}{\Gamma(\frac{1}{q-1})},$$

$\min\{-\alpha_i\} < \mathbf{R}(s) < \min\{\frac{(2-q)\alpha_i}{q-1}\}$, i=1,2, $1 < q < 2$, α_i, $\lambda_i > 0$. From (7) we have

$$F_1 \leq F_2 \text{ when } \alpha_1 = \alpha_2 \text{ and } \lambda_1 < \lambda_2 \text{ for } x \geq 0 \tag{14}$$

and

$$F_1 \leq F_2 \text{ when } \lambda_1 = \lambda_2 \text{ and } \alpha_1 < \alpha_2 \text{ for } x \geq 0. \tag{15}$$

Now let $D_{\phi_1}(s) = (-\infty, \frac{\alpha_1(2-q)}{(q-1)})$ and $D_{\phi_2}(s) = (-\infty, \frac{\alpha_2(2-q)}{(q-1)})$. Since $\alpha_1 \leq \alpha_2$, from (14) and (15), we have $D_{\phi_1}(s) \subseteq D_{\phi_2}(s)$ and

$$\lim_{s \to \frac{\alpha_1(2-q)}{(q-1)}} \phi_1(s) = \lim_{s \to \frac{\alpha_1(2-q)}{(q-1)}} \frac{(2-q)}{{\lambda_1}^s (q-1)^{\frac{s}{\alpha_1}+1}} \frac{\Gamma(\frac{s}{\alpha_1}+1)\Gamma(\frac{1}{q-1}-\frac{s}{\alpha_1}-1)}{\Gamma(\frac{1}{q-1})} \to \infty. \tag{16}$$

Similarly, when $\lambda_1 = \lambda_2$ and $\alpha_1 < \alpha_2$, we have

$$\lim_{s \to \frac{\alpha_1(2-q)}{(q-1)}} \phi_2(s) = \lim_{s \to \frac{\alpha_1(2-q)}{(q-1)}} \frac{(2-q)}{{\lambda_2}^s (q-1)^{\frac{s}{\alpha_2}+1}} \frac{\Gamma(\frac{s}{\alpha_2}+1)\Gamma(\frac{1}{q-1}-\frac{s}{\alpha_2}-1)}{\Gamma(\frac{1}{q-1})} > 0. \tag{17}$$

From (16) and (17), we have

$$\lim_{s \to \frac{\alpha_1(2-q)}{(q-1)}} \frac{\phi_2(s)}{\phi_1(s)} = 0. \tag{18}$$

Hence by Theorem 3.1 the identifiability of WM_q is proved.

4- Special Cases :

The following known mixture models are particular cases of WM_q:

4.1 Exponential-Weibull mixture

If $q \to 1$, $\alpha_1 = 1$, the WM_q reduces to

$$f_3(x; \Psi_3) = \begin{cases} \pi\lambda_1 \mathrm{e}^{-(\lambda_1 x)} + (1-\pi)\alpha_2\lambda_2^{\alpha_2} x^{\alpha_2-1}\mathrm{e}^{-(\lambda_2 x)^{\alpha_2}} \\ 0, \ \text{otherwise} \end{cases}$$

for $\lambda_i > 0$, $\alpha_2 > 0$ $i = 1, 2$. This exponential-Weibull model was considered and studied by Erisoğlu et al. (2011) in their work on mixture model of two different distributions approach to model heterogeneous survival data. They illustrated that the mixture of different distributions is the appropriate model for the heterogeneous survival times. They proposed three models and among them one was the exponential-Weibull mixture. They have successfully applied the exponential-Weibull model for modeling failure times of oral irrigators dataset.

4.2 Weibull-Weibull mixture

If $q \to 1$, the WM_q reduces to

$$f_4(x; \Psi_4) = \begin{cases} \pi\alpha_1\lambda_1^{\alpha_1} x^{\alpha_1-1}\mathrm{e}^{-(\lambda_1 x)^{\alpha_1}} + (1-\pi)\alpha_2\lambda_2^{\alpha_2} x^{\alpha_2-1}\mathrm{e}^{-(\lambda_2 x)^{\alpha_2}} \\ 0, \ \text{otherwise} \end{cases}$$

for $\alpha_i > 0$. $\lambda_i > 0$, $i = 1, 2$. This Weibull-Weibull model was considered by Jiang and Murthy (1995). They characterized the 2-fold Weibull mixture models in terms of the Weibull probability plotting, and examined the graphical plotting approach to determine if a given data set can be modeled by such models.

4.3 Exponential-Exponential mixture

If $q \to 1$, $\alpha_1 = 1$, and $\alpha_2 = 1$, the WM_q reduces to

$$f_5(x;\Psi_5) = \begin{cases} \pi\lambda_1 e^{-(\lambda_1 x)} + (1-\pi)\lambda_2 e^{-(\lambda_2 x)} \\ 0, \text{ otherwise} \end{cases}$$

for $\lambda_i > 0$, $i = 1, 2$. This exponential-exponential model was considered by Jaheen (2005) in connection with the problem of estimating the parameters of the finite mixture of two exponential distributions based on record statistics. They used the maximum likelihood method and Bayes method of estimation for estimating the parameters. They have also given a comparison of the Bayes estimates with their corresponding maximum likelihood estimates based on a Monte Carlo simulation study.

4.4 Rayleigh-Rayleigh mixture

If $q \to 1$, $\alpha_1 = 2$, and $\alpha_2 = 2$, the WM_q reduces to

$$f_6(x;\Psi_6) = \begin{cases} 2\pi{\lambda_1}^2 x e^{-(\lambda_1 x)^2} + 2(1-\pi){\lambda_2}^2 x e^{-(\lambda_2 x)^2} \\ 0, \text{ otherwise} \end{cases}$$

for $\lambda_i > 0$, $i = 1, 2$. This Rayleigh-Rayleigh model was studied by Soliman (2006) for solving the problem of estimating the parameters and some lifetime parameters such as reliability and hazard functions using progressively type-II censored samples from a heterogeneous population. They used both classical and Bayesian approach and a comparison between the estimates obtained by these two methods is also given.

5- Application in stress-strength analysis :

Let Y represent the strength of a component which is subject to a stress X, then $R = Pr\{X < Y\}$ is the measure of system performance or reliability of the system which arises in the context of mechanical reliability. The system fails if and only if at any time the applied stress is greater than its strength. More details on stress-strength analysis can be found in Kotz et al. (2007).

Let X and Y be independently and identically distributed as WM_q with parameters $(\alpha_1,\ \alpha_2\ ,\lambda_1,\ \lambda_2,\ q)$. Also let $f_1^*(.)$ and $f_2^*(.)$ be the probability density functions of X, Y and $F_1(.)$ and $F_2(.)$ be the corresponding distribution functions.Then

$$R = Pr\{X < Y\} = \int_0^\infty \int_0^y f_1^*(x) f_2^*(y) \mathrm{d}x \wedge \mathrm{d}y,$$

where $\wedge$ is the wedge product discussed in Mathai (1997).

$$\Rightarrow R = \int_0^\infty F_1(y) f_2^*(y) \mathrm{d}y,$$

where $F_1(y) = \pi[1 - [1 + (q-1)(\lambda_1 y)^{\alpha_1}]^{\frac{q-2}{q-1}}] + (1-\pi)\left[1 - [1 + (q-1)(\lambda_2 y)^{\alpha_2}]^{\frac{q-2}{q-1}}\right]$ $y \geq 0,\ \alpha_i > 0,\ \lambda_i > 0,\ 1 < q < 2,\ i = 1, 2.$

$$\begin{aligned}
R &= \int_0^\infty \left[1 - \pi(1 + (q-1)(\lambda_1 y)^{\alpha_1})^{\frac{q-2}{q-1}} - (1-\pi)(1 + (q-1)(\lambda_2 y)^{\alpha_2})^{\frac{q-2}{q-1}}\right] f_2^*(y) \mathrm{d}y \\
&= 1 - \int_0^\infty \left[\pi^2 \alpha_1 \lambda_1^{\alpha_1}(2-q) y^{\alpha_1 - 1}[1 + (q-1)(\lambda_1 y)^{\alpha_1}]^{\frac{q-3}{q-1}}\right] \mathrm{d}y \\
&\quad - \int_0^\infty \left[(1-\pi)^2 \alpha_2 \lambda_2^{\alpha_2}(2-q) y^{\alpha_2 - 1}[1 + (q-1)(\lambda_2 y)^{\alpha_2}]^{\frac{q-3}{q-1}}\right] \mathrm{d}y \\
&\quad - \int_0^\infty \left[\pi^2(1-\pi)\alpha_1 \lambda_1^{\alpha_1}(2-q) y^{\alpha_1 - 1}[1 + (q-1)(\lambda_1 y)^{\alpha_1}]^{-\frac{1}{q-1}}[1 + (q-1)(\lambda_2 y)^{\alpha_2}]^{\frac{q-2}{q-1}}\right] \mathrm{d}y \\
&\quad - \int_0^\infty \left[\pi^2(1-\pi)\alpha_2 \lambda_2^{\alpha_2}(2-q) y^{\alpha_2 - 1}[1 + (q-1)(\lambda_2 y)^{\alpha_2}]^{-\frac{1}{q-1}}[1 + (q-1)(\lambda_1 y)^{\alpha_1}]^{\frac{q-2}{q-1}}\right] \mathrm{d}y \\
&= 1 - [I_1 + I_2 + I_3 + I_4]
\end{aligned}$$

where I_1, I_2, I_3, I_4 are the first, second, third and fourth integrals respectively. By direct integration we get

$$I_1 = \frac{\pi^2}{2} \text{ and } I_2 = \frac{(1-\pi)^2}{2}.$$

The integrals I_3 and I_4 can be integrated by using the Mellin Convolution technique. Consider

$$I_3 = \pi(1-\pi)\alpha_1 \lambda_1^{\alpha_1}(2-q) \int_0^\infty \left[y^{\alpha_1 - 1}[1 + (q-1)(\lambda_1 y)^{\alpha_1}]^{-\frac{1}{q-1}}[1 + (q-1)(\lambda_2 y)^{\alpha_2}]^{\frac{q-2}{q-1}}\right] \mathrm{d}y$$

The integrand can be taken as a product of two integrable functions. Consider the transformation $u = \frac{x_1}{x_2}$ and $v = x_2$, where x_1 and x_2 be two independently distributed real scalar positive random variables with density functions $h_1(x_1)$ and $h_2(x_2)$.
$u = \frac{x_1}{x_2} \Rightarrow x_1 = uv$ and $x_2 = v$, $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = v\mathrm{d}u \wedge \mathrm{d}v$. Then the joint density of u and v is $g(u, v) = vh_1(uv)h_2(v)$. Now the marginal density of u, denoted by $g_1(u)$ is given by $g_1(u) = \int_v vh_1(uv)h_2(v)\mathrm{d}v$. Let

$$h_1(x_1) = \begin{cases} c_1[1 + (q-1)(x_1)^{\alpha_2}]^{\frac{q-2}{q-1}}, & x_1 \geq 0,\ 1 < q < 2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h_2(x_2) = \begin{cases} c_2 x_2^{\alpha_1 - 2}(1 + (q-1)(\lambda_1 x_2)^{\alpha_1})^{-\frac{1}{q-1}}, & x_2,\ \alpha_1,\ \lambda_1 \geq 0,\ 1 < q < 2, \\ 0, & \text{otherwise} \end{cases}$$

where c_1 and c_2 are normalizing constants. Then

$$g_1(u) = c_1c_2\int_0^\infty v^{\alpha_1-1}(1+(q-1)(\lambda_1 v)^{\alpha_1})^{-\frac{1}{q-1}}[1+(q-1)(uv)^{\alpha_2}]^{\frac{q-2}{q-1}}\mathrm{d}v,\ u=\lambda_2. \quad (19)$$

Now

$$\begin{aligned}\mathbf{E}(x_1^{s-1}) &= c_1\int_0^\infty x_1{}^{s-1}[1+(q-1)(x_1)^{\alpha_2}]^{\frac{q-2}{q-1}}\mathrm{d}x_1 = \frac{c_1}{\alpha_2(q-1)^{\frac{s}{\alpha_2}}}\frac{\Gamma(\frac{s}{\alpha_2})\Gamma(\frac{2-q}{q-1}-\frac{s}{\alpha_2})}{\Gamma(\frac{2-q}{q-1})},\\ &\quad 0<\mathbf{R}(\mathbf{s})<\frac{\alpha_2(2-q)}{q-1} \text{ and}\\ \mathbf{E}(x_2^{1-s}) &= c_2\int_0^\infty x_2{}^{\alpha_1-s-1}(1+(q-1)(\lambda_1 x_2)^{\alpha_1})^{-\frac{1}{q-1}}\mathrm{d}x_2\\ &= \frac{c_2}{\alpha_1(\lambda_1)^{\alpha_1-s}(q-1)^{1-\frac{s}{\alpha_1}}}\frac{\Gamma(1-\frac{s}{\alpha_1})\Gamma(\frac{1}{q-1}+\frac{s}{\alpha_1}-1)}{\Gamma(\frac{1}{q-1})},\\ &\quad \frac{(q-2)\alpha_1}{q-1}<\mathbf{R}(s)<\alpha_1.\end{aligned}$$

Now

$$\begin{aligned}\mathbf{E}(u^{s-1}) &= \mathbf{E}(x_1^{s-1})\mathbf{E}(x_2^{1-s})\\ &= \frac{c_1c_2}{\alpha_1\alpha_2\lambda_1{}^{\alpha_1-s}(q-1)^{1-\frac{s}{\alpha_1}+\frac{s}{\alpha_2}}}\frac{\Gamma(\frac{s}{\alpha_2})\Gamma(\frac{2-q}{q-1}-\frac{s}{\alpha_2})}{\Gamma(\frac{2-q}{q-1})}\frac{\Gamma(1-\frac{s}{\alpha_1})\Gamma(\frac{1}{q-1}+\frac{s}{\alpha_1}-1)}{\Gamma(\frac{1}{q-1})},\\ &\quad 0<\mathbf{R}(\mathbf{s})<\frac{\alpha_2(2-q)}{q-1},\ \frac{(q-2)\alpha_1}{q-1}<\mathbf{R}(s)<\alpha_1.\end{aligned}$$

Now the density of u is obtained by the inverse Mellin transform. The detailed existence conditions for Mellin and Inverse Mellin transforms are available in Mathai (1993). Thus

$$\begin{aligned}g_1(u) &= \frac{c_1c_2}{\alpha_1\alpha_2\lambda_1{}^{\alpha_1}(q-1)\Gamma(\frac{2-q}{q-1})\Gamma(\frac{1}{q-1})}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big[\Gamma(\frac{s}{\alpha_2})\Gamma(\frac{2-q}{q-1}-\frac{s}{\alpha_2})\Gamma(1-\frac{s}{\alpha_1}) \quad (20)\\ &\quad \Gamma(\frac{1}{q-1}+\frac{s}{\alpha_1}-1)\left(\frac{u}{\lambda_1(q-1)^{(\frac{1}{\alpha_1}-\frac{1}{\alpha_2})}}\right)^{-s}\mathrm{d}s\Big],\\ &\quad 0<\mathbf{R}(\mathbf{s})<\frac{\alpha_2(2-q)}{q-1},\ \frac{(q-2)\alpha_1}{q-1}<\mathbf{R}(s)<\alpha_1.\end{aligned}$$

Equating (19) and (20) we get

$$c_1c_2\int_0^\infty v^{\alpha_1-1}(1+(q-1)(\lambda_1 v)^{\alpha_1})^{-\frac{1}{q-1}}[1+(q-1)(uv)^{\alpha_2}]^{\frac{q-2}{q-1}}\mathrm{d}v=$$

$$\frac{c_1c_2}{\alpha_1\alpha_2\lambda_1{}^{\alpha_1}(q-1)\Gamma(\frac{2-q}{q-1})\Gamma(\frac{1}{q-1})}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big[\Gamma(\frac{s}{\alpha_2})\Gamma(\frac{2-q}{q-1}-\frac{s}{\alpha_2})\Gamma(1-\frac{s}{\alpha_1})\Gamma(\frac{1}{q-1}+\frac{s}{\alpha_1}-1)\times$$

$$\left(\frac{u}{\lambda_1(q-1)(\frac{1}{\alpha_1}-\frac{1}{\alpha_2})}\right)^{-s} \mathrm{d}s\Big],$$

$0 < \mathbf{R}(\mathbf{s}) < \frac{\alpha_2(2-q)}{q-1}, \ \frac{(q-2)\alpha_1}{q-1} < \mathbf{R}(s) < \alpha_1.$

$$\Rightarrow \int_0^\infty v^{\alpha_1-1}(1+(q-1)(\lambda_1 v)^{\alpha_1})^{-\frac{1}{q-1}}[1+(q-1)(uv)^{\alpha_2}]^{\frac{q-2}{q-1}}\mathrm{d}v$$
$$= \frac{1}{\alpha_1\alpha_2\lambda_1{}^{\alpha_1}(q-1)\Gamma(\frac{2-q}{q-1})\Gamma(\frac{1}{q-1})} \times H^{2,2}_{2,2}\left[\frac{1}{\lambda_1(q-1)(\frac{1}{\alpha_1}-\frac{1}{\alpha_2})}\Big|^{(\frac{2q-3}{q-1},\frac{1}{\alpha_2}),\ (0,\frac{1}{\alpha_1})}_{(0,\frac{1}{\alpha_2}),\ (\frac{2-q}{q-1},\frac{1}{\alpha_1})}\right],$$

$\alpha_1 > 0,\ \lambda_1 > 0,\ \alpha_2 > 0,\ 1 < q < 2,$ whereH(.)denotes the H-function (see Mathai and Saxena (1978)).

$$\text{Then } I_3 = \frac{\pi(1-\pi)(2-q)}{\alpha_2(q-1)\Gamma(\frac{2-q}{q-1})\Gamma(\frac{1}{q-1})}H^{2,2}_{2,2}\left[\frac{1}{\lambda_1(q-1)(\frac{1}{\alpha_1}-\frac{1}{\alpha_2})}\Big|^{(\frac{2q-3}{q-1},\frac{1}{\alpha_2}),\ (0,\frac{1}{\alpha_1})}_{(0,\frac{1}{\alpha_2}),\ (\frac{2-q}{q-1},\frac{1}{\alpha_1})}\right], \alpha_1 > 0,$$
$$\lambda_1 > 0,\ \alpha_2 > 0,\ 1 < q < 2.$$

Proceeding in the same way we get,

$$I_4 = \frac{\pi(1-\pi)(2-q)}{\alpha_1(q-1)\Gamma(\frac{2-q}{q-1})\Gamma(\frac{1}{q-1})}H^{2,2}_{2,2}\left[\frac{1}{\lambda_2(q-1)(\frac{1}{\alpha_2}-\frac{1}{\alpha_1})}\Big|^{(\frac{2q-3}{q-1},\frac{1}{\alpha_1}),\ (0,\frac{1}{\alpha_2})}_{(0,\frac{1}{\alpha_1}),\ (\frac{2-q}{q-1},\frac{1}{\alpha_2})}\right], \alpha_1 > 0,\ \lambda_2 > 0,$$
$$\alpha_2 > 0,\ 1 < q < 2.$$

Substituting the values of I_1, I_2, I_3, and I_4 we get,

$$R = Pr\{X < Y\} = \frac{1}{2} + \pi(1-\pi) - \frac{\pi(1-\pi)(2-q)}{(q-1)\Gamma(\frac{2-q}{q-1})\Gamma(\frac{1}{q-1})} \times$$
$$\left[\frac{1}{\alpha_1}H^{2,2}_{2,2}\left[\frac{1}{\lambda_2(q-1)(\frac{1}{\alpha_2}-\frac{1}{\alpha_1})}\Big|^{(\frac{2q-3}{q-1},\frac{1}{\alpha_1}),\ (0,\frac{1}{\alpha_2})}_{(0,\frac{1}{\alpha_1}),\ (\frac{2-q}{q-1},\frac{1}{\alpha_2})}\right] + \frac{1}{\alpha_2}H^{2,2}_{2,2}\left[\frac{1}{\lambda_1(q-1)(\frac{1}{\alpha_1}-\frac{1}{\alpha_2})}\Big|^{(\frac{2q-3}{q-1},\frac{1}{\alpha_2}),\ (0,\frac{1}{\alpha_1})}_{(0,\frac{1}{\alpha_2}),\ (\frac{2-q}{q-1},\frac{1}{\alpha_1})}\right]\right],$$

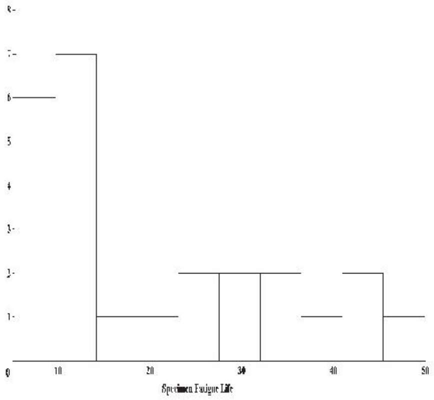
$\alpha_1 > 0,\ \lambda_1 > 0,\ \alpha_2 > 0,\ \lambda_2 > 0,\ 1 < q < 2.$

6- Data Analysis :

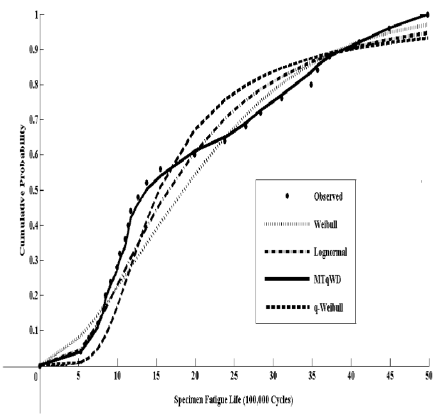
The data used in this section is the fatigue lives data set which consists of 25 specimens of two different types listed in ascending order (Ling and Pan, 1998). The pdf of Weibull model, pathway Weibull model, lognormal model and WM_q are applied to the data. A graphical comparison of the fitted empirical cumulative distribution function (cdf) with cdf of the Weibull model, pathway Weibull model, lognormal model and the WM_q for fatigue lives data are given in Figure 5(ii). Histogram of the data is given in Figure 5(i). Keeping the parameters $\pi,\ \alpha_i,\ \lambda_i,\ i = 1,2$ fixed, by using the method of moments, the estimate of the pathway parameter q is obtained as follows:

$$\hat{q} = 1.57434.$$

The graphical representation shown in Figure 5(ii) clearly shows that the WM_q is the best representation of the sample data among these models. We calculated the Kolmogorov-Smirnov (K-S) statistic for these four different probability models. For Weibull model, the value of K-S statistic is obtained as 0.1737. For the lognormal model, the value of the statistic is 0.1343, for the pathway Weibull model, the value of the K-S statistic is 0.1582 and for the WM_q, the value is 0.0200. The table value of the K-S statistic for $n = 25$ and a significance level of 0.05 is 0.270. All the four models are consistent with the data. But the distance measure of the statistic of WM_q is less than that of the other three probability models which can be clearly understood from the figure. Hence we can conclude that our new model fits the data better than the other three. Here, the mathematical software MATLAB is used for the data analysis.



(i)



(ii)

Figure 5 - (i): Histogram of the data. (ii): A comparison of the theoretical cdfs of failure and the observed failure probability for different densities.

Conclusions :

In this paper, we illustrate the efficiency of pathway idea in modeling heterogeneous real data sets through a finite mixture of two pathway Weibull models (WM_q) and discuss some of its statistical properties. We made a path from the WM_q for obtaining some known mixture models that exists in the literature. In addition, the identifiability property of the WM_q is proved. A theorem which is useful in stress-strength analysis is also given. Further, a comparison study which shows the efficiency of the new model over the existing ones in the literature is demonstrated through graphs.

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