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TENSOR PRODUCT OF INTUITIONISTIC FUZZY MODULES

P. K. Sharma and Chandni*

P.G. Department of Mathematics,
D.A.V. College, Jalandhar, Punjab, INDIA

E-mail : pksharma@davjalandhar.com

*Department of Mathematics,
Lovely Professional University, Jalandhar, Punjab, INDIA

E-mail : chandni16041986@gmail.com

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Abstract: In this paper, we introduce the concept of tensor product between intuitionistic fuzzy submodules. We establish a formal framework for the tensor product operation, examining its properties and applications within the context of intuitionistic fuzzy modules. We then establish a relationship between the Hom functor and the tensor product in the category of intuitionistic fuzzy modules. The connection between tensor products and hom-functors in some algebraic structures, such as modules, is made possible via a natural isomorphism known as the Hom-Tensor adjunction and it establishes a relationship between $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B \otimes A, C)$ and $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B, C))$. An application of tensor product of intuitionistic fuzzy modules can be used in decision-making processes by embracing ambiguity and vagueness, making it a valuable tool when exact data is lacking.

Keywords and Phrases: Hom functor, Tensor product, Category, Intuitionistic fuzzy R-homomorphism.

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1. Introduction

The tensor product is a fundamental construction in algebra and module theory. It provides a way to extend the notion of the product of modules, allowing for a richer understanding of module structures. Hom-functors and tensor products are concepts from category theory, often used in algebra and algebraic topology. For any two objects X and Y in a category C, there is a Hom-functor denoted as $Hom(A, B)$. It assigns to each object X in the category C the set of morphisms from X to Y. In other words, $Hom(X, Y)$ represents the set of all morphisms from object X to object Y. This concept is fundamental in understanding maps between objects in a category.

The tensor product is a construction that generalizes the concept of the product of modules to other algebraic structures. In the context of modules over a ring, the tensor product of two modules M and N over a ring R, denoted as $M \otimes N$, is itself a module. It has the property that bilinear maps from $M \times N$ to another module K uniquely factor through $M \otimes N$.

Connecting the two concepts, in certain algebraic structures like modules, there is a natural isomorphism known as the Hom-Tensor adjunction, linking Hom-functors and tensor products. It establishes a relationship between $Hom(M \otimes N, K)$ and $Hom(M, Hom(N, K))$. In the context of intuitionistic fuzzy modules, the tensor product can be applied to extend the module operations and provide a framework for dealing with uncertainty. We extend the concept of tensor product from classical modules to intuitionistic fuzzy modules. We study the tensor product of two intuitionistic fuzzy modules, incorporating the notions of intuitionistic fuzzy sets and their operations. We develop algebraic operations on the tensor product of intuitionistic fuzzy modules. This involves defining addition and scalar multiplication in a way that accommodates the inherent uncertainty and vagueness present in intuitionistic fuzzy sets. We study module homomorphisms and isomorphisms involving the tensor product and explore how mappings between intuitionistic fuzzy modules interact with the tensor product, preserving the uncertainty characteristics of the involved modules. We generalize classical results from module theory to the realm of intuitionistic fuzzy modules through the tensor product. We explore how classical module properties and theorems extend or adapt in the context of intuitionistic fuzzy modules. With this, we explore interdisciplinary applications where intuitionistic fuzzy modules and their tensor product can be used. This might include areas such as decision sciences, optimization, pattern recognition, or any field where uncertainty needs to be explicitly considered in a modular structure.

In [6], B. Davaaz and others published the perception of the intuitionistic fuzzy submodule of a module. Later on, numerous mathematicians in [4, 7, 12, 13, 16] made significant contributions to the study of intuitionistic fuzzy submodules. In the category of fuzzy modules, H.X. Lui [9] established the relationship between fuzzy projective module and Hom functor. Authors in [15] studied the relationship between the intuitionistic fuzzy projective (injective) modules with retraction, coretraction morphisms in the category of intuitionistic fuzzy modules.

In our earlier paper [14], we defined a category ($\mathbf{C_{R\text{-}IFM}}$) of intuitionistic fuzzy modules over the ring R where the classes of all intuitionistic fuzzy modules and intuitionistic fuzzy R-homomorphisms constitute objects and morphisms. Hom functors and tensor functors play an important role in ring theory. In section 3, we study the concept of tensor product in the category of intuitionistic fuzzy modules(C_{R-IFM}). Furthermore, we established the existence of the tensor product of two intuitionistic fuzzy R-modules exists. The connection that exists between Hom functor and tensor functor is then investigated.

2. Preliminary

For conceptual concepts about Category theory and related areas, we follow Tom Leinster [8] and O. Wyler [18] and concepts about intuitionistic fuzzy modules and related results we follow Basnet [4] and Davvaz [6]. Throughout the paper, R is a commutative ring with unity 1 and $1 \neq 0$. M is a unitary R-module, θ is a zero element of M and I represents the unit interval $[0, 1]$.

Definition 2.1. [14] *A mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an intuitionistic fuzzy set(IFS) on X where the mappings $\mu_A : X \to I$ and $\nu_A : X \to I$ denotes the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A, respectively with the condition that $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.*
An intuitionistic fuzzy set A in X can be represented as an object of the form

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}.$$

Remark 2.2.
(i) When $\mu_A(x) + \nu_A(x) = 1$, $\forall x \in X$ i.e., $\nu_A(x) = 1 - \mu_A(x)$. Then A is called a fuzzy set.
(ii) We denote the IFS $A = \{< x, \mu_A(x), \nu_A(x) >; \forall x \in X\}$ by $A = (\mu_A, \nu_A)$.

Definition 2.3. [14] *An IFS A of an R-module M is called an intuitionistic fuzzy submodule (IFSM) of M, if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:*
(i) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)$;
(ii) $\mu_A(rx) \geq \mu_A(x)$ and $\nu_A(rx) \leq \nu_A(x)$;
(iii) $\mu_A(\theta) = 1$ and $\nu_A(\theta) = 0$, where θ is a zero element of M.
Condition (i) and (ii) can be combined to a single condition $\mu_A(rx + sy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(rx + sy) \leq \nu_A(x) \vee \nu_A(y)$, where $r, s \in R$ and $x, y \in M$.

Remark 2.4.

(i) The set of intuitionistic fuzzy submodules of R-module M is denoted by IFSM(M).
(ii) We denote the IFSM A of an R-module M by $(\mu_A, \nu_A)_M$.

Definition 2.5. [14] *Let K as a submodule of an R-module M. The intuitionistic fuzzy characteristic function of K is denoted by χ_K and is described by $\chi_K(a) = (\mu_{\chi_K}(a), \nu_{\chi_K}(a))$, where*

$$\mu_{\chi_K}(a) = \begin{cases} 1, & \text{if } a \in K \\ 0, & \text{if } a \notin K \end{cases}; \quad \nu_{\chi_K}(a) = \begin{cases} 0, & \text{if } a \in K \\ 1, & \text{if } a \notin K. \end{cases}$$

Clearly, χ_K is an IFSM of M. The IFSMs $\chi_{\{\theta\}}, \chi_M$ are called trivial IFSMs of module M. Any IFSM of the module M apart from this is called proper IFSM.

Definition 2.6. [14] *Let $A = (\mu_A, \nu_A)_M, B = (\mu_B, \nu_B)_N$ are IFSM of R-modules M and N respectively. Then the map $f : A \to B$ is called an intuitionistic fuzzy R-homomorphism (or IF R-hom) from A to B if*

(i) $f : M \to N$ is R-homomorphism and

(ii) $\mu_B(f(a)) \geq \mu_A(a)$ and $\nu_B(f(a)) \leq \nu_A(a), \forall a \in M$.

To avoid confusion between an R-homomorphism $f : M \to N$ and an intuitionistic fuzzy R-homomorphism $f : A \to B$. We denote the latter by $\bar{f} : A \to B$. So, given an IF R-homomorphism $\bar{f} : A \to B$, $f : M \to N$ is the underlying R-homomorphism of $\bar{f}$. The set of all IF R-homomorphisms from A to B is denoted by $Hom(A, B)$.

Definition 2.7. [14] *A category of R-modules denoted by $\boldsymbol{C_{R\text{-}M}}$ has R-modules as objects and R-homomorphisms as morphisms, with composition of morphisms defined as composition of mappings.*

Definition 2.8. [8] *Let $C = (Ob(C), Hom(C), id, o)$ and $D = (Ob(D), Hom(D), id, o)$ be two categories and let $F_1 : Ob(C) \to Ob(D)$ and $F_2 : Hom(C) \to Hom(D)$ be maps. Then the quadruple $F = (C, D, F_1, F_2)$ is a functor provided:*
(i) $X \in Ob(C)$ implies $F_1(X) \in Ob(D)$;
(ii) $f \in Hom(X, Y)$ implies $F_2(f) \in Hom(F_1(X), F_1(Y))$, $\forall\ X, Y \in Ob(C)$;
(iii) F_2 preserves composition, i.e., $F_2(gof) = F_2(g)oF_2(f)$, $\forall\ f \in Hom(X, Y)$ and $g \in Hom(Y, Z)$;
(iv) F preserves identities, i.e., $F_2(e_X) = e_{F_1(X)}$, $\forall\ X \in Ob(C)$.

Remark 2.9. [8]
(i) Instead of $F_1(X)$ we write $F(X)$.

(ii) In preference to $F_2(f)$ we write $F(f)$.
(iii) We call $F : C \to D$ a functor from C to D.
(iv) A functor defined above is called a covariant functor that preserves both:

- *The domains, the codomains and identities.*
- *The composition of arrows, especially it retains the path of the arrows.*

(v) A contravariant functor F is similar to the covariant functor in addition to the other side of the arrow, $F(f) : F(Y) \to F(X)$ and $F(gof) = F(f)oF(g), \forall f \in Hom(X, Y), g \in Hom(Y, Z)$.
Thus a contravariant functor $F : C \to D$ is the same as a covariant functor $F : C^{op} \to D$.

Proposition 2.10. [14] *Let $Hom_{\boldsymbol{C_{R\text{-}IFM}}}(A, B)$ be the set of all IF R-homomorphisms from the IFSM A of R-module M into the IFSM B of R-module N. Then $Hom_{\boldsymbol{C_{R\text{-}IFM}}}$ (A, B) is an abelian additive group. Moreover, it is a unitary R-module when R is a commutative ring with unity.*

Theorem 2.11. [14] *Let $A = (\mu_A, \nu_A)_M$ and $B = (\mu_B, \nu_B)_N$ are two IF modules of R-modules M and N respectively. Then the function $\beta : Hom(A, B) \to I \times I$ on R-module $Hom(A, B)$ defined by*

$$\beta(\bar{f}) = (\mu_{\beta(\bar{f})}, \nu_{\beta(\bar{f})})$$

where $\mu_{\beta(\bar{f})} = \wedge\{\mu_B(\bar{f}(a)) : a \in M\}$ and $\nu_{\beta(\bar{f})} = \vee\{\nu_B(\bar{f}(a)) : a \in M\}$ is an intuitionistic fuzzy submodule of $Hom(A, B)$.

Definition 2.12. [16] *Let A, B and C be IFSMs of R-modules M,N and P respectively. A sequence of the form*

$$\bar{0} \longrightarrow A \xrightarrow{\bar{f}} B \xrightarrow{\bar{g}} C \longrightarrow \bar{0}$$

is said to be an intutionistic fuzzy short exact sequence if $\bar{f}$ is an IF-monomorphism, $\bar{g}$ is an IF-epimorphism and $Im(\bar{f}) = ker(\bar{g})$. We abbreviate the intuitionistic fuzzy short exact sequence by IFSE sequence.

Example 2.13. [16] Let $A = \chi_Z$, $B = \chi_{nZ}$ and $C = \chi_{\frac{Z}{nZ}}$ then

$$\bar{0} \longrightarrow A \xrightarrow{\bar{i}} B \xrightarrow{\bar{\pi}} C \longrightarrow \bar{0}$$

is a IFSE sequence where $\bar{i}$ and $\bar{\pi}$ are IF-inclusion map and natural IF-epimorphism respectively.

Example 2.14. [16] Let $M = 6Z$. Define the IFSs A and B of Z-module M by

$$\mu_A(x) = \begin{cases} 0.1, & \text{if } x \neq 0 \\ 1, & \text{if } x \doteq 0 \end{cases}; \quad \nu_A(x) = \begin{cases} 0.4, & \text{if } x \neq 0 \\ 0, & \text{if } x \doteq 0. \end{cases}$$

and

$$\mu_B(y) = \begin{cases} 0.6, & \text{if } y \neq 0 \\ 1, & \text{if } y \doteq 0 \end{cases}; \quad \nu_B(y) = \begin{cases} 0.3, & \text{if } y \neq 0 \\ 0, & \text{if } y \doteq 0. \end{cases}$$

Clearly, A and B are IF Z- modules and 0_M is a zero IFSM of Z defined by $0_M(x) = (0, 1)$ for all $x \in M$. Then

$$\bar{0} \longrightarrow 0_M \xrightarrow{\bar{f}} A \xrightarrow{\bar{g}} B \longrightarrow \bar{0}$$

is a IFSE sequence, where $\bar{f}$ is an IF-inclusion map and $\bar{g}$ is the natural IF-epimorphism.

Remark 2.15. *In* [16], *it is shown that* $Hom_{\boldsymbol{C_{R\text{-}IFM}}} : \boldsymbol{C_{R\text{-}IFM}} \to \boldsymbol{C_{R\text{-}IFM}}$ *is an invariant functor in the sense that it is both a covariant and a contravariant functor. Also,* $Hom_{\boldsymbol{C_{R\text{-}IFM}}}(A, -)$ *is not left exact in* $\boldsymbol{C_{R\text{-}IFM}}$. *However,* $Hom_{\boldsymbol{C_{R\text{-}IFM}}}(A, -)$ *and* $Hom_{\boldsymbol{C_{R\text{-}IFM}}}(-, A)$ *preserve the IFSEs*

$\bar{0} \longrightarrow Hom_{\boldsymbol{C_{R\text{-}IFM}}}(A, B) \xrightarrow{F\bar{f}=\bar{f}_*} Hom_{\boldsymbol{C_{R\text{-}IFM}}}(A, C) \xrightarrow{F\bar{g}=\bar{g}_*} Hom_{\boldsymbol{C_{R\text{-}IFM}}}(A, D)$

and

$B \xrightarrow{\bar{f}} C \xrightarrow{\bar{g}} D \longrightarrow \bar{0}$ *respectively, provided that* $\bar{f}$ *is an IF-split.*

Now, we will discuss about the basic concept of tensor product in the Category of R-modules (C_{R-M}) as discussed in [1].

Definition 2.16. [1] *Let* M *be an* R*-module and* S *is subset of* M. *The set of all finite linear combinations of the elements of* S *denoted by* $L(S)$, *i.e.,* $L(S) = \{\sum_{i=1}^{n} r_i x_i : r_i \in R, x_i \in S, n \in \mathbb{N}\}$, *is a submodule of* M *and it is the smallest submodule* M *that contains* S.

Definition 2.17. [1] *Let* M, N *and* P *are* R*-modules. An* R*-homomorphism* $f : M \times N \to P$ *is said to be an* R*-biadditive provided that for all* $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ *and* $r \in R$,

(i) $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$;

(ii) $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$;

(iii) $f(xr, y) = f(x, ry) = rf(x, y)$.

Definition 2.18. [1] *A tensor product of M and N over R is denoted by $M \otimes N$ and defined as*

$$M \otimes N = (M \times N)/L(S)$$

Being the quotient module of R-module by its submodule, the tensor product $M \otimes N$ is also an R-module. Then there exists an R-homomorphism $\tau : M \times N \to M \otimes N$ such that

$$\tau(m, n) = (m, n) + L(S),$$

for all $m \in M, n \in N$. we will denote $\tau(m, n)$ by $m \otimes n$.

Definition 2.19. [1] *A tensor product of M and N over R is an R-module $M \otimes N$ which is equipped with an R-biadditive $\tau : M \times N \to M \otimes N$ such that for each R-module P and each R-biadditive $\psi : M \times N \to P$, there is an unique R-homomorphism $\phi : M \otimes N \to P$ such that $\phi \circ \tau = \psi$, i.e., the following diagram commutes:*

$$\begin{tikzcd} M \times N \arrow[r, "\tau"] \arrow[dr, "\psi"'] & M \otimes N \arrow[d, "\phi"] \\ & P \end{tikzcd}$$

Tensor product of M and N is denoted by the pair $(M \otimes N, \tau)$.

Theorem 2.20. [1] *The tensor product of two R-modules exist and it is unique upto isomorphism.*

3. Tensor Products in $\mathbf{C_{R\text{-}IFM}}$

The construction of tensor products gives a most characteristic strategy for joining two modules. This section covers the structure and properties of tensor products derived from two intuitionistic fuzzy modules in the $C_{\mathbf{R\text{-}IFM}}$ category. In addition, we investigate whether there is a connection between Hom functor and tensor product.

Definition 3.1. *Let $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ be IFSMs of R-modules M and N respectively. We defined the cartesian product $A \times B$ on $L(M \times N)$ is an IFS defined as*

$$\mu_{A \times B}(\sum(x_i, y_i)) = (\mu_A \times \mu_B)(\sum(x_i, y_i)) = \wedge\{\vee\{\mu_A(x_i), \mu_B(y_i)|i \in I\}\}$$

and

$$\nu_{A\times B}(\sum(x_i,y_i)) = (\nu_A \times \nu_B)(\sum(x_i,y_i)) = \vee\{\wedge\{\nu_A(x_i),\nu_B(y_i)|i\in I\}\}$$

Proposition 3.2. *Let $A=(\mu_A,\nu_A)$, $B=(\mu_B,\nu_B)$ be IFSMs of R-modules M and N respectively. Then $A\times B$ is an IFSM on $L(M\times N)$.*

Proof. We will claim that

(i) $\mu_{A\times B}(\theta_1,\theta_2)=1$ and $\nu_{A\times B}(\theta_1,\theta_2)=0$, where θ_1 and θ_2 are zero elements of M and N respectively.

(ii) $\mu_{A\times B}(\sum(x_i,y_i)+\sum(x_i^{'},y_i^{'}))\geq \mu_{A\times B}(\sum(x_i,y_i))\wedge\mu_{A\times B}(\sum(x_i^{'},y_i^{'}))$
$\nu_{A\times B}(\sum(x_i,y_i)+\sum(x_i^{'},y_i^{'}))\leq \nu_{A\times B}(\sum(x_i,y_i))\vee\nu_{A\times B}(\sum(x_i^{'},y_i^{'}))$;

(iii) $\mu_{A\times B}(r(x,y))\geq\mu_{A\times B}(x,y)$ and $\nu_{A\times B}(r(x,y))\leq\nu_{A\times B}(x,y)$, $\forall(x,y)\in M\times N, r\in R$.

Consider

$$\mu_{A\times B}(\theta_1,\theta_2)=\mu_A(\theta_1)\vee\mu_B(\theta_2)=1$$

and

$$\nu_{A\times B}(\theta_1,\theta_2)=\nu_A(\theta_1)\vee\nu_B(\theta_2)=0$$

Now $\forall(x_1,y_1),(x_2,y_2)\in M\times N$

$$\begin{aligned}
\mu_{A\times B}(\sum(x_i,y_i)+\sum(x_i^{'},y_i^{'})) &= \mu_{A\times B}(\sum(x_i+x_i^{'},y_i+y_i^{'}))\\
&= (\mu_A\times\mu_B)(\sum(x_i+x_i^{'},y_i+y_i^{'}))\\
&= \wedge\{\vee\{\mu_A(x_i+x_i^{'}),\mu_B(y_i+y_i^{'})|i\in I\}\}\\
&\geq [\wedge\{\vee\{\mu_A(x_i,x_i^{'})\}\wedge\{\mu_B(y_i,y_i^{'})\}\}|\in I]\\
&= [\wedge\{\vee\{\mu_{A\times B}(x_i,y_i)\}\wedge\{\mu_{A\times B}(x_i^{'},y_i^{'})\}\}|\in I\}]\\
&= [\mu_{A\times B}(\sum(x_i,y_i))]\wedge[\mu_{A\times B}(\sum(x_i^{'},y_i^{'}))].
\end{aligned}$$

This implies that $\mu_{A\times B}(\sum(x_i,y_i)+\sum(x_i^{'},y_i^{'}))\geq[\mu_{A\times B}(\sum(x_i,y_i))]\wedge[\mu_{A\times B}(\sum(x_i^{'},y_i^{'}))]$. Similarly, we can prove
$\nu_{A\times B}(\sum(x_i,y_i)+\sum(x_i^{'},y_i^{'}))\leq[\nu_{A\times B}(\sum(x_i,y_i))]\vee[\nu_{A\times B}(\sum(x_i^{'},y_i^{'}))]$. Now,

$$\begin{aligned}
\mu_{A\times B}(r(\sum(x_i,y_i))) &= \mu_{A\times B}(\sum(rx_i,ry_i))=(\mu_A\times\mu_B)(\sum(rx_i,ry_i))\\
&= \wedge\{\vee\{\mu_A(rx_i),\mu_B(ry_i)|i\in I\}\}\\
&\geq \wedge\{\vee\{\mu_A(x_i),\mu_B(y_i)|i\in I\}\}\\
&= \mu_{A\times B}(\sum(x_i,y_i))\\
&= \mu_{A\times B}(\sum(x_i,y_i)).
\end{aligned}$$

This implies that $\mu_{A\times B}(r(\sum(x_i, y_i))) \geq \mu_{A\times B}(\sum(x_i, y_i))$. Similarly, we can prove that
$\nu_{A\times B}(r(\sum(x_i, y_i))) \leq \nu_{A\times B}(\sum(x_i, y_i))$. Hence $A \times B$ is an IFSM on $L(M \times N)$.

Definition 3.3. *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ be IFSMs of R-modules M, N and P respectively. A mapping $\bar{\phi} : A \times B \to C$ is said to be an intuitionistic fuzzy bi-additive (IF-biadditive) if the following conditions are satisfied $\forall \sum(x_i, y_i) \in L(M \times N)$*

(i) R-homomorphism $\phi : M \to N$ is R-biadditive and

(ii) $\mu_C(\bar{\phi}(\sum(x_i, y_i))) \geq (\mu_A \times \mu_B)(\sum(x_i, y_i))$ and $\nu_C(\bar{\phi}(\sum(x_i, y_i))) \leq (\nu_A \times \nu_B)(\sum(x_i, y_i))$.

Definition 3.4. *Let A be an IFSM of an R-module M and N be a submodule of M. Define an IFS $A_N = (\mu_{A_N}, \nu_{A_N})$ on the R-module M/N by*

$$\mu_{A_N}([x]) = \vee\{\mu_A(z)|z \in [x]\}$$

and

$$\nu_{A_N}([x]) = \wedge\{\nu_A(z)|z \in [x]\}$$

where $[x] = x + N \in M/N$. Then A_N is an IFSM of M/N.

Definition 3.5. *Let A be an IFSM of an R-module $L(M\times N)$ and P be a submodule of $L(M \times N)$. Then the IF-tensor product of R-modules M and N, $M \otimes N$, is an IFS T of $L(M \times N)/P$ defined as $T = \{\langle[(m, n)], \mu_T([(m, n)]), \nu_T([(m, n)]\rangle : (m, n) \in M \otimes N, m \in M, n \in N\}$ such that*

$$\mu_T([(m, n)]) = \vee\{\mu_A(x, y)|(x, y) \in [(m, n)]\}$$

and

$$\nu_T([(m, n)]) = \wedge\{\nu_A(x, y)|(x, y) \in [(m, n)]\}$$

where $[(m, n)] = (x, y) + P \in L(M \times N)/P$, $m, x \in M$ and $n, y \in N$.

Theorem 3.6. *Let T be an IF-tensor product of R-module $M \otimes N$. Then T is an IFSM of R-module $L(M \times N)/P$.*

Proof. Let A be an IFSM of an R-module $L(M \times N)$ and P be a submodule of $L(M \times N)$. Then the IF-tensor product of R-modules M and N, $M \otimes N$ is an IFS T of $L(M \times N)/P$ defined as $T = \{\langle[m, n], \mu_T([m, n]), \nu_T([m, n]\rangle : (m, n) \in M \otimes N, m \in M, n \in N\}$ such that

$$\mu_T([(m, n)]) = \vee\{\mu_A(x, y)|(x, y) \in [(m, n)]\}$$

and

$$\nu_T([(m,n)]) = \wedge\{\nu_A(x,y)|(x,y) \in [(m,n)]\}$$

where $[(m,n)] = (x,y) + P \in L(M \times N)/P$, $m, x \in M$ and $n, y \in N$.
(i) Consider $\mu_T([(0,0)]) = \vee\{\mu_A(x,y)|(x,y) \in [(0,0)]\} = \mu_A((0,0)) = 1$ and $\nu_T([(0,0)]) = \wedge\{\nu_A(x,y)|(x,y) \in [(0,0)]\} = \nu_A((0,0)) = 0$.
(ii) For $[(m_1,n_1)], [(m_2,n_2)] \in M \otimes N$, Consider

$$\begin{aligned}
&\mu_T([(m_1,n_1)] + [(m_2,n_2)]) = \vee\{\mu_A(x,y)|(x,y) \in ([(m_1,n_1)] + [(m_2,n_2)])\} \\
&\quad = \vee\{\mu_A((x_1,y_1) + (x_2,y_2))|((x_1,y_1) + (x_2,y_2)) \in ([(m_1,n_1)] + [(m_2,n_2)])\} \\
&\quad \geq \vee\{\mu_A((x_1,y_1) + (x_2,y_2))|(x_1,y_1) \in [(m_1,n_1)], (x_2,y_2) \in [(m_2,n_2)]\} \\
&\quad \geq \vee\{\mu_A(x_1,y_1) \wedge \mu_A(x_2,y_2)|(x_1,y_1) \in [(m_1,n_1)], (x_2,y_2) \in [(m_2,n_2)]\} \\
&\quad = (\vee\{\mu_A(x_1,y_1)|(x_1,y_1) \in [(m_1,n_1)]\}) \wedge (\vee\{\mu_A(x_2,y_2)|(x_2,y_2) \in [(m_2,n_2)]\}) \\
&\quad = \mu_A([(m_1,n_1)]) \wedge \mu_A([(m_2,n_2)])
\end{aligned}$$

Similarly, we can prove that
$\nu_T([(m_1,n_1)] + [(m_2,n_2)]) \leq \nu_T([(m_1,n_1)]) \vee \nu_T([(m_2,n_2)])$.
(iii) For $r \in R$, $[(m,n)] \in M \otimes N$, Consider

$$\begin{aligned}
\mu_T(r[(m,n)]) &= \vee\mu_T([r(m,n)]) \\
&= \vee\{\mu_A(x,y)|(x,y) \in [r(m,n)]\} \\
&= \vee\{\mu_A(r(x,y) + p)|p \in P\} \\
&\geq \vee\{\mu_A(r(x,y) + rp_1)|p_1 \in P\}[\text{ where } p = rp_1] \\
&= \vee\{\mu_A(r((x,y) + p_1)|p_1 \in P\} \\
&\geq \vee\{\mu_A((x,y) + p_1)|p_1 \in P\} \\
&= \vee\{\mu_A(x_1,y_1)|(x_1,y_1) \in [(m,n)]\} \\
&= \mu_T([(m,n)])
\end{aligned}$$

Similarly , we can prove that $\nu_T(r[(m,n)]) \leq \nu_T([(m,n)])$.
Hence, T is an IFSM of R-module $L(M \times N)/P$.

Definition 3.7. *Let A and B are IFSMs of R-modules M and N. A tensor product of A and B is an IFSM $A \otimes B$ which is equipped with an IF-biadditive map $\bar{\tau} : A \times B \to A \otimes B$ such that for each IFSM C of an R-module P and for every IF-biadditive map $\bar{\psi} : A \times B \to C$ there exists a unique IF R-homomorphism $\bar{\phi} : A \otimes B \to C$ such that the following diagram commutes,*

$$\begin{tikzcd} A \times B \arrow[r, "\bar{\tau}"] \arrow[dr, "\bar{\psi}"] & A \otimes B \arrow[d, "\bar{\phi}"] \\ & C \end{tikzcd}$$

that is $\bar{\phi} \circ \bar{\tau} = \bar{\psi}$ *and* $\mu_C(\bar{\phi}(x \otimes y)) \geq (\mu_A \times \mu_B)(x, y)$ *and* $\nu_C(\bar{\phi}(x \otimes y)) \leq (\nu_A \times \nu_B)(x, y)$.

Theorem 3.8. *The tensor product of two intuitionistic fuzzy modules exists and it is unique upto isomorphism.*

Proof. Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ be IFSM's of R-modules M, N and P respectively. Let $\bar{\tau} : A \times B \to A \otimes B$ be the tensor product of A and B.

Define the map : $A \otimes B : M \otimes N \to I \times I$ as

$$(\mu_A \otimes \mu_B)(\sum(x_i \otimes y_i)) = \vee\{(\mu_A \times \mu_B)(\sum(x_i^{'}, y_i^{'}))| \sum(x_i^{'} \otimes y_i^{'}) = \sum(x_i \otimes y_i)\}$$

and

$$(\nu_A \otimes \nu_B)(\sum(x_i \otimes y_i)) = \wedge\{(\nu_A \times \nu_B)(\sum(x_i^{'}, y_i^{'}))| \sum(x_i^{'} \otimes y_i^{'}) = \sum(x_i \otimes y_i)\}$$

From this, it is easy to see that $\bar{\tau}$ is an IF-biadditive. Let $\bar{\psi} : A \times B \to C$ be IF-biadditive. Since the tensor product of two R-modules exists and unique upto isomorphism in C_{R-M}. Then by Definition (2.18), for every R-biadditive map $\psi : M \times N \to P$, there exists an R-module homomorphism $\phi : M \otimes N \to P$ such that $\phi \circ \tau = \psi$.

$$\begin{tikzcd} M \times N \arrow[r, "\tau"] \arrow[dr, "\psi"] & M \otimes N \arrow[d, "\phi"] \\ & P \end{tikzcd}$$

We only need to show that $\bar{\phi} : A \otimes B \to C$ is an IF R-homomorphism, i.e., we want to claim that, $\forall \sum(x_i \otimes y_i) \in M \times N$

$$\mu_C(\bar{\phi}(\sum(x_i \otimes y_i))) \geq (\mu_A \otimes \mu_B)(\sum(x_i \otimes y_i))$$

$$\nu_C(\bar{\phi}(\sum(x_i \otimes y_i))) \leq (\nu_A \otimes \nu_B)(\sum(x_i \otimes y_i))$$

Let $\sum(x_i^{'} \otimes y_i^{'}) = \sum(x_i \otimes y_i)$. Consider

$$\begin{aligned}
\mu_C(\phi(\sum(x_i^{'} \otimes y_i^{'}))) &= \mu_C(\sum \phi(x_i^{'} \otimes y_i^{'})) \\
&\geq \wedge\{\mu_C(\phi(x_i^{'} \otimes y_i^{'}))\} \\
&= \wedge\{\mu_C(\phi \circ \tau)(x^{'}, y^{'}))\} \\
&= \wedge\{\mu_C(\psi)(x^{'}, y^{'}))\} \\
&\geq \wedge\{(\mu_A \times \mu_B)(x^{'}, y^{'})\} \\
&= (\mu_A \times \mu_B)(\sum(x_i^{'} \otimes y_i^{'})) \\
\Rightarrow \mu_C(\bar{\phi}(\sum(x_i \otimes y_i))) &\geq (\mu_A \otimes \mu_B)(\sum(x_i \otimes y_i)).
\end{aligned}$$

Similarly, we can prove that $\nu_C(\bar{\phi}(\sum(x_i \otimes y_i))) \leq (\nu_A \otimes \nu_B)(\sum(x_i \otimes y_i))$.
Hence $(A \otimes B, \bar{\tau})$ is the tensor product of A and B.

If $M \in Ob(C_{R-M})$, we have $R \otimes M \cong M$. Using this fact, we have the following result:

Proposition 3.9. *Let $A \in Ob(C_{R-IFM})$. Then, we have $\bar{0} \otimes A \cong \bar{0}$.*

For $M \in Ob(C_{R-M})$, $M \otimes -$ is right exact. In C_{R-IFM}, we have the following result:

Proposition 3.10. *Let $A \in Ob(C_{R-IFM})$. Then $A \otimes -$ preserves epimorphisms in C_{R-IFM}.*

Proof. Let $B \xrightarrow{\bar{g}} C \longrightarrow \bar{0}$. be an IFSE sequence in C_{R-IFM} and so $\bar{g}$ is an IF-epimorphism. Since $M \otimes -$ is right exact, we have

$$A \otimes B \xrightarrow{I_A \otimes \bar{g}} A \otimes C \longrightarrow \bar{0}.$$

Now, we only need to show that $I_A \otimes \bar{g}$ is an IF R-homomorphism.
For every $\sum(x_i \otimes y_i) \in M \times N$, we have
$(\mu_A \otimes \mu_C)((I_A \otimes \bar{g})(\sum(x_i \otimes y_i)))$
$= (\mu_A \otimes \mu_C)(\sum(x_i \otimes \bar{g}(y_i)))$
$= \vee\{(\mu_A \times \mu_C)(\sum(x_i^{'}, z_i^{'}))| \sum(x_i^{'} \otimes z_i^{'}) = \sum(x_i \otimes \bar{g}(y_i))\}$
$\geq \vee\{(\mu_A \times \mu_B)(\sum(x_i^{'}, y_i^{'}))|\bar{g}(y_i) = z_i \text{ and} \sum(\mathrm{x_i^{'}} \otimes \mathrm{z_i^{'}}) = \sum(\mathrm{x_i} \otimes \bar{\mathrm{g}}(\mathrm{y_i}))\}$
$\geq \vee\{(\mu_A \times \mu_B)(\sum(x_i^{''}, y_i^{''}))| \sum(x_i^{''} \otimes y_i^{''}) = \sum(x_i \otimes y_i)\}$
$= (\mu_A \otimes \mu_B)(\sum(x_i \otimes y_i))$.
Thus $(\mu_A \otimes \mu_C)((I_A \otimes \bar{g})(\sum(x_i \otimes y_i))) \geq (\mu_A \otimes \mu_B)(\sum(x_i \otimes y_i))$.
Similarly, we can show that $(\nu_A \otimes \nu_C)((I_A \otimes \bar{g})(\sum(x_i \otimes y_i))) \leq (\nu_A \otimes \nu_B)(\sum(x_i \otimes y_i))$.

Hence, we get the desired results.

Proposition 3.11. *Let $A \in Ob(C_{R-IFM})$. Then $- \otimes A$ preserves epimorphisms in C_{R-IFM}.*

We now investigate the relationship between the tensor product and the Hom functor. Linking the two concepts is a natural isomorphism called the Hom-Tensor adjunction, which connects tensor products and hom-functors in some algebraic structures like modules. It establishes a relationship between $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B \otimes A, C)$ and $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B, C))$.

Theorem 3.12. (Adjoint Isomorphism) *In C_{R-IFM}, there exists an IF quasi-isomorphisms*

$$\tau : \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B \otimes A, C) \cong_Q \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B, C));$$

$$\tau' : \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A \otimes B, C) \cong_Q \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B, C)).$$

Proof. We merely show the first quasi-isomorphism. For $A, B, C \in Ob(\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}})$ and by existence of tensor product in $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}$, there exists an unique IF R-homomorphism
$\phi \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B \otimes A, C)$ such that

$$\mu_C(\phi(y \otimes x)) \geq (\mu_B \otimes \mu_A)(y \otimes x)$$

$$\nu_C(\phi(y \otimes x)) \leq (\mu_B \otimes \mu_A)(y \otimes x).$$

With due reference to Theorem 2.75 [20], we will define the following IF R- homomorphisms:
For $x \in M$ and $y \in N$, define $\phi_x : B \to C$ as $\phi_x(y) = \phi(y \otimes x)$,
$\bar{\phi} : A \to \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B, C)$ as $\bar{\phi}(y) = \phi_x$ and
$\tau : \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B \otimes A, C) \to \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B, C))$ as $\tau(\phi) = \bar{\phi}$.
It is therefore necessary to prove that ϕ_x, $\bar{\phi}$ are IF R-homomorphisms and τ are IF R-isomorphism.
(i) To begin with, we will show that ϕ_x is an IF R-homomorphism. For $y \in N$, we have

$$\mu_C(\phi_x(y)) = \mu_C(\phi(y \otimes x)) \geq (\mu_B \otimes \mu_A)(y \otimes x) \geq (\mu_B \times \mu_A)(y, x)$$

$$= \vee\{\mu_B(y), \mu_A(x)\} \geq \mu_B(y).$$

Likewise, we can exhibit that $\nu_C(\phi_x(y)) \leq \nu_B(y)$. Thus, ϕ_x is an IF R-homomorphism.
(ii) Secondly, we will show that $\bar{\phi}$ is an IF R-homomorphism. For $x \in M$, we have

$$\begin{aligned}
\mu_{\mathbf{Hom_{C_{R\text{-}IFM}}}(B,C)}(\bar{\phi}(x)) &= \mu_{\mathbf{Hom_{C_{R\text{-}IFM}}}(B,C)}(\phi_x) \\
&= \wedge\{\mu_C(\phi_x(y))|y \in N\} \\
&= \wedge\{\mu_C(\phi(y \otimes x))|y \in N\} \\
&\geq \wedge\{(\mu_B \otimes \mu_A)(y \otimes x)|x \in M, y \in N\} \\
&\geq \wedge\{\vee\{\mu_B(y), \mu_A(x)\}|x \in M, y \in N\} \\
&\geq \mu_A(x).
\end{aligned}$$

Likewise, we can exhibit that $\nu_{\mathbf{Hom_{C_{R\text{-}IFM}}}(B,C)}(\bar{\phi}(x)) \leq \nu_A(x)$.
(iii) Finally, we will show that τ is an IF R-isomorphism. For $\phi \in \mathbf{Hom_{C_{R\text{-}IFM}}}(B \otimes A, C)$, we have

$$\begin{aligned}
\mu_{\mathbf{Hom_{C_{R\text{-}IFM}}}(A,\mathbf{Hom_{C_{R\text{-}IFM}}}(B,C))}(\tau(\phi)) &= \mu_{\mathbf{Hom_{C_{R\text{-}IFM}}}(A,\mathbf{Hom_{C_{R\text{-}IFM}}}(B,C))}(\bar{\phi}) \\
&= \wedge\{\mu_{\mathbf{Hom_{C_{R\text{-}IFM}}}(B,C)}(\bar{\phi}(x))|x \in M\} \\
&= \wedge\{\mu_{\mathbf{Hom_{C_{R\text{-}IFM}}}(B,C)}(f_x)|x \in M\} \\
&= \wedge\{\wedge\{\mu_C(\phi_x(y))|y \in N\}|x \in M\} \\
&= \wedge\{\wedge\{\mu_C(\phi(y \otimes x))|y \in N\}|x \in M\} \\
&= \wedge\{\mu_C(\phi(y \otimes x))|y \in N, x \in M\} \\
&= \mu_{\mathbf{Hom_{C_{R\text{-}IFM}}}(B\otimes A,C)}(\phi).
\end{aligned}$$

Similarly, we can prove
$\nu_{\mathbf{Hom_{C_{R\text{-}IFM}}}(A,\mathbf{Hom_{C_{R\text{-}IFM}}}(B,C))}(\tau(\phi)) = \nu_{\mathbf{Hom_{C_{R\text{-}IFM}}}(B\otimes A,C)}(\phi)$.
This is called adjoint isomorphism.

Remark 3.13. *The adjoint isomorphism theorem (3.12) gives natural isomorphism*

$$\boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(B \otimes A, C) \cong_Q \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(A, \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(B, C))$$

Thus, $B \otimes \Box$ is the right adjoint of $Hom(\Box, B)$.
Fixing any two IF-modules A, B, C, each $\tau = \tau_{A,B,C}$ is a natural isomorphism:

$$\boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(B \otimes \Box, C) \cong_Q \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(\Box, \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(B, C));$$

$$\boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(B \otimes A, \Box) \cong_Q \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(A, \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(B, \Box));$$

$$\boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(\Box \otimes A, C) \cong_Q \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(A, \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(\Box, C))$$

If $f : A \to A'$, the following diagram is commutative

$$\begin{array}{ccc}
\boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(B \otimes A', C) & \xrightarrow{\tau_{A',B,C}} & \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(A', \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(B, C)) \\
\Big\downarrow{\scriptstyle (I_B \otimes f)^*} & & \Big\downarrow{\scriptstyle f*} \\
\boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(B \otimes A, C) & \xrightarrow{\tau_{A,B,C}} & \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(A, \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(B, C))
\end{array}$$

4. Application of tensor product of intuitionistic fuzzy modules in decision making

The tensor product of intuitionistic fuzzy modules finds application in decision making by providing a framework to handle uncertainty and vagueness in the decision-making process. It allows for the representation of complex relationships and dependencies among decision criteria. This approach is particularly useful when dealing with incomplete or imprecise information in decision-making scenarios.

By employing the tensor product of intuitionistic fuzzy modules, decision-makers can model and analyze various factors simultaneously, considering both degree of membership and degree of non-membership. This helps in capturing the nuances of decision criteria and their interdependencies, leading to more comprehensive and robust decision models. Let's consider a decision-making scenario where a company needs to evaluate potential suppliers based on multiple criteria such as cost, reliability, and quality. Each criterion can be represented as an intuitionistic fuzzy module.

(i) **Cost Module** $= (\mu_{\text{Cost}}, \nu_{\text{Cost}})$, where $\mu_{\text{Cost}} = 0.8$ and $\nu_{\text{Cost}} = 0.2$.

(ii) **Reliability Module** $= (\mu_{\text{Reliability}}, \nu_{\text{Reliability}})$, where $\mu_{\text{Reliability}} = 0.6$ and $\nu_{\text{Reliability}} = 0.2$.

(iii) **Quality Module** $= (\mu_{\text{Quality}}, \nu_{\text{Quality}})$, where $\mu_{\text{Quality}} = 0.9$ and $\nu_{\text{Quality}} = 0.4$.

To make an overall decision, we can use the tensor product of these intuitionistic fuzzy modules. The resulting module would represent the combined intuitionistic fuzzy information considering all criteria. After the tensor product operation, we might find that a particular supplier has a higher overall degree of membership, indicating a better fit for the company's needs. This approach allows decision-makers to integrate and analyze fuzzy information from multiple sources, making the decision-making process more robust and reflective of real-world uncertainties. Now, let's calculate the tensor product using the following formulas:

Integrated degree of membership is given by :
$\mu_{\text{integrated}} = \mu_{\text{Cost}} \otimes \mu_{\text{Reliability}} \otimes \mu_{\text{Quality}} = \mu_{\text{Cost}} \times \mu_{\text{Reliability}} \times \mu_{\text{Quality}} = 0.8 \times 0.6 \times 0.9 = 0.432$
and integrated degree of non-membership is given by :
$\nu_{\text{integrated}} = \nu_{\text{Cost}} \otimes \nu_{\text{Reliability}} \otimes \nu_{\text{Quality}} = \nu_{\text{Cost}} \times \nu_{\text{Reliability}} \times \nu_{\text{Quality}} = 0.2 \times 0.2 \times 0.4 = 0.016$.
So, the integrated intuitionistic fuzzy module $= (\mu_{\text{integrated}}, \nu_{\text{integrated}}) = (0.432, 0.016)$

These values represent the combined information considering cost, reliability, and quality. The high integrated degree of membership suggests a strong overall alignment with the company's needs, while the degree of non-membership reflect the uncertainty and potential areas of disagreement in the decision-making process.

5. Conclusion

The tensor product of intuitionistic fuzzy modules provides a mathematical framework to handle uncertainty and imprecision in module theory, offering a more flexible and realistic approach to modeling and solving problems in various domains. We established the existence of the tensor product of two intuitionistic fuzzy R-modules. We then investigated the relation between Hom functor and tensor functor in the category of intuitionistic fuzzy modules. The application of the tensor product of intuitionistic fuzzy modules enhances decision-making processes by accommodating uncertainty and vagueness, making it a valuable tool in situations where precise information is lacking.

Further research in this area is possible and our findings in this study already provide a framework for future discussions regarding the tensor product.

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