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AN APPROACH TO THE STUDY OF FIXED POINT THEORY IN HILBERT SPACE

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Abstract: The purpose of this article is to extend Banach's contraction principle through a new rational expression in the contractive condition to establish the existence and uniqueness of fixed point of a closed subset of Hilbert space to a self mapping. The result is extended to a pair of self mappings and positive integers powers of a pair mapping and further extended to a sequence of mappings in the space. The presented results extend and generalized various known comparable results from the current literature.

Keywords and Phrases: Fixed point, Common fixed point, Hilbert space, Closed subset.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

The most celebrated contraction mapping principle, formulated and proved in the Ph.D. dissertation of Banach [1] in 1920, which was published in 1922, is one of the most important theorems in classical functional analysis. This contraction mapping principle has been generalized in various directions. One of the most interesting of them is the generalization obtained by Kannan [4] who investigated the extension of Banach fixed point theorem by removing the completeness of the space with different sufficient conditions. Later, Reich [15] also discussed some generalization of Banach's fixed point theorem and some remarks on it. Zamfirescu [22] obtained various results similar to the well-known contraction theorem of Banach and some of these results were sufficient enough to include the theorem of Kannan. Also, a generalization of Banach fixed point theorem was given by Jaggi [3] which involved a continuous map satisfying certain inequality involving rational expression. Fisher [2] developed the approach of Kannan and proved analogous results involving two mappings on a complete metric space. In this direction several authors obtained further results [6, 7, 8, 12, 13, 14, 16, 19, 22].

Koparde and Waghmode [5] extended the Banach fixed point theorem to obtain the existence and uniqueness of a fixed point for a sequence of mappings on a Hilbert space satisfying Kannan's type conditions. Pandhare and Waghmode [10] developed this approach of Koparde and Waghmode [5] and proved the fixed point theorem for a self mapping on a closed subset of a Hilbert space satisfying certain condition. The same is extended to a family of self mappings in [9, 11, 17, 21] by taken a sequence of mappings, which converges point wise to a limit mapping and show that if this limit mapping has a fixed point then this fixed point is also the limit of fixed points of the mappings of the sequence.

Sharma et al. [20] considered a pair of self mappings of a closed subset of a Hilbert space, satisfying certain rational inequalities and proved a common fixed point theorem for self mappings.

Recently, Seshagiri and Kalyani [18] proved the existence and uniqueness of a common fixed point for a pair of self mappings, positive integer's powers of a pair of self mapping and a sequence of self mappings over a closed subset of a Hilbert space satisfying various conditions involving rational expressions.

In view of the above considerations, we proved that a self mapping T satisfying certain rational contraction condition has a unique fixed point on a closed subset X of Hilbert space and again the same result is then extended to a pair of mappings T_1, T_2, some positive integers powers r, s of a pair mappings T_1^r, T_2^s and then further generalized to a sequence of mappings in the space.

2. Preliminary Theorems

Here are the lists of some of the results that motivated our results.

Theorem 2.1. [5] *Let C be a non empty closed subset of Hilbert space H. Let $T : C \to C$ be a self mapping satisfying the Kannan type condition*

$$\|Tx - Ty\|^2 \leq \alpha \left\{ \|x - Tx\|^2 + \|y - Ty\|^2 \right\} \tag{2.1}$$

for all $x, y \in S$, $x \neq y$, $\alpha \in [0, \frac{1}{2})$. *Then,* T *has a fixed point in* X.

Theorem 2.2. [5] *Let* C *be a non empty closed sunset of a Hilbert space* H . *Let* $T : T \to C$ *be a family of self-mappings satisfying the Kannan type condition*

$$\|T_i x - T_i y\|^2 \leq \alpha \left\{ \|x - T_i x\|^2 + \|y - T_i y\|^2 \right\} \tag{2.2}$$

for all $x, y \in S$, $x \neq y$, $\alpha \in [0, \frac{1}{2})$. *Then,* T *has a fixed point in* X.

Theorem 2.3. [20] *Let* S *be a non empty closed subset of Hilbert space* H. *Let* $T : S \to S$ *be self-mappings satisfying the following conditions.*

$$\|Tx - Ty\| \leq \alpha \frac{\|x - Tx\|^2 + \|y - Ty\|^2}{\|x - Tx\| + \|y - Ty\|} + \beta \|x - y\| \tag{2.3}$$

for all $x, y \in S$, $x \neq y$,$0 \leq \alpha < \frac{1}{2}$, $0 \leq \beta$, $2\alpha + \beta < 1$. *Then,* T *has a fixed point in* X.

Theorem 2.4. [18] *Let* X *be a closed subset of a Hilbert space and* T_1, T_2 *be two continuous self-mappings on* X *satisfying contraction condition, then* T_1 *and* T_2 *have unique common fixed point in* X:

$$\begin{aligned} \|T_1 x - T_2 y\|^2 \leq \alpha & \frac{\|x - T_2 y\|^2 [1 + \|y - T_1 x\|^2}{1 + \|x - y\|^2} \\ & + \beta \frac{\|x - y\|^2 [1 + \|x - T_2 y\|^2}{1 + \|x - y\|^2} \\ & + \gamma \frac{\|x - T_1 x\|^2 + \|y - T_2 y\|^2 + \|y - T_1 x\|^2}{1 + \|x - T_1 x\|^2 \|x - T_2 y\|^2 \|y - T_1 x\|^2} \\ & + \delta [\|y - T_1 x\|^2 + \|x - T_2 y\|^2] + \epsilon \|x - y\|^2 \end{aligned} \tag{2.4}$$

for all $x, y \in X$, $x \neq y$, *where* $\alpha, \beta, \gamma, \delta, \epsilon$ *are positive real's with* $2\alpha + \beta + 4\gamma + 2\delta + \epsilon < 1$.

3. Main Theorems

In this section, we state and prove our main result as follows:

Theorem 3.1. *Let* T *be a closed subset of a Hilbert space and a self mapping* $T : X \to X$ *satisfying the following.*

$$\begin{aligned} d(Tx, Ty) \leq & \alpha \frac{\|x - Tx\| \|x - Ty\| + \|y - Tx\| \|y - Ty\|}{\|y - Tx\| + \|x - Ty\|} \\ & + \beta \frac{\|x - Tx\| \|y - Ty\|}{\|x - y\| + \|x - Ty\| + \|y - Tx\|} \\ & + \gamma \frac{\|x - Tx\| \|y - Ty\|}{\|x - y\|} + \delta \|x - y\|, \end{aligned} \tag{3.1}$$

for all distinct $x, y \in X$ and $\alpha, \beta, \gamma, \delta$ are non negative real numbers with $0 \leq \alpha + \beta + \gamma + \delta < 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\}$ in X as

$$x_1 = Tx_0, x_2 = Tx_1, ..., X_{n+1} = Tx_n, \forall\ n = 0, 1, 2, ... \tag{3.2}$$

Next, we show that the sequence $\{x_n\}$ is Cauchy in X. with (3.1) and (3.2), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|Tx_n - Tx_{n-1}\| \\ &\leq \alpha \frac{\|x_n - Tx_n\|\|x_n - Tx_{n-1}\| + \|x_{n-1} - Tx_n\|\|x_{n-1} - Tx_{n-1}\|}{\|x_{n-1} - Tx_n\| + \|x_n - Tx_{n-1}\|} \\ &+ \beta \frac{\|x_n - Tx_n\|\|x_{n-1} - Tx_{n-1}\|}{\|x_n - x_{n-1}\| + \|x_n - Tx_{n-1}\| + \|x_{n-1} - Tx_n\|} \\ &+ \gamma \frac{\|x_n - Tx_n\|\|x_{n-1} - Tx_{n-1}\|}{\|x_n - x_{n-1}\|} + \delta \|x_n - x_{n-1}\| \end{aligned} \tag{3.3}$$

$$\|x_{n+1} - x_n\| \leq (\frac{\alpha + \beta + \delta}{1 - \gamma})\|x_n - x_{n-1}\|, \tag{3.4}$$

implies

$$\|x_{n+1} - x_n\| \leq r(n)\|x_n - x_{n-1}\|, \tag{3.5}$$

where

$$r(n) = \frac{\alpha + \beta + \delta}{1 - \gamma},$$

for all $n = 0, 1, 2, ...$. Clearly, $r(n) < 1$, for all $0 \leq \alpha + \beta + \gamma + \delta < 1$.
Repeating the above process in the same manner as in (3.5), we find some $r < 1$, such that

$$\|x_{n+1} - x_n\| \leq r^n \|x_1 - x_0\|. \tag{3.6}$$

On taking $n \to \infty$ in (3.6), we get $\|x_{n+1} - x_n\| \to 0$. Hence, the sequence $\{x_n\}$ is Cauchy in X. But X is a closed subset of Hilbert space and so by completeness of X, there exist a point $u \in X$ such that $x_n \to u$ as $n \to \infty$. Consequently, $\{x_{n+1}\} = \{Tx_n\}$ is a subsequence of $\{x_n\}$ and hence has the same limit as u.
Since T is continuous, we get

$$\begin{aligned} T(u) &= T(\lim_{n\to\infty} x_n) \\ &= \lim_{n\to\infty} Tx_n \\ &= \lim_{n\to\infty} x_{n+1} \\ &= u. \end{aligned} \tag{3.7}$$

Thus, u is the fixed point of T in X.
Finally, to prove the uniqueness of a fixed point u, we let $v(u \neq v)$ be another fixed point of T. Then, from (3.1), we get

$$\begin{aligned} \|u-v\| &= \|Tu-Tv\| \\ &\leq \alpha \frac{\|u-Tu\|\|u-Tv\|+\|v-Tu\|\|v-Tv\|}{\|v-Tu\|+\|u-Tv\|} \\ &+ \beta \frac{\|u-Tu\|\|v-Tv\|}{\|u-v\|+\|u-Tv\|+\|v-Tu\|} \\ &+ \gamma \frac{\|u-Tu\|\|v-Tv\|}{\|u-v\|} + \delta\|u-v\| \end{aligned} \tag{3.8}$$

$$\|u-v\| \leq \delta\|u-v\|, \tag{3.9}$$

a contradiction, for $\delta < 1$. Implies $u = v$. Hence, u is a unique fixed point of T in X.

Example 3.2. Let $T : [0,1] \to [0,1]$ be defined by $Tx = \frac{x^3}{3}$ for all $x \in [0,1]$. Clearly 0 is the fixed point of T with the usual norm $\|x-y\| = |x-y|$.

Now, we consider a pair of self mappings of a closed subset of a Hilbert space.

Theorem 3.3. *Let X be a closed subset of a Hilbert space and $T_1T_2 : X \to X$ be self mappings satisfying the following*

$$\begin{aligned} & d(T_1x, T_2y) \\ &\leq \alpha \frac{\|x-T_1x\|\|x-T_2y\|+\|y-T_1x\|\|y-T_2y\|}{\|v-T_1x\|+\|x-T_2y\|} \\ &+ \beta \frac{\|x-T_1x\|\|y-T_2y\|}{\|x-y\|+\|x-T_2y\|+\|y-T_1x\|} \\ &+ \gamma \frac{\|x-T_1x\|\|y-T_2y\|}{\|x-y\|} + \delta\|x-y\|, \end{aligned} \tag{3.10}$$

for all distinct $x, y \in X$ and $\alpha, \beta, \gamma, \delta$ are non negative real numbers with $0 \leq \alpha+\beta+\gamma+\delta < 1$. Then T_1, T_2 have a common fixed point in X.

Proof. Let us construct a sequence $\{x_n\}$ in X for an arbitrary point $x_0 \in X$ as

$$x_{2n+1} = T_1x_{2n}, x_{2n+2} = T_2x_{2n+1}, \forall n = 0, 1, 2, ... \tag{3.11}$$

We show that the sequence $\{x_n\}$ is Cauchy in X. With (3.10) and (3.11), we have

$$\begin{aligned}\|x_{2n+1}-x_{2n}\| &= \|T_1x_{2n}-T_2x_{2n-1}\|\\ &\le \alpha\frac{\|x_{2n}-T_1x_{2n}\|\|x_{2n}-T_2x_{2n-1}\|+\|x_{2n-1}-T_1x_{2n}\|\|x_{2n-1}-T_2x_{2n-1}\|}{\|x_{2n-1}-T_1x_{2n}\|+\|x_{2n}-T_2x_{2n-1}\|}\\ &+\beta\frac{\|x_{2n}-T_1x_{2n}\|\|x_{2n-1}-T_2x_{2n-1}\|}{\|x_{2n}-x_{2n-1}\|+\|x_{2n}+T_2x_{2n-1}\|+\|x_{2n-1}-T_1x_{2n}\|}\\ &+\gamma\frac{\|x_{2n}-T_1x_{2n}\|\|x_{2n-1}-T_2x_{2n-1}\|}{\|x_{2n}-x_{2n-1}\|}+\delta\|x_{2n}-x_{2n-1}\|\end{aligned} \tag{3.12}$$

$$\|x_{2n+1}-x_{2n}\| \le (\frac{\alpha+\beta+\delta}{1-\gamma})\|x_{2n}-x_{2n-1}\|, \tag{3.13}$$

implies

$$\|x_{2n+1}-x_{2n}\| \le \lambda(n)\|x_{2n}-x_{2n-1}\|, \tag{3.14}$$

where

$$\lambda(n) = \frac{\alpha+\beta+\delta}{1-\gamma},$$

for all $n = 0, 1, 2, ...$ Clearly, $t = \lambda(n) < 1$, for all $0 \le \alpha+\beta+\gamma+\delta < 1$.
In general, we have

$$\|x_{n+1}-x_n\| \le t\|x_n-x_{n-1}\|. \tag{3.15}$$

Continuing the process in (3.15), we get

$$\|x_{n+1}-x_n\| \le t^n\|x_1-x_0\|, n \ge 1. \tag{3.16}$$

On taking $n \to \infty$ in (3.16), we get $\|x_{n+1}-x_n\| \to 0$. Hence, the sequence $\{x_n\}$ is Cauchy in X and has a limit u in X.

Since the sequences $\{x_{2n+1}\} = \{T_1x_{2n}\}$ and $\{x_{2n+2}\} = \{T_2x_{2n+1}\}$ are subsequences of $\{x_n\}$ and hence the subsequences have the same limit u in X.
Now, we show that u is the common fixed point of T_1 and T_2 in X. Then, from (3.10), we get

$$\begin{aligned}\|u-T_1u\| &= \|(u-x_{2n+2})+(x_{2n+2}-T_1u)\|\\ &\le \|u-x_{2n+2}\|+\|T_1u-T_2x_{2n+1}\|\\ &\le \alpha\frac{\|u-T_1u\|\|u-T_2x_{2n+1}\|+\|x_{2n+1}-T_1u\|\|x_{2n+1}-T_2x_{2n+1}\|}{\|x_{2n+1}-T_1u\|+\|u-T_2x_{2n+1}\|}\\ &+\beta\frac{\|u-T_1u\|\|x_{2n+1}-T_2x_{2n+1}\|}{\|u-x_{2n+1}\|+\|u-T_2x_{2n+1}\|+\|x_{2n+1}-T_1u\|}\\ &+\gamma\frac{\|u-T_1u\|\|x_{2n+1}-T_2x_{2n+1}\|}{\|u-x_{2n+1}\|}+\delta\|u-x_{2n+1}\|+\|u-x_{2n+2}\|.\end{aligned} \tag{3.17}$$

Letting $n \to \infty$ in (3.17), we get $\|u - T_1 u\| \leq (\alpha + \delta)\|u - T_1 u\|$. Since $\alpha + \delta < 1$, it follows that $T_1 u = u$.
Similarly, from, from (3.10), we get $T_2 u = u$ by using the following

$$\|u - T_2 u\| = \|(u - x_{2n+1}) + (x_{2n+1} - T_2 u).\|$$

Thus, u is a common fixed point of T_1 and T_2 in X.
To complete the proof, we show that u is a unique fixed point of $T_1 T_2$. Now, let us suppose that $v(u \neq v)$ be another common fixed point of T_1 and T_2. Then, from (3.10), we get

$$\begin{aligned} \|u - v\| &= \|T_1 u - T_2 v\| \\ &\leq \alpha \frac{\|u - T_1 u\|\|u - T_2 v\| + \|v - T_1 u\|\|v - T_2 v\|}{\|v - T_1 u\| + \|u - T_2 v\|} \\ &+ \beta \frac{\|u - T_1 u\|\|v - T_2 v\|}{\|u - v\| + \|u - T_2 v\| + \|v - T_1 u\|} \\ &+ \gamma \frac{\|u - T_1 u\|\|v - T_2 v\|}{\|u - v\|} + \delta\|u - v\| \end{aligned} \tag{3.18}$$

implies

$$\|u - v\| \leq \delta\|u - v\|, \tag{3.19}$$

a contradiction, for $\delta < 1$. Implies $u = v$. Hence, u is a unique common fixed point of T_1 and T_2 in X.

Example 3.4. Let $T_1 T_2 : [0,1] \to [0,1]$ defined by $T_1 x = \frac{x}{3}$ and $T_2 x = \frac{x}{4}$, for all $x \in [0,1]$. Clearly 0 is the common fixed point of T_1 and T_2 and usual norm $\|x - y\| = |x - y|$.
In this theorem, we consider a pair taking positive integers power of the self mappings in the contraction inequality.

Theorem 3.5. *Let X be a closed subset of a Hilbert space and $T_1 T_2 : X \to X$ be self mappings satisfying the following.*

$$\begin{aligned} &d(T_1^r x, T_2^s y) \\ &\leq \alpha \frac{\|x - T_1^r x\|\|x - T_2^s y\| + \|y - T_1^r x\|\|y - T_2^s y\|}{\|y - T_1^r x\| + \|x - T_2^s y\|} \\ &+ \beta \frac{\|x - T_1^r x\|\|y - T_2^s y\|}{\|x - y\| + \|x - T_2^s y\| + \|y - T_1^r x\|} \\ &+ \gamma \frac{\|x - T_1^r x\|\|y - T_2^s y\|}{\|x - y\|} + \delta\|x - y\|, \end{aligned} \tag{3.20}$$

for all distinct $x, y \in X$ and $\alpha, \beta, \gamma, \delta$ are non negative real numbers with $0 \leq \alpha + \beta + \gamma + \delta < 1$. Then T_1, T_2 have a unique common fixed point in X.

Proof. In view of Theorem 3.3, T_1^r and T_2^s have a unique common fixed point $u \in X$, so that $T_1^r u = u$ and $T_2^s u = u$.
By considering $T_1^r(T_1 u) = T_1(T_1^r u) = T_1 u$
We have that $T_1 u$ is a fixed point of T_1^r.
But u is a unique fixed point of T_1^r.
Therefore, $T_1 u = u$.
Similarly, we get $T_2 u = u$.
Hence, u is a common fixed point of T_1 and T_2.
To show uniqueness, we let v be another fixed point of T_1 and T_2, so that $T_1 v = T_2 v = v$. With (3.20), we have

$$\begin{aligned} \|u - v\| &= \|T_1^r u - T_2^s v\| \\ &\leq \alpha \frac{\|u - T_1^r u\| \|u - T_2^s v\| + \|v - T_1^r u\| \|v - T_2^s v\|}{\|v - T_1^r u\| + \|u - T_2^s v\|} \\ &+ \beta \frac{\|u - T_1^r u\| \|v - T_2^s v\|}{\|u - v\| + \|u - T_2^s v\| + \|v - T_1^r u\|} \\ &+ \gamma \frac{\|u - T_1^r u\| \|v - T_2^s v\|}{\|u - v\|} + \delta \|u - v\|, \end{aligned} \tag{3.21}$$

$$\|u - v\| \leq \delta \|u - v\|. \tag{3.22}$$

Implies $u = v$, since $\delta < 1$. Hence, u is a unique common fixed point of T_1 and T_2 in X.
In the following Theorem, we consider a sequence of mappings on a closed subset of a Hilbert space which converges point wise to a limit mapping and show that if this limit mapping has a fixed point then this fixed point is also the limit of fixed points of the mappings of the sequence.

Theorem 3.6. *Let X be a closed subset of a Hilbert space and $\{T_i\}$ be a sequence of self mappings on X converging point wise to T and satisfying the following*

$$\begin{aligned} &d(T_i x, T_i y) \\ &\leq \alpha \frac{\|x - T_i x\| \|x - T_i y\| + \|y - T_i x\| \|y - T_i y\|}{\|y - T_i x\| + \|x - T_i y\|} \\ &+ \beta \frac{\|x - T_i x\| \|y - T_i y\|}{\|x - y\| + \|x - T_i y\| + \|y - T_i x\|} \\ &+ \gamma \frac{\|x - T_i x\| \|y - T_i y\|}{\|x - y\|} + \delta \|x - y\|, \end{aligned} \tag{3.23}$$

for all distinct $x, y \in X$ and $\alpha, \beta, \gamma, \delta$ are non negative real numbers with $0 \leq \alpha + \beta + \gamma + \delta < 1$, if each T_i has a fixed point u_i and T has a fixed point u, then the sequence $\{u_i\}$ converges to u.

Proof. In light of Theorem 3.3, u_i is a fixed point of T_i. Then, from (3.23), we have

$$\begin{aligned} \|u - u_n\| = \|Tu - T_n u_n\| &= \|(Tu - T_n u) + (T_n u - T_n u_n)\| \\ &\leq \|Tu - T_n u\| + \|T_n u - T_n u_n\| \\ &\leq \alpha \frac{\|u - T_n u\|\|u - T_n u_n\| + \|u_n - T_n u\|\|u_n - T_n u_n\|}{\|u_n - T_n u\| + \|u - T_n u_n\|} \\ &+ \beta \frac{\|u - T_n u\|\|u_n - T_n u_n\|}{\|u - u_n\| + \|u - T_n u_n\| + \|u_n - T_n u\|} \\ &+ \gamma \frac{\|u - T_n u\|\|u_n - T_n u_n\|}{\|u - u_n\|} \\ &+ \delta\|u - u_n\| + \|Tu - T_n u\|. \end{aligned} \tag{3.24}$$

On taking $n \to \infty$ in (3.24), $T_n u \to Tu$, $T_n u_n \to u_n$ and $Tu = u$, we get

$$\lim_{n\to\infty} \|u - u_n\| \leq \delta \lim_{n\to\infty} \|u - u_n\|.$$

Implies

$$\lim_{n\to\infty} \|u - u_n\| = 0,$$

since $\delta < 1$.
Hence, $u_n \to u$ as $n \to \infty$.

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