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SOME ELLIPTICAL INTEGRALS ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS

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Abstract: The main object of this paper is to establish seven elliptical integrals associated with hypergeometric functions and suggest new way to compute their numerical values. The results presented in this article are presumably new and not present in the scientific literature.

Keywords and Phrases: Elliptic integral, hypergeometric function.

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1. Introduction and Definitions

A generalized hypergeometric function ${}_\alpha F_\beta(a_1, ..., a_\alpha; b_1, ..., b_\beta; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written as:

$$\frac{c_{\zeta+1}}{c_\zeta} = \frac{P(\zeta)}{Q(\zeta)} = \frac{(\zeta + a_1)(\zeta + a_2)...(\zeta + a_\alpha)}{(\zeta + b_1)(\zeta + b_2)...(\zeta + b_\beta)(\zeta + 1)}\ z. \tag{1.1}$$

Where $\zeta + 1$ in the denominator is present for historical reasons of notation (see [1]; [6], p. 12 (2.9)), and the resulting generalized hypergeometric function is written as [4]:

$$ {}_{\alpha}F_{\beta}\left[\begin{array}{ccc} a_1, a_2, \cdots, a_{\alpha} & ; & \\ & & z \\ b_1, b_2, \cdots, b_{\beta} & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{\alpha})_k z^k}{(b_1)_k (b_2)_k \cdots (b_{\beta})_k k!} \tag{1.2} $$

where the parameters $b_1, b_2, \cdots, b_{\beta}$ are positive integers.

The complete elliptic integral of the first kind K is defined as [4]:

$$ K(\eta) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\eta^2 \sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\eta^2 t^2)}}, \tag{1.3} $$

In power series

$$ K(\eta) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \eta^{2n} = \frac{\pi}{2} \sum_{n=0}^{\infty} \Big[P_{2n}(0)\Big]^2 \eta^{2n}, \tag{1.4} $$

where P_n is the Legendre polynomial.
In terms of the Gauss hypergeometric function, the complete elliptic integral of the first kind can be expressed as

$$ K(\eta) = \frac{\pi}{2}\ {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; \eta^2\Big) \tag{1.5} $$

The complete elliptic integral of the second kind E is defined as [4]:

$$ E(\eta) = \int_0^{\frac{\pi}{2}} \sqrt{1-\eta^2\sin^2\theta}\ d\theta = \int_0^1 \frac{\sqrt{1-\eta^2 t^2}}{\sqrt{1-t^2}}\ dt. \tag{1.6} $$

It can be expressed as a power series

$$ E(\eta) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \frac{\eta^{2n}}{1-2n}. \tag{1.7} $$

In terms of the Gauss hypergeometric function, the complete elliptic integral of second kind is defined as

$$ E(\eta) = \frac{\pi}{2}\ {}_2F_1\Big(\frac{1}{2}, -\frac{1}{2}; 1; \eta^2\Big). \tag{1.8} $$

2. Main formulae of the elliptical integrals associated with hypergeometric functions

In this section, we establish set of seven elliptical integrals associated with hypergeometric functions, as follows:

Theorem 2.1. *Each of the following assertion holds true:*

$$\int_0^1 \frac{x\ E(c\ x)}{\sqrt{1-x^2}}\ dx = -\frac{1}{32}\pi\Big[\pi\ c\ {}_4F_3\Big(\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{5}{4};1,\frac{3}{2},2;c^2\Big)-$$
$$-16\ {}_4F_3\Big(-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},\frac{3}{2};c^2\Big)\Big] \quad \textit{for } Im(c)\neq 0\ \vee Re(c)<0. \tag{2.1}$$

$$\int_0^1 \frac{x\ E(\frac{x}{c})}{\sqrt{1-x^2}}\ dx = \frac{1}{32}\pi\Big[16\ {}_4F_3\Big(-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{1}{c^2}\Big)-$$
$$-\frac{1}{c}\Big(\pi\ {}_4F_3\Big(\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{5}{4};1,\frac{3}{2},2;\frac{1}{c^2}\Big)\Big)\Big] \quad \textit{for } Im(c)\neq 0\ \vee Re(c)<0. \tag{2.2}$$

$$\int_0^1 \frac{x\ E(c\ x^2)}{\sqrt{1-x^2}}\ dx = \frac{\pi[\sqrt{-(c-1)c}+\sin^{-1}\sqrt{c}]}{4\ \sqrt{c}} \quad \textit{for } Im(c)\neq 0\ \vee Re(c)<0. \tag{2.3}$$

$$\int_0^1 \frac{x\ E(c\ x^4)}{\sqrt{1-x^2}}\ dx = \frac{1}{2}\pi\ {}_3F_2\Big(-\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{4},\frac{5}{4};c\Big) \quad \textit{for } Im(c)\neq 0\ \vee Re(c)<0. \tag{2.4}$$

$$\int_0^1 \frac{x\ E(c\ x^6)}{\sqrt{1-x^2}}\ dx = \frac{1}{2}\pi\ {}_3F_2\Big(-\frac{1}{2},\frac{1}{3},\frac{2}{3};\frac{5}{6},\frac{7}{6};c\Big) \quad \textit{for } Im(c)\neq 0\ \vee Re(c)<0. \tag{2.5}$$

$$\int_0^1 \frac{x\ E(c\ x^8)}{\sqrt{1-x^2}}\ dx = \frac{1}{2}\pi\ {}_5F_4\Big(-\frac{1}{2},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4};\frac{3}{8},\frac{5}{8},\frac{7}{8},\frac{9}{8};c\Big) \textit{for } Im(c)\neq 0 \vee Re(c)<0. \tag{2.6}$$

$$\int_0^1 \frac{x\ E(c\ x^{10})}{\sqrt{1-x^2}}\ dx = \frac{1}{2}\pi\ {}_5F_4\Big(-\frac{1}{2},\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};\frac{3}{10},\frac{7}{10},\frac{9}{10},\frac{11}{10};c\Big) \tag{2.7}$$

for $Im(c)\neq 0\ \vee Re(c)<0$, *provided that each member of the assertions (2.1) to (2.7) exists.*

3. Some results derived from main formulae

In this section, we suggest corresponding results for the assertions (2.1) to (2.7), and also compute numerical values, as given below:
Putting $c=1$ in (2.1), we get

$$\int_0^1 \frac{x\ E(x)}{\sqrt{1-x^2}}\ dx = -\frac{1}{32}\pi\Big[\pi\ {}_4F_3\Big(\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{5}{4};1,\frac{3}{2},2;1\Big)-$$

$$-16\ {}_4F_3\Big(-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},\frac{3}{2};1\Big)\Big] \approx 1.16569. \tag{3.1}$$

Putting $c = 1$ in (2.2), we get

$$\int_0^1 \frac{x\ E(x)}{\sqrt{1-x^2}}\ dx = \frac{1}{32}\pi\Big[16\ {}_4F_3\Big(-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{1}{1}\Big)-$$

$$-\frac{1}{1}\Big(\pi\ {}_4F_3\Big(\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{5}{4};1,\frac{3}{2},2;\frac{1}{1}\Big)\Big)\Big] \approx 1.16569. \tag{3.2}$$

Putting $c = 1$ in (2.3), we get

$$\int_0^1 \frac{x\ E(x^2)}{\sqrt{1-x^2}}\ dx = \frac{\pi^2}{8} \approx 1.2337. \tag{3.3}$$

Putting $c = 1$ in (2.4), we get

$$\int_0^1 \frac{x\ E(x^4)}{\sqrt{1-x^2}}\ dx = \frac{1}{2}\pi\ {}_3F_2\Big(-\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{4},\frac{5}{4};1\Big). \approx 1.30498 \tag{3.4}$$

Putting $c = 1$ in (2.5), we get

$$\int_0^1 \frac{x\ E(x^6)}{\sqrt{1-x^2}}\ dx = \frac{1}{2}\pi\ {}_3F_2\Big(-\frac{1}{2},\frac{1}{3},\frac{2}{3};\frac{5}{6},\frac{7}{6};1\Big). \approx 1.34428 \tag{3.5}$$

Putting $c = 1$ in (2.6), we get

$$\int_0^1 \frac{x\ E(x^8)}{\sqrt{1-x^2}}\ dx = \frac{1}{2}\pi\ {}_5F_4\Big(-\frac{1}{2},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4};\frac{3}{8},\frac{5}{8},\frac{7}{8},\frac{9}{8};1\Big) \approx 1.37007 \tag{3.6}$$

Putting $c = 1$ in (2.7), we get

$$\int_0^1 \frac{x\ E(x^{10})}{\sqrt{1-x^2}}\ dx = \frac{1}{2}\pi\ {}_5F_4\Big(-\frac{1}{2},\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};\frac{3}{10},\frac{7}{10},\frac{9}{10},\frac{11}{10};1\Big). \approx 1.38868 \tag{3.7}$$

4. Derivation of the Main Formulae

We first prove our first assertion (2.1). Consider its left hand side, and by substituting $x = sin\ \theta$, such that $dx = cos\ \theta\ d\theta$, and also apply suitable limits, we have:

$$\int_0^1 \frac{x\ E(c\ x)}{\sqrt{1-x^2}}\ dx$$

$$= \int_0^{\pi/2} \frac{sin\ \theta\ E(c\ sin\ \theta)}{\sqrt{1 - sin^2\ \theta}} cos\ \theta\ d\theta$$

$$= \int_0^{\pi/2} sin\ \theta\ E(c\ sin\ \theta)\ d\theta$$

$$= \frac{1}{2}\pi \int_0^{\pi/2} \sin\ \theta \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k(c\ \sin\ \theta)^k}{(k!)^2}\ d\theta$$

$$= \frac{1}{2}\pi\ \sin \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k\ c^k}{(k!)^2} \int_0^{\pi/2} sin^{k+1}\theta\ d\theta = \frac{1}{4}\pi^{\frac{3}{2}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k\ c^k}{(k!)^2} \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+3}{2})}$$

Now after simplification, we arrived

$$\int_0^1 \frac{x\ E(c\ x)}{\sqrt{1-x^2}}\ dx = -\frac{1}{32}\pi\Big[\pi\ c\ {}_4F_3\Big(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}; 1, \frac{3}{2}, 2; c^2\Big)$$

$$-16\ {}_4F_3\Big(-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; c^2\Big)\Big].$$

This completes our demonstration of the first assertion (2.1).
Assertion (2.2) is easily, and can be proved similarly as (2.1), hence we left for the readers.
Next, we prove our third assertion (2.3). Consider its left hand side, and by substituting $x = sin\ \theta$, such that $dx = cos\ \theta\ d\theta$, and also apply suitable limits, we have:

$$\int_0^1 \frac{x\ E(c\ x^2)}{\sqrt{1-x^2}}\ dx$$

$$= \int_0^{\pi/2} \frac{sin\ \theta\ E(c\ sin^2\ \theta)}{\sqrt{1 - sin^2\ \theta}} cos\ \theta\ d\theta$$

$$= \int_0^{\pi/2} sin\ \theta\ E(c\ sin^2\ \theta)\ d\theta$$

$$= \frac{1}{2}\pi \int_0^{\pi/2} \sin\ \theta \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k(c\ \sin^2\ \theta)^k}{(k!)^2}\ d\theta$$

$$= \frac{1}{2}\pi\ \sin \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k\ c^k}{(k!)^2} \int_0^{\pi/2} sin^{2k+1}\theta\ d\theta = \frac{1}{4}\pi^{\frac{3}{2}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k\ c^k}{(k!)^2} \frac{\Gamma(k+1)}{\Gamma\Big(\frac{2k+3}{2}\Big)}.$$

Now after simplification, we got:

$$\int_0^1 \frac{x\ E(c\ x^2)}{\sqrt{1-x^2}}\ dx = \frac{\pi[\sqrt{-(c-1)c} + \sin^{-1}\sqrt{c}]}{4\ \sqrt{c}}.$$

This completes our demonstration of the first assertion (2.3).
Assertion (2.4) is easily, and can be proved similarly as (2.3), hence we left for the readers.
Next, we prove our fifth assertion (2.5). Consider its left hand side, and by substituting $x = sin\ \theta$,such that $dx = cos\ \theta\ d\theta$, and also apply suitable limits, we have:

$$\int_0^1 \frac{x\ E(c\ x^6)}{\sqrt{1-x^2}}\ dx$$

$$= \int_0^{\pi/2} \frac{sin\ \theta\ E(c\ sin^6\ \theta)}{\sqrt{1-sin^2\ \theta}} cos\ \theta\ d\theta$$

$$= \int_0^{\pi/2} sin\ \theta\ E(c\ sin^6\ \theta)\ d\theta$$

$$= \frac{1}{2}\pi \int_0^{\pi/2} \sin\ \theta \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k(c\ \sin^6\ \theta)^k}{(k!)^2}\ d\theta$$

$$= \frac{1}{2}\pi \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k\ c^k}{(k!)^2} \int_0^{\pi/2} sin^{6k+1}\theta\ d\theta$$

$$= \frac{1}{4}\pi^{\frac{3}{2}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k\ c^k}{(k!)^2} \frac{\Gamma(3k+1)}{\Gamma\left(\frac{6k+3}{2}\right)}.$$

Now after simplification, we obtained:

$$\int_0^1 \frac{x\ E(c\ x^6)}{\sqrt{1-x^2}}\ dx = \frac{1}{2}\pi\ {}_3F_2\Big(-\frac{1}{2}, \frac{1}{3}, \frac{2}{3}; \frac{5}{6}, \frac{7}{6}; c\Big).$$

This completes our demonstration of the first assertion (2.5).
Assertion (2.6) is easily, and can be proved similarly as (2.5), hence we left for the readers.
Next, we prove our seventh assertion (2.7). Consider its left hand side, and by

substituting $x = sin\ \theta$,such that $dx = cos\ \theta\ d\theta$, and also apply suitable limits, we have:

$$\int_0^1 \frac{x\ E(c\ x^{10})}{\sqrt{1-x^2}}\ dx$$

$$= \int_0^{\pi/2} \frac{sin\ \theta\ E(c\ sin^{10}\ \theta)}{\sqrt{1-sin^2\ \theta}} cos\ \theta\ d\theta$$

$$= \int_0^{\pi/2} sin\ \theta\ E(c\ sin^{10}\ \theta)\ d\theta$$

$$= \frac{1}{2}\pi \int_0^{\pi/2} \sin\ \theta \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k(c\ \sin^{10}\ \theta)^k}{(k!)^2}\ d\theta$$

$$= \frac{1}{2}\pi \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k\ c^k}{(k!)^2} \int_0^{\pi/2} sin^{10k+1}\theta\ d\theta$$

$$= \frac{1}{4}\pi^{\frac{3}{2}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1}{2})_k\ c^k}{(k!)^2} \frac{\Gamma(5k+1)}{\Gamma\Big(\frac{10k+3}{2}\Big)}.$$

Now after simplification, we arrived:

$$\int_0^1 \frac{x\ E(c\ x^{10})}{\sqrt{1-x^2}}\ dx = \frac{1}{2}\pi\ {}_5F_4\Big(-\frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \frac{3}{10}, \frac{7}{10}, \frac{9}{10}, \frac{11}{10}; c\Big).$$

This completes our demonstration of the first assertion (2.7).
This obviously completes our proof of Theorem 1.

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