J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 11, No. 1 (2023), pp. 79-106

DOI: 10.56827/JRSMMS.2023.1101.6 **ISSN (Online): 2582-5461**

ISSN (Print): 2319-1023

P-VALENTLY BAZILEVIČ FUNCTIONS DEFINED WITH HIGHER ORDER DERIVATIVES

Mohamed K. Aouf, Adela O. Mostafa and Teodor Bulboacă*

Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, EGYPT

E-mail : mkaouf@mans.edu.eg, osmanz@mans.edu.eg

*Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 400084 Cluj-Napoca, ROMANIA

E-mail : teodor.bulboaca@ubbcluj.ro

Dedicated to the memory of Professor Petru T. Mocanu (1931–2016)

(**Received:** *Dec. 21, 2023* **Accepted:** *Dec. 28, 2023* **Published:** *Dec. 30, 2023)*

Abstract: The object of this paper is to derive various properties and characters for a class of p-valently Bazilevič functions defined with higher order derivatives. Using the technique of Briot-Bouquet differential subordination we obtained several results, some of them presented here generalize and improve a few earlier results, and several others are new ones involving differential inequalities. These last implications deal with the symmetric open disc and symmetric conic domains with respect to the real axe. Our results extends and improves many other previous ones, and some of them are the best possible under the given assumptions.

Keywords and Phrases: p-valent functions, Bazilevič functions, Gauss hypergeometric function, differential subordination, p-valently close-to-convex functions of order β, higher order derivative, Briot-Bouquet differential equations and differential subordinations.

2020 Mathematics Subject Classification: 30C45, 30C80.

1. Introduction

Denote by $\mathcal{A}(p)$ the class of multivalent analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},\ p \in \mathbb{N} := \{1, 2, \dots\}, \quad (1.1)$$

and let $\mathcal{A}(1) =: \mathcal{A}$.

If f and g are analytic functions in $\mathbb{D}$, we say that f is *subordinate* to g, denoted by $f(z) \prec g(z)$, or g is superordinate to f, if there exists a Schwarz function w, that is w is analytic in $\mathbb{D}$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \mathbb{D}$. Furthermore, if g is univalent in $\mathbb{D}$, then the next equivalence holds (see [11] and [21]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

Differentiating the relation (1.1) q-times, we have

$$f^{(q)}(z) = \delta(p, q) z^{p-q} + \sum_{k=p+1}^{\infty} \delta(k, q) a_k z^{k-q},\ z \in \mathbb{D},\ q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where

$$\delta(p, q) := \frac{p!}{(p-q)!},\ p > q.$$

Definition 1.1. *For $f \in \mathcal{A}(p)$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, with $p > q$, and $0 \le \beta < p - q$, we say that $f \in \mathbb{S}^*_{p,q}(\beta)$ if it satisfies the inequality*

$$\operatorname{Re} \frac{z f^{(1+q)}(z)}{f^{(q)}(z)} > \beta,\ z \in \mathbb{D},$$

and it is in the class $\mathbb{K}_{p,q}(\beta)$ if it satisfies

$$\operatorname{Re} \left(1 + \frac{z f^{(2+q)}(z)}{f^{(1+q)}(z)}\right) > \beta,\ z \in \mathbb{D}.$$

The classes $\mathbb{S}^*_{p,q}(\beta)$ and $\mathbb{K}_{p,q}(\beta)$ were introduced and studied by Aouf in [4, 6, 7]. Also, the classes $\mathbb{S}^*_{p,0}(\beta) =: \mathbb{S}^*_p(\beta)$ and $\mathbb{K}_{p,0}(\beta) =: \mathbb{K}_p(\beta)$ have been extensively studied by Aouf [1, 2, 10] and Owa [28].

Definition 1.2. *For $0 < \beta \le p - q$, with $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, and $p > q$, we say that the function $f \in \mathcal{A}(p)$ is in the class $C_{p,q}(\beta)$, if there exists a function $g \in \mathbb{S}^*_{p,q}(0) =: \mathbb{S}^*_{p,q}$ such that*

$$\operatorname{Re} \frac{z f^{(1+q)}(z)}{g^{(q)}(z)} > \beta,\ z \in \mathbb{D}.$$

We note that $C_{p,0}(\beta) =: C_p(\beta)$ *represents the class of p-valently close-to-convex functions of order* β *(see Aouf* [3, 8]*).*

Definition 1.3. *A function* $f \in \mathcal{A}(p)$*, with* $f^{(q)}(z) \neq 0$ *for all* $z \in \mathbb{D} \setminus \{0\}$*, is said to be p-valently Bazilevič of type* μ *and order* β *with higher order derivatives, if there exists a function* $g \in \mathbb{S}^*_{p,q}$ *such that*

$$\operatorname{Re}\left[\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu}\right] > \beta,\ z \in \mathbb{D} \tag{1.2}$$

for some μ*, with* $\mu \geq 0$*, and* β*, with* $0 \leq \beta < p - q$*, and we denote by* $B_{p,q}(\mu, \beta)$ *the class of these functions.*

We remark that $B_{p,0}(\mu, \beta) =: B_p(\mu, \beta)$ (see Patel and Cho [31]) and $B_{p,0}(1, \beta) =: C_p(\beta)$, while for arbitrary real numbers A and B, with $-1 \leq B < A \leq 1$, let

$$\mathbb{S}^*_p(A, B) := \left\{ f \in \mathcal{A}(p) : \frac{zf'(z)}{f(z)} \prec p\frac{1 + Az}{1 + Bz} \right\},$$

One can see that $\mathbb{S}^*_p\left(1 - \dfrac{2\alpha}{p}, -1\right) =: \mathbb{S}^*_p(\alpha)$, where $0 \leq \alpha < p$, and the class $\mathbb{S}^*_p(A, B)$ was introduced by Goel and Sohi [15], and Aouf [2].

Now, for $\mu \geq 0$, $\lambda > 0$, $-1 \leq B < A \leq 1$, and $g \in \mathbb{S}^*_{p,q}$, we define the class $\mathcal{M}_{p,q}(\lambda, \mu, A, B)$ as follows:

Definition 1.4. *A function* $f \in \mathcal{A}(p)$*, with* $f^{(q)}(z) \neq 0$ *for all* $z \in \mathbb{D} \setminus \{0\}$*, is said to be in the class* $\mathcal{M}_{p,q}(\lambda, \mu, A, B)$ *if*

$$\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} + \lambda\left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right] \prec (p - q)\frac{1 + Az}{1 + Bz},$$

for some $g \in \mathbb{S}^*_{p,q}$*, where the powers are the principal ones.*

For convenience, if $0 \leq \beta < p - q$, we write

$$\begin{aligned}\mathcal{M}_{p,q}(\lambda, \mu, \beta) &:= \mathcal{M}_{p,q}\left(\lambda, \mu, 1 - \frac{2\beta}{p - q}, -1\right)\\ &= \left\{ f \in \mathcal{A}(p) : \operatorname{Re}\left\{\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} + \lambda\left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \right.\right.\right.\\ &\qquad \left.\left.\left. - \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right]\right\} > \beta \right\}.\end{aligned}$$

We note that the class $\mathcal{M}_{p,0}(\lambda,\mu,A,B) =: \mathcal{M}_p(\lambda,\mu,A,B)$ was introduced by Patel and Cho [31] (see also Guo and Liu [16], and Wang and Jiang [34]), the class $\mathcal{M}_{p,0}(\lambda,\mu,\beta) =: \mathcal{M}_p(\lambda,\mu,\beta)$ $(0 \le \beta < p)$ was defined by Patel and Cho [31] (see also [34]), while the class $\mathcal{M}_1(\lambda,\mu,\beta) =: \mathcal{M}(\lambda,\mu,\beta)$ $(0 \le \beta < 1)$ was introduced and studied by Guo and Liu [16].

To prove our main results we shall require the following lemmas. The first one deals with the *Briot-Bouquet* differential equations and differential subordinations:

Lemma 1.1. [20, Corollary 3.2] *If* $-1 \le B < A \le 1$, $\beta > 0$, *and the complex number* γ *satisfies* $\operatorname{Re}\gamma \ge -\dfrac{\beta(1-A)}{1-B}$, *then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = \frac{1+Az}{1+Bz},$$

has a univalent solution in $\mathbb{D}$ *given by*

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\frac{\beta(A-B)}{B}}}{\beta\displaystyle\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\frac{\beta(A-B)}{B}}dt} - \dfrac{\gamma}{\beta}, & \text{if } \ B \ne 0, \\ \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta\displaystyle\int_0^z t^{\beta+\gamma-1}\exp(\beta At)dt} - \dfrac{\gamma}{\beta}, & \text{if } \ B = 0. \end{cases}$$

If $\phi(z) = 1 + c_1z + c_2z^2 + \dots$ *is analytic in* $\mathbb{D}$ *and satisfies*

$$\phi(z) + \frac{z\phi'(z)}{\beta\phi(z)+\gamma} \prec \frac{1+Az}{1+Bz}, \tag{1.3}$$

then

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz}$$

and q *is the best dominant of* (1.3).

Lemma 1.2. [36, Lemma 2] *Let* ν *be a positive measure on the interval* $[0,1]$. *Let* $h(z,t)$ *be a complex valued function defined on* $\mathbb{D}\times[0,1]$ *such that* $h(\cdot,t)$ *is analytic in* $\mathbb{D}$ *for each* $t \in [0,1]$, *and such that* $h(z,\cdot)$ *is* ν*-integrable on* $[0,1]$ *for all* $z \in \mathbb{D}$. *In addition, suppose that* $\operatorname{Re} h(z,t) > 0$, $h(-r,t)$ *is real and*

$$\operatorname{Re}\frac{1}{h(z,t)} \ge \frac{1}{h(-r,t)},\ |z| \le r < 1,\ t \in [0,1].$$

If G *is defined by*

$$G(z)=\int_0^1 h(z,t)d\nu(t),$$

then $\operatorname{Re}\dfrac{1}{G(z)}\geq\dfrac{1}{G(-r)},\ |z|\leq r<1.$

Lemma 1.3. [18] *Let* F *be an analytic convex function in* $\mathbb{D}$*. If* $f,g\in\mathcal{A}$ *and* $f(z),\ g(z)\prec F(z)$*, then*

$$\lambda f(z)+(1-\lambda)g(z)\prec F(z),\ 0\leq\lambda\leq 1.$$

Lemma 1.4. [17] *If* $-1\leq B_1\leq B_2<A_2\leq A_1\leq 1$*, then*

$$\frac{1+A_2z}{1+B_2z}\prec\frac{1+A_1z}{1+B_1z}.$$

Lemma 1.5. [23, 24] *Let* $p(z)=1+p_1z+p_2z^2+\ldots$ *be analytic in* $\mathbb{D}$ *and* $p(z)\neq 0$ *for all* $z\in\mathbb{D}$*. If there exists a point* $z_0\in\mathbb{D}$ *such that*

$$|\arg p(z)|<\frac{\pi}{2}\eta,\ \textit{for}\ |z|<|z_0|\,,$$

and

$$|\arg p(z_0)|=\frac{\pi}{2}\eta,\ 0<\eta\leq 1.$$

Then, we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\eta,$$

where

$$\begin{aligned} k&\geq\frac{1}{2}\left(a+\frac{1}{a}\right), &&\textit{if}\quad \arg p(z_0)=\frac{\pi}{2}\eta,\\ k&\leq-\frac{1}{2}\left(a+\frac{1}{a}\right), &&\textit{if}\quad \arg p(z_0)=-\frac{\pi}{2}\eta, \end{aligned}\tag{1.4}$$

and

$$(p(z_0))^{\frac{1}{\eta}}=\pm ia,\ (a>0).$$

Lemma 1.6. [35] *For real or complex numbers* a*,* b *and* c $(c\neq 0,-1,-2,\ldots)$*, the Gauss hypergeometric function* ${}_2F_1(a,b;c;z)$ *satisfies the identities:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z),\tag{1.5}$$

$$z\in\mathbb{C}\setminus(1,+\infty),\ \operatorname{Re}c>\operatorname{Re}b>0,$$

$${}_2F_1(a,b;c;z)={}_2F_1(b,a;c;z),$$

and

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right),\ z \in \mathbb{C} \setminus (1, +\infty). $$

Lemma 1.7. [21, Theorem 3.4h.] *Let g be univalent in $\mathbb{D}$ and let Θ and ϕ be analytic in a domain $\mathbb{D}$ containing $g(\mathbb{D})$, with $\phi(w) \neq 0$ when $w \in g(\mathbb{D})$. Set*

$$ Q(z) = zg'(z)\phi(g(z)),\ h(z) = \Theta(g(z)) + Q(z), $$

and suppose that:

$$ (i) \qquad Q \text{ is starlike in } \mathbb{D}; $$

$$ (ii) \qquad \operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\frac{\Theta'(g(z))}{\phi(g(z))} + \frac{zQ'(z)}{Q(z)}\right] > 0,\ z \in \mathbb{D}. $$

If P is analytic in $\mathbb{D}$, $P(0) = g(0)$, $P(\mathbb{D}) \subset D$, and

$$ \Theta(P(z)) + zP'(z)\phi(P(z)) \prec \Theta(g(z)) + zg'(z)\phi(g(z)), \tag{1.6} $$

then

$$ P(z) \prec g(z), $$

and g is the best dominant of (1.6).

2. Main Results

Unless otherwise stated we assume that $f \in \mathcal{A}(p)$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ with $p > q$, $\lambda > 0$, $\mu \geq 0$, $-1 \leq B < A \leq 1$, and all the powers are considered the principal ones. Our first result gives the best dominant for the LHS of (1.2) for functions of $\mathcal{M}_{p,q}(\lambda, \mu, A, B)$ and the inclusion of these classes in $B_{p,q}(\mu, \rho)$ where the value of ρ is the best possible (i.e. it is sharp).

Theorem 2.1. *Let $f \in \mathcal{M}_{p,q}(\lambda, \mu, A, B)$.*
(i) Then,

$$ \frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} \prec \frac{\lambda}{(p-q)Q(z)} =: q(z), \tag{2.1} $$

where

$$ Q(z) = \begin{cases} \dfrac{\lambda}{p-q}\ {}_2F_1\left(1, \dfrac{(p-q)(B-A)}{\lambda B}; \dfrac{p-q}{\lambda} + 1; \dfrac{Bz}{Bz+1}\right), & \text{if } B \neq 0, \\ \displaystyle\int_0^1 s^{\frac{p-q}{\lambda}-1} \exp \frac{(p-q)Az(s-1)}{\lambda} ds, & \text{if } B = 0, \end{cases} \tag{2.2} $$

and q is the best dominant of (2.1).
(ii) Furthermore, if $A \leq \dfrac{-\lambda B}{p-q}$ with $-1 \leq B < 0$, then

$$\mathcal{M}_{p,q}(\lambda, \mu, A, B) \subset B_{p,q}(\mu, \rho), \tag{2.3}$$

where

$$\rho := \rho(\lambda, \mu, A, B) = (p-q)\left[{}_2F_1\left(1, \frac{(p-q)(B-A)}{\lambda B}; \frac{p-q}{\lambda}+1; \frac{B}{B-1}\right)\right]^{-1}.$$

The result is the best possible.
Proof. Let $f \in \mathcal{M}_{p,q}(\lambda, \mu, A, B)$ and define the function

$$\phi(z) := \frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu}, z \in \mathbb{D}. \tag{2.4}$$

Then, ϕ is analytic in $\mathbb{D}$ with $\phi(0) = 1$, and differentiating (2.4) we have

$$(p-q)\phi(z) + \lambda\frac{z\phi'(z)}{\phi(z)} = \frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu}$$
$$+\lambda\left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right] \prec (p-q)\frac{1+Az}{1+Bz}. \tag{2.5}$$

Thus, ϕ satisfies the differential subordination (1.3), and by applying Lemma 1.1 for $\beta = \dfrac{p-q}{\lambda}$ and $\gamma = 0$ we get

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where q is given by (2.1) and (2.2), and is the best dominant of (2.5), which proves the assertion (2.1).

Next we will show that

$$\inf\{\operatorname{Re} q(z) : |z| < 1\} = q(-1). \tag{2.6}$$

If we set $a = \dfrac{(p-q)(B-A)}{\lambda B}$, $b = \dfrac{p-q}{\lambda}$, $c = \dfrac{p-q}{\lambda} + 1$, then $c > b > 0$. From (2.2), and using Lemma 1.6 we see that for $B \neq 0$ we get

$$Q(z) = (1+Bz)^{\frac{(p-q)(B-A)}{\lambda B}} \int_0^1 s^{\frac{p-q}{\lambda}-1}(1+Bzs)^{-\frac{(p-q)(B-A)}{\lambda B}} ds$$
$$= \frac{\lambda}{p-q}\,{}_2F_1\left(1, \frac{(p-q)(B-A)}{\lambda B}; \frac{p-q}{\lambda}+1; \frac{Bz}{Bz+1}\right). \tag{2.7}$$

To prove (2.6) we need to show that $\operatorname{Re}\dfrac{1}{Q(z)} \geq \dfrac{1}{Q(-1)}$, $z \in \mathbb{D}$. Since $A < -\dfrac{\lambda B}{p-q}$ implies $c > a > 0$, by using (1.5) and (2.7) we have

$$Q(z) = \int_0^1 h(z,s)d\nu(s),$$

where

$$h(z,s) = \frac{1+Bz}{1+(1-s)Bz} \; (0 \leq s \leq 1) \quad \text{and} \quad d\nu(s) = \frac{\lambda}{p-q} s^{\frac{(p-q)(B-A)}{\lambda B}} (1-s)^{\frac{(p-q)A}{\lambda B}} ds,$$

which is a positive measure on $[0,1]$. For $-1 \leq B < 0$, it may be noted that $\operatorname{Re} h(z,s) > 0$, $h(-r,s)$ is is real for $|z| \leq r$, $0 \leq r < 1$ and $0 \leq s \leq 1$, and

$$\operatorname{Re}\frac{1}{h(z,s)} = \operatorname{Re}\frac{1+(1-s)Bz}{1+Bz} \geq \frac{1-(1-s)Br}{1-Br} = \frac{1}{h(-r,s)}.$$

Therefore, by Lemma 1.2 we get

$$\operatorname{Re}\frac{1}{Q(z)} \geq \frac{1}{Q(-r)}, |z| \leq r < 1,$$

and by letting $r \to 1^-$ we obtain $\operatorname{Re}\dfrac{1}{Q(z)} \geq \dfrac{1}{Q(-1)}$.

Further, taking $A \to -\dfrac{\lambda B}{p-q}$ for the case $A = -\dfrac{\lambda B}{p-q}$ and by (2.1), we get that (2.3) holds. The result is the best possible since q is the best dominant of (2.1), which completes the proof.

The following three corollaries represent particular cases of the above theorem obtained for different choices of the parameter. We emphasize that these results represent generalizations and extensions of some previous ones obtained by different authors.

Thus, putting $\mu = 1$, $A = 1 - \dfrac{2\beta}{p-q}$, with $\dfrac{p-q-\lambda}{2} \leq \beta < p-q$, and $B = -1$ in the second part of Theorem 2.1 we have:

Corollary 2.1. *If $f \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re}\left\{\frac{zf^{(1+q)}(z)}{g^{(q)}(z)} + \lambda\left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - \frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right]\right\} > \beta, \; z \in \mathbb{D}, \; (\lambda > 0),$$

for some $g \in \mathbb{S}^*_{p,q}$, *then* $f \in B_{p,q}\left(\Psi(p,q,\lambda,\beta)\right)$, *where*

$$\Psi(p,q,\lambda,\beta) := (p-q)\left[{}_2F_1\left(1, \frac{2(p-q-\beta)}{\lambda}; \frac{p-q}{\lambda}+1; \frac{1}{2}\right)\right]^{-1}. \tag{2.8}$$

The result is the best possible.

Taking $\mu = 0$, $A = 1 - \dfrac{2\beta}{p-q}$, with $\dfrac{p-q-\lambda}{2} \leq \beta < p-q$, and $B = -1$ in Theorem 2.1 we obtain:

Corollary 2.2. *If* $f \in \mathcal{A}(p)$ *satisfies*

$$\operatorname{Re}\left[(1-\lambda)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + \lambda\left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right)\right] > \beta,\ z \in \mathbb{D},\ (\lambda > 0),$$

then $f \in \mathbb{S}^*_{p,q}\left(\Psi(p,q,\lambda,\beta)\right)$, *where* $\Psi(p,q,\lambda,\beta)$ *is given by* (2.8). *The result is the best possible.*

Remark 2.1. Our results of Theorem 2.1 and Corollary 2.1 and Corollary 2.2 but for $q = 0$ were also obtained by Patel [30, Theorem 3.1 and Corollaries 3.2 and 3.3, respectively].

Putting $\lambda = 1$ in Corollary 2.2 we get the next special case:

Corollary 2.3. *If* $\dfrac{p-q-1}{2} \leq \beta < p$, *then*

$$\mathbb{K}_{p,q}(\beta) \subset \mathbb{S}^*_{p,q}(\rho(p,q,\beta)),$$

where $\rho(p,q,\beta) := (p-q)\left\{{}_2F_1\left(1, 2(p-q-\beta); p-q+1; \frac{1}{2}\right)\right\}^{-1}$. *The result is the best possible.*

Remark 2.2. (i) Putting $q = 0$ in Corollary 2.3 we obtain the result of Srivastava et al. [33, Corollary 7], and Patel [30, Corollary 3.4];

(ii) Putting $p = 1$ and $q = 0$ in Corollary 2.3 we obtain the results obtained by MacGreogor [19], and Wilken and Feng [36].

In the next result we found the best radius where the reverse inclusion of Theorem 2.1 (ii) holds, and we underline that result is the best possible under the given assumptions.

Theorem 2.2. *If* $f \in B_{p,q}(\mu,\beta)$ *for some* μ, *with* $\mu \geq 0$, *and* β, *with* $0 \leq \beta < p-q$, *then $f \in M_{p,q}(\lambda, \mu, \beta)$ for $|z| < R(p, q, \lambda, \beta)$, where $\lambda > 0$, and*

$$R(p,q,\lambda,\beta) := \begin{cases} \dfrac{(p-q+\lambda-\beta) - \sqrt{(p-q+\lambda-\beta)^2 - (p-q)(p-q-2\beta)}}{p-q-2\beta}, & \\ & if \quad \beta \neq \dfrac{p-q}{2}, \\ \dfrac{p-q}{p-q+2\lambda}, \quad if \quad \beta = \dfrac{p-q}{2}, & \end{cases} \tag{2.9}$$

and the bound $R(p, q, \lambda, \beta)$ is the best possible.

Proof. For $f \in B_{p,q}(\mu, \beta)$, according to (1.2) let define the function h by

$$\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} = \beta + (p-q-\beta)h(z), \ z \in \mathbb{D}, \tag{2.10}$$

where $h(z) = 1 + h_1 z + h_2 z^2 + \dots$ is analytic and have positive real part in $\mathbb{D}$. Differentiating (2.10), we obtain

$$\begin{aligned} &\operatorname{Re}\left\{\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} + \lambda\left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\right.\right. \\ &\qquad\qquad\qquad\qquad \left.\left. - \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right]\right\} - \beta \\ &\quad = (p-q-\beta)\operatorname{Re}\left[h(z) + \lambda\frac{zh'(z)}{\beta + (p-q-\beta)h(z)}\right] \\ &\quad \geq (p-q-\beta)\operatorname{Re}\left[h(z) - \lambda\frac{|zh'(z)|}{|\beta + (p-q-\beta)h(z)|}\right], \ z \in \mathbb{D}. \end{aligned} \tag{2.11}$$

Using in (2.11) the well-known estimates [19]

$$|zh'(z)| \leq \frac{2r}{1-r^2}\operatorname{Re} h(z), \quad \text{and} \quad \operatorname{Re} h(z) \geq \frac{1-r}{1+r}, \quad |z| = r < 1,$$

we get

$$\begin{aligned} &\operatorname{Re}\left\{\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} + \lambda\left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\right.\right. \\ &\qquad\qquad\qquad\qquad \left.\left. - \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right]\right\} - \beta \\ &\geq (p-q-\beta)\operatorname{Re} h(z)\left[1 - \frac{2\lambda r}{\beta(1-r^2) + (p-q-\beta)(1-r)^2}\right], \ |z| = r < 1. \end{aligned} \tag{2.12}$$

Since $0 \leq \beta < p-q$ and $0 \leq r < 1$, the inequality

$$1 - \frac{2\lambda r}{\beta(1-r^2)+(p-q-\beta)(1-r)^2} \geq 0 \tag{2.13}$$

is equivalent to

$$\varphi(r) := (p-q-2\beta)r^2 - 2(p-q+\lambda-\beta)r + p - q \geq 0.$$

Using the fact that $\varphi(0) = p-q > 0$ and $\varphi(1) = -\lambda < 0$, the inequality (2.13) holds for $r \in [0, r_*]$, where $r_* = R(p,q,\lambda,\beta)$ is the smallest positive root of φ. Therefore, the right-hand side of (2.12) is positive whenever $r < R(p,q,\lambda,\beta)$, where $R(p,q,\lambda,\beta)$ is given by (2.9).

To show that the bound $R(p,q,\lambda,\beta)$ is the best possible, consider the function $f \in \mathcal{A}(p)$ defined by

$$\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} = \beta + (p-q-\beta)\frac{1-z}{1+z},\ 0 \leq \beta < p-q,$$

for some $g \in \mathbb{S}^*_{p,q}$. Noting that

$$\begin{aligned}\operatorname{Re}\Bigg\{ \frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} + \lambda\Bigg[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \\ - \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\Bigg]\Bigg\} - \beta \\ = (p-q-\beta)\left[\frac{1-z}{1+z} + \frac{2\lambda z}{\beta(1+z)^2 + (p-q-\beta)(1-z^2)}\right] = 0\end{aligned}$$

for $z = R(p,q,\lambda,\beta)$, we conclude that the bound $R(p,q,\lambda,\beta)$ is the best possible, which completes our proof.

The next two corollaries are special cases of the above theorem, that on their turn generalise some previous results of obtained in different articles.

Thus, putting $\mu = 0$ and $\lambda = 1$ in Theorem 2.2 we get the next special case:

Corollary 2.4. *If $f \in \mathbb{S}^*_{p,q}(\beta)$, with $0 \leq \beta < p-q$, then $f \in \mathbb{K}_{p,q}(\beta)$ in $|z| < R^*(p,q,\beta)$, where*

$$R^*(p,q,\beta) := \begin{cases} \dfrac{(p-q+1-\beta) - \sqrt{\beta^2 + 2(p-q-\beta)+1}}{p-q-2\beta}, & \text{if } \beta \neq \dfrac{p-q}{2} \\ \dfrac{p-q}{p-q+2}, & \text{if } \beta = \dfrac{p-q}{2}, \end{cases}$$

and the bound $R^*(p,q,\beta)$ *is the best possible.*

For $q=0$, Corollary 2.4 reduces to the next result:

Corollary 2.5. *If* $f\in\mathbb{S}_p^*(\beta)$, *with* $0\leq\beta<p$, *then* $f\in\mathbb{K}_p(\beta)$ *in* $|z|<R_1(p,\beta)$, *where*

$$R_1(p,\beta):=\begin{cases}\dfrac{(p-\beta+1)-\sqrt{\beta^2+2(p-\beta)+1}}{p-2\beta}, & \text{if } \beta\neq\dfrac{p}{2},\\ \dfrac{p}{p+2}, & \text{if } \beta=\dfrac{p}{2},\end{cases}$$

and the bound $R_1(p,\beta)$ *is the best possible.*

Remark 2.3. The result of Corollary 2.5 was previously obtained by Patel [30, Corollary 3.7], Patel and Cho [31, Corollary 3.3], and Aouf et al. [9, Corollary 3.8].

In the next theorem we give sufficient conditions for a function f that belongs to a subclass of $B_{p,q}(\mu,\beta)$ to be in $\mathbb{K}_{p,q}:=\mathbb{K}_{p,q}^*(0)$ for a sufficient small disc included in $\mathbb{D}$.

Theorem 2.3. *If* $f\in\mathcal{A}(p)$ *and satisfies the following conditions*

$$\operatorname{Re}\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}>0,\ z\in\mathbb{D}, \tag{2.14}$$

$$\left|\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu}-(p-q)\right|<\nu(p-q),\ z\in\mathbb{D}, \tag{2.15}$$

with $0\leq\mu\leq1$ *and* $0<\nu\leq1$, *for some* $g\in\mathbb{S}_{p,q}^*$, *then* $f\in\mathbb{K}_{p,q}:=\mathbb{K}_{p,q}^*(0)$ *in* $|z|<R^*(p,q,\mu,\nu)$, *where*

$$R^*(p,q,\mu,\nu):=\min\{r_*;1\},$$

and r_* *is the smallest positive root of*

$$[(p-q)(2\mu-1)-\nu]r^2-[2(p-q-1)\mu+\nu+2]r+p-q=0.$$

Proof. Letting

$$h(z)=\frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu}-1,\ z\in\mathbb{D}, \tag{2.16}$$

from (2.15) it follows that h is analytic in $\mathbb{D}$, with $h(0)=0$ and $|h(z)|<\nu$, $z\in\mathbb{D}$. Therefore, the function h has the form $h(z)=\nu\Phi(z)$, where Φ ia analytic in $\mathbb{D}$, with $\Phi(0) = 0$ and $|\Phi(z)| < 1$ for all $z \in \mathbb{D}$, Thus, from the definition relation (2.16) we have

$$zf^{(1+q)}(z) = (p-q)\left(f^{(q)}(z)\right)^{1-\mu}\left(g^{(q)}(z)\right)^{\mu}(1+\nu\Phi(z)),\ z \in \mathbb{D}, \tag{2.17}$$

and differentiating (2.17) with respect to z, we get

$$1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} = (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)} + \frac{\nu z\Phi'(z)}{1+\nu\Phi(z)},\ z \in \mathbb{D}. \tag{2.18}$$

Putting $K(z) := \dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}$, $z \in \mathbb{D}$, then $K(0) = 1$, and from (2.14) we have $\operatorname{Re} K(z) > 0$ for all $z \in \mathbb{D}$. Also,

$$\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} = p - q + \frac{zK'(z)}{K(z)},\ z \in \mathbb{D}, \tag{2.19}$$

and from (2.19) and (2.18) we obtain

$$\begin{aligned} 1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} =& (1-\mu)(p-q) + (1-\mu)\frac{zK'(z)}{K(z)} \\ &+ \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)} + \frac{\nu z\Phi'(z)}{1+\nu\Phi(z)},\ z \in \mathbb{D}. \end{aligned} \tag{2.20}$$

Using the well-known estimates [19], since $K(0) = 1$ and $\operatorname{Re} K(z) > 0$ for all $z \in \mathbb{D}$, we have

$$\left|\frac{zK'(z)}{K(z)}\right| \leq \frac{2r}{1-r^2},\ |z| = r < 1,$$

hence

$$\operatorname{Re}\frac{zK'(z)}{K(z)} \geq -\frac{2r}{1-r^2},\ |z| = r < 1. \tag{2.21}$$

Denoting $H(z) := \dfrac{zg^{(1+q)}(z)}{(p-q)g^{(q)}(z)}$, then $H(0) = 1$, and from $g \in \mathbb{S}^*_{p,q}$ it follows that $\operatorname{Re} H(z) > 0$ for all $z \in \mathbb{D}$. According to the well-known estimates [19], we get

$$\operatorname{Re} H(z) \geq \frac{1-r}{1+r},\ |z| = r < 1,$$

that is

$$\operatorname{Re}\frac{zg^{(1+q)}(z)}{g^{(q)}(z)} \geq \frac{(p-q)(1-r)}{1+r},\ |z| = r < 1. \tag{2.22}$$

Since Φ is analytic in $\mathbb{D}$, with $\Phi(0)=0$ and $|\Phi(z)|<1$ for all $z\in\mathbb{D}$, from [12] we have

$$|\Phi'(z)|\le\frac{1-|\Phi(z)|^2}{1-|z|^2},\ z\in\mathbb{D},$$

and since $0<\nu\le1$, this implies

$$\left|\frac{\nu z\Phi'(z)}{1+\nu\Phi(z)}\right|\le\frac{\nu|z|}{1-|z|^2}\frac{1-|\phi(z)|^2}{|1+\nu\Phi(z)|}\le\frac{\nu|z|}{1-|z|^2}\frac{1-|\Phi(z)|^2}{1-\nu|\Phi(z)|}$$
$$\le\frac{\nu|z|}{1-|z|^2}\frac{1-|\Phi(z)|^2}{1-|\Phi(z)|}=\frac{\nu|z|}{1-|z|^2}\left(1+|\Phi(z)|\right),\ z\in\mathbb{D}.$$

Moreover, from the Schwarz's lemma we have $|\Phi(z)|\le|z|$ for all $z\in\mathbb{D}$, then from the above inequality we obtain

$$\left|\frac{\nu z\Phi'(z)}{1+\nu\Phi(z)}\right|\le\frac{\nu|z|\left(1+|z|\right)}{1-|z|^2}=\frac{\nu r}{1-r},\ |z|=r<1,$$

hence

$$\operatorname{Re}\frac{\nu z\Phi'(z)}{1+\nu\Phi(z)}\ge-\frac{\nu r}{1-r},\ |z|=r<1.\tag{2.23}$$

Using the inequalities (2.21), (2.22), and (2.23) in (2.20), we obtain

$$\operatorname{Re}\left[1+\frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right]\ge\frac{1}{1-r^2}\Big\{[(p-q)(2\mu-1)-\nu]r^2$$
$$-[2(p-q-1)\mu+\nu+2]r+p-q\Big\},\ |z|=r<1.$$

To solve the inequality

$$\psi(r):=[(p-q)(2\mu-1)-\nu]r^2-[2(p-q-1)\mu+\nu+2]r+p-q\ge0,\tag{2.24}$$

we see that $\psi(0)=p-q\ge0$, hence the inequality (2.24) holds whenever $0\le r\le R^*(p,q,\mu,\nu):=\min\{r_*;1\}$, where r_* is the smallest positive root of ψ.

Putting $\nu=1$ in Theorem 2.3 we get the next corollary, that also generalise and extends some previous results as could we see in the following remark:

Corollary 2.6. *If $f\in\mathcal{A}(p)$ and satisfies the following conditions*

$$\operatorname{Re}\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}>0,\ z\in\mathbb{D},$$

$$\left|\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu}-(p-q)\right|<p-q,\ z\in\mathbb{D},$$

with $0 \leq \mu \leq 1$, *for some* $g \in \mathbb{S}^*_{p,q}$, *then* $f \in \mathbb{K}_{p,q}$ *in* $|z| < R^*_1(p,q,\mu) := R^*(p,q,\mu,1)$, *where*

$$R^*_1(p,q,\mu) := \min\{r_*; 1\},$$

and r_* *is the smallest positive root of*

$$[(p-q)(2\mu-1)-1]r^2 - [2(p-q-1)\mu+3]r + p - q = 0. \tag{2.25}$$

The result is the best possible.

It is easy to see that the bound $R^*_1(p,q,\mu)$ is sharp for $f, g \in \mathcal{A}(p)$ defined by

$$\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} = \frac{1}{1-z},\ 0 \leq \mu \leq 1,$$

and

$$g^{(q)}(z) = \frac{\delta(p,q)z^{p-q}}{(1-z)^{2(p-q)}}.$$

Remark 2.4. Putting $q = 0$ in Corollary 2.6 our result improve those obtained by Wang and Jiang [34, Corollary 3.4], and corrects the result obtained by Patel [30, Theorem 3.8].

Putting $\mu = 0$ in Corollary 2.6 we may easily see that the equation (2.25) has a root in $[0,1]$, hence we get the next result:

Corollary 2.7. *If* $f \in \mathcal{A}(p)$ *and satisfies the following conditions*

$$\operatorname{Re}\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} > 0,\ z \in \mathbb{D},$$

$$\left|\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - (p-q)\right| < p-q,\ z \in \mathbb{D},$$

then $f \in \mathbb{K}_{p,q}$ *in* $|z| < R^*_2(p,q) := R^*(p,q,0,1)$, *where*

$$R^*_2(p,q) := \frac{\sqrt{9+4(p-q+1)(p-q)}-3}{p-q+1}.$$

The result is the best possible.

Using Theorem 2.1, the following theorem represents an inclusion result between the classes $M_{p,q}(\lambda,\mu,A,B)$ if the parameters λ, A and B satisfy some simple ordering relations.

Theorem 2.4. *If* $\mu \geq 0$, $\lambda_2 \geq \lambda_1 \geq 0$, *and* $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, *then*

$$M_{p,q}(\lambda_2, \mu, A_2, B_2) \subset M_{p,q}(\lambda_1, \mu, A_1, B_1). \tag{2.26}$$

Proof. Supposing that $f \in M_{p,q}(\lambda_2, \mu, A_2, B_2)$, then

$$\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} + \lambda_2\left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right] \prec (p-q)\frac{1+A_2z}{1+B_2z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, it follows from Lemma 1.4 that

$$\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} + \lambda_2\left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right] \prec (p-q)\frac{1+A_2z}{1+B_2z} \prec (p-q)\frac{1+A_1z}{1+B_1z}, \tag{2.27}$$

that is that $f \in M_{p,q}(\lambda_2, \mu, A_1, B_1)$. Thus, the assertion (2.26) holds for $\lambda_2 = \lambda_1 = 0$.

If $\lambda_2 > \lambda_1 \geq 0$, by Theorem 2.1 and (2.27) we have $f \in M_{p,q}(0, \mu, A_1, B_1)$, that is

$$\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} \prec (p-q)\frac{1+A_1z}{1+B_1z}. \tag{2.28}$$

At the same time, we have

$$\begin{aligned}&\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} + \lambda_1\left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right]\\ &= \left(1 - \frac{\lambda_1}{\lambda_2}\right)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} + \frac{\lambda_1}{\lambda_2}\left\{\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu}\right.\\ &\left.+ \lambda_2\left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right]\right\}.\end{aligned} \tag{2.29}$$

Moreover, since $0 \leq \dfrac{\lambda_1}{\lambda_2} < 1$, and $\dfrac{1+A_1z}{1+B_1z}$, with $-1 \leq B_1 < A_1 \leq 1$, is an analytic convex function in $\mathbb{D}$, combining (2.29), (2.28), (2.27) and Lemma 1.3 we conclude that

$$\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu} + \lambda_1\left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - \mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right] \prec (p-q)\frac{1+A_1z}{1+B_1z},$$

that is that $f \in M_{p,q}(\lambda_1, \mu, A_1, B_1)$, hence $M_{p,q}(\lambda_2, \mu, A_2, B_2) \subset M_{p,q}(\lambda_1, \mu, A_1, B_1)$, and the proof is complete.

The importance Theorem 2.4 is shown in the next two remarks and corollary, that represent particular cases of this theorem but for $q = 0$ these results were earlier obtained in a few articles.

Remark 2.5. (i) Putting $q = 0$, $A_1 = A_2 = A$, and $B_1 = B_2 = B$ in Theorem 2.4 we obtain the result of Guo and Liu [16, Theorem 3.5];

(ii) Putting $q = 0$ in Theorem 2.4 we get the result obtained by Wang and Jiang [34, Theorem 3.1].

Putting $B_1 = B_2 = -1$, $A_1 = 1 - \dfrac{2\beta_1}{p}$, with $0 \le \beta_1 < p$, $A_2 = 1 - \dfrac{2\beta_2}{p}$, with $0 \le \beta_2 < p$, $p > \beta_2 \ge \beta_1 \ge 0$, and $q = 0$ in Theorem 2.4 we get the next special case:

Corollary 2.8. *If $\mu \ge 0$, $\lambda_2 \ge \lambda_1 \ge 0$ and $p > \beta_2 \ge \beta_1 \ge 0$, then*

$$M_p(\mu, \lambda_2, \beta_2) \subset M_p(\mu, \lambda_1, \beta_1).$$

Remark 2.6. For the special case $p = 1$, Corollary 2.8 reduces to the result of Guo and Liu [16, Theorem 3.3], and of Wang and Jiang [34, Corollary 3.2].

The next theorem gives a simple sufficient condition for a function $f \in \mathcal{A}(p)$ to be in the class $B_{p,q}(\mu, 0)$.

Theorem 2.5. *Let $\gamma > 0$, and $\lambda > 0$ or $\lambda \le -2(p-q)\gamma$. If $f \in \mathcal{A}(p)$ satisfies*

$$\gamma \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \left(\frac{f^{(q)}(z)}{g^{(q)}(z)} \right)^{\mu} + \lambda \left[1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (1-\mu) \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - \mu \frac{zg^{(1+q)}(z)}{g^{(q)}(z)} \right] \neq ib, \ z \in \mathbb{D} \tag{2.30}$$

for some $\mu \ge 0$, and $g \in \mathbb{S}^*_{p,q}$, where b is a real number with

$$|b| \ge \sqrt{\lambda[\lambda + 2(p-q)\gamma]}, \tag{2.31}$$

then

$$\operatorname{Re} \left[\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \left(\frac{f^{(q)}(z)}{g^{(q)}(z)} \right)^{\mu} \right] > 0, \ z \in \mathbb{D}.$$

Proof. Letting

$$P(z) = \frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)} \left(\frac{f^{(q)}(z)}{g^{(q)}(z)} \right)^{\mu}, \ z \in \mathbb{D},$$

then $P(0)=1$. Assuming that there exists $z_* \in \mathbb{D}\setminus\{0\}$ such that $f^{(q)}(z_*)=0$ or $f^{(1+q)}(z_*)=0$, then z_* will be a pole for the function

$$\gamma\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu}+\lambda\left[1+\frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}-(1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}-\mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right],$$

which contradicts the assumption (2.30), hence P is analytic in $\mathbb{D}$.

We will prove that under our assumptions $P(z)\neq 0$ for all $z\in\mathbb{D}$. If P has a zero of order $m\in\mathbb{N}$ at $z_1\in\mathbb{D}\setminus\{0\}$, then P can be written as

$$P(z)=(z-z_1)^m\eta(z),\ z\in\mathbb{D},$$

where η is an analytic function in $\mathbb{D}$, and $\eta(z_1)\neq 0$. Hence, we have

$$\gamma\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\mu}+\lambda\left[1+\frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}-(1-\mu)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}-\mu\frac{zg^{(1+q)}(z)}{g^{(q)}(z)}\right]$$
$$=(p-q)\gamma P(z)+\lambda\frac{zP'(z)}{P(z)}=(p-q)\gamma(z-z_1)^m\eta(z)+\lambda\frac{mz}{z-z_1}+\lambda\frac{z\eta'(z)}{\eta(z)},\quad (2.32)$$

in a neighborhood of z_1 that doesn't contains any zero of η. From (2.32) it follows that the point $z_1\in\mathbb{D}$ will be a pole for the function of the left-hand side of (2.30), which contradicts (2.30).

Thus, if there exists a point $z_0\in\mathbb{D}\setminus\{0\}$ such that $\operatorname{Re}P(z)>0$ for $|z|<|z_0|$, with $\operatorname{Re}P(z_0)>0$ and $P(z_0)=i(\pm a)$, $a>0$, then we have $P(z_0)\neq 0$.

From Lemma 1.5 and the first equality of (2.32) we have

$$F(z_0):=(p-q)\gamma P(z_0)+\lambda\frac{z_0P'(z_0)}{P(z_0)}=i[(p-q)\gamma(\pm a)+\lambda k],$$

where k satisfies the inequalities (1.4) with $\eta=1$, therefore $F(z_0)$ has a pure imaginary value.

Assuming that $\lambda>0$, according to Lemma 1.5 for $P(z_0)=ia$ it follows that

$$\operatorname{Im}F(z_0)=(p-q)\gamma a+\lambda k\geq(p-q)\gamma a+\frac{\lambda}{2}(a+a^{-1})$$
$$=\frac{1}{2}\left\{\frac{\lambda}{a}+[\lambda+2(p-q)\gamma]a\right\}\geq\sqrt{\lambda[\lambda+2(p-q)\gamma]},$$

while for $P(z_0)=-ia$ we have

$$\operatorname{Im}F(z_0)=-(p-q)\gamma a+\lambda k\leq-(p-q)\gamma a-\frac{\lambda}{2}\left(a+a^{-1}\right)$$
$$=-\frac{1}{2}\left\{\frac{\lambda}{a}+[\lambda+2(p-q)\gamma]a\right\}\leq-\sqrt{\lambda[\lambda+2(p-q)\gamma]},$$

which contradicts the assumptions (2.30) and (2.31). Therefore, we have $\operatorname{Re} P(z) > 0$.

The next corollaries and remarks emphasize the high level of generality of this theorem, since for some particular choices of the parameters we get may known particular cases as follows.

Assuming that $\lambda \leq -2(p-q)\gamma$, using the same method we similarly obtain the same conclusion as above, which completes our proof.

Putting $\gamma = 1$ and $\mu = 0$ in Theorem 2.5 we obtain:

Corollary 2.9. *Let $\lambda > 0$ or $\lambda \leq -2(p-q)$. If $f \in \mathcal{A}(p)$ satisfies*

$$(1-\lambda)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + \lambda\left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right) \neq ib,$$

where b is a real number with $|b| \geq \sqrt{\lambda[\lambda + 2(p-q)]}$, then $f \in \mathbb{S}^*_{p,q}$.

Remark 2.7. (i) Putting $q = 0$ in Corollary 2.9 we obtain the result of Dinggong [14, Theorem 1];

(ii) Putting $p = 1$ and $q = 0$ in Corollary 2.9 we obtain the results of Cho and Kim [13, Corollary 1], and Nunokawa [25, Corollary 1].

From Corollary 2.9 we obtain immediately the following results:

Corollary 2.10. *Let $\lambda > 0$ or $\lambda \leq -2(p-q)$. If $f \in \mathcal{A}(p)$ satisfies*

$$\left|\operatorname{Im}\left[(1-\lambda)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + \lambda\left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right)\right]\right| < \sqrt{\lambda[\lambda + 2(p-q)]},\ z \in \mathbb{D},$$

then $f \in \mathbb{S}^*_{p,q}$.

Corollary 2.11. *Let $\lambda > 0$ or $\lambda \leq -2(p-q)$. If $f \in \mathcal{A}(p)$ satisfies*

$$\left|(1-\lambda)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + \lambda\left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right) - (p-q)\right| < p - q + \lambda,\ z \in \mathbb{D},$$

then $f \in \mathbb{S}^*_{p,q}$.

Putting $\lambda = 1$ in Corollary 2.9 we have:

Corollary 2.12. *If $f \in \mathcal{A}(p)$ satisfies the condition*

$$1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \neq ib,\ z \in \mathbb{D},$$

where b is real with $|b| \geq \sqrt{1 + 2(p-q)}$, then $f \in \mathbb{S}^*_{p,q}$.

Remark 2.8. For the special case $q = 0$, Corollary 2.9, Corollary 2.10, Corollary 2.11 and Corollary 2.12, reduces to the results obtained by Dinggong [14, Theorem 1, Corollary 1, Corollary 2 and Corollary 3], respectively.

If we take $q = 0$ in Corollary 2.12 we have (see also Dinggong [14, Corollary 3]):

Corollary 2.13. *If $f \in \mathcal{A}(p)$ satisfies the condition*

$$1 + \frac{zf''(z)}{f'(z)} \neq ib, \; z \in \mathbb{D},$$

where b is real with $|b| \geq \sqrt{1 + 2p}$, then $f \in \mathbb{S}^*_p := \mathbb{S}^*_{p,0}$.

Remark 2.9. We note that our results in Corollary 2.13 corrects the result obtained by Nunokawa [26], because this last one it is not true for $p \geq 2$ (see Dinggong [14]).

If we take $\lambda = 1$ in Corollary 2.11 we get:

Corollary 2.14. *If $f \in \mathcal{A}(p)$ satisfies*

$$\left|1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (p - q)\right| < p - q + 1, \; z \in \mathbb{D},$$

then $f \in \mathbb{S}^*_{p,q}$.

Remark 2.10. Putting $q = 0$ in Corollary 2.14 we obtain the result of Dinggong [14], and our result of Corollary 2.13 for the special case $p = 1$ and $q = 0$ is an improvement of that of Singh and Singh [32, Corollary 3].

Using a particular case of Lemma 1.7, the next theorem gives us an implication that involves differential inequalities; in particular, these implications give simple sufficient starlikeness conditions, and some of them were earlier obtained by using different methods and techniques.

Theorem 2.6. *If $\lambda > 0$ and $f \in \mathcal{A}(p)$ satisfies*

$$\left|(1 - \lambda)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + \lambda\left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right) - (p - q)\right| < p - q + \frac{\lambda}{2}, \; z \in \mathbb{D}, \tag{2.33}$$

then

$$\left|\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - (p - q)\right| < p - q, \; z \in \mathbb{D}. \tag{2.34}$$

Proof. If the function $f \in \mathcal{A}(p)$ and $\lambda > 0$, according to Corollary 2.11 the assumption (2.33) implies $f \in \mathbb{S}^*_{p,q}$. Defining the function P by

$$P(z) := \frac{zf^{(1+q)}(z)}{f^{(q)}(z)}, \; z \in \mathbb{D},$$

by using (2.32), the assumption (2.33) becomes

$$\left|P(z)+\lambda\frac{zP'(z)}{P(z)}-(p-q)\right|<p-q+\frac{\lambda}{2},\ z\in\mathbb{D}. \tag{2.35}$$

Setting in Lemma 1.7 the functions $g(z)=(p-q)(1+z)$, $\theta(w)=w$, $\phi(w)=\dfrac{\lambda}{w}$, and $D=\mathbb{C}^*:=\mathbb{C}\setminus\{0\}$, we get

$$Q(z)=\frac{\lambda zg'(z)}{g(z)}=\frac{\lambda z}{1+z},\ z\in\mathbb{D},$$
$$h(z)=g(z)+Q(z)=(p-q)(1+z)+\frac{\lambda z}{1+z},\ z\in\mathbb{D},$$

and because $p>q$, $\lambda>0$, we have

$$\operatorname{Re}\frac{zh'(z)}{Q(z)}=\operatorname{Re}\left[\frac{p-q}{\lambda}(1+z)+\frac{1}{1+z}\right]>\frac{1}{2},\ z\in\mathbb{D},$$
$$|h(z)-(p-q)|\geq p-q+\lambda\operatorname{Re}\frac{1}{1+z}=p-q+\frac{\lambda}{2},\ z\in\mathbb{D}.$$

Therefore, from the above last inequality and (2.35) it follows

$$P(z)+\lambda\frac{zP'(z)}{P(z)}\prec h(z),$$

and by Lemma 1.7 we conclude that $P(z)\prec(p-q)(1+z)$, which gives (2.34).

Remark 2.11. Putting $q=0$ in Theorem 2.6 we obtain the result of Dinggong [14, Theorem 2], and for $p=1$ and $q=0$ we get that of Mocanu [22, Theorem 3] who proved it by using a different method.

For $\lambda=1$ and $q=0$, Theorem 2.6 reduces to the next special case:

Corollary 2.15. *If* $f\in\mathcal{A}(p)$ *satisfies*

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|<p+\frac{1}{2},\ z\in\mathbb{D},$$

then $f\in\mathbb{S}_p^*$.

Putting $p=1$ in Corollary 2.15 we get the following result which was also obtained by Singh and Singh [32, Corollary 3], and Owa [29, Corollary 1]:

Corollary 2.16. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3}{2},\ z \in \mathbb{D},$$

then $f \in S^*$.

The last main result deals with argument inequalities for the expressions that appeared in the previous theorem, that could be connected with subordinations to conic domains with the apex in origin and symmetric to the real axe, and the main tool used for this result as Lemma 1.4. As we will see in the corresponding corollary and remark, this theorem generalize a few previous results of different authors.

Theorem 2.7. *If $f \in \mathcal{A}(p)$ satisfies*

$$\left|\arg\left[(1-\lambda)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + \lambda\left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right)\right]\right| < \frac{\pi}{2}\,\delta,\ z \in \mathbb{D}, \tag{2.36}$$

with $0 < \delta \leq 1$, then

$$\left|\arg\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\right| < \frac{\pi}{2}\,\beta,\ z \in \mathbb{D},$$

where β, with $0 < \beta < 1$, is the solution of the equation

$$\delta = \beta + \frac{2}{\pi}\tan^{-1}\left[\frac{\lambda\beta\sin\frac{\pi(1-\beta)}{2}}{(p-q)(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}} + \lambda\beta\cos\frac{\pi(1-\beta)}{2}}\right].$$

Proof. Letting

$$p(z) = \frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)},\ z \in \mathbb{D}, \tag{2.37}$$

then $p(0) = 1$. Assuming that there exists $z_* \in \mathbb{D} \setminus \{0\}$ such that $f^{(q)}(z_*) = 0$ or $f^{(1+q)}(z_*) = 0$, then z_* will be a pole for the function

$$F(z) = \frac{1}{p-q}\left[(1-\lambda)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + \lambda\left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right)\right],$$

which contradicts the assumption (2.36). Therefore, p is analytic in $\mathbb{D}$, and $p(z) \neq 0$ for all $z \in \mathbb{D}$.

Differentiating (2.37) we have

$$p(z)\left[1 + \frac{\lambda}{p-q}\frac{zp'(z)}{p^2(z)}\right] = \frac{1}{p-q}\left[(1-\lambda)\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + \lambda\left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right)\right],\ z \in \mathbb{D}.$$

Suppose that there exists a point $z_0 \in \mathbb{D}$ such that $|\arg p(z)| < \frac{\pi}{2}\beta$ for $|z| < |z_0|$, and $|\arg p(z_0)| = \frac{\pi}{2}\beta$. Then, according to Lemma 1.5 we can write $\frac{zp'(z_0)}{p(z_0)} = ik\beta$, where $(p(z_0))^{\frac{1}{\beta}} = \pm ia$, with $a > 0$, and k satisfies the inequalities (1.4).

If $\arg p(z_0) = \frac{\pi}{2}\beta$, then we have $(p(z_0))^{\frac{1}{\beta}} = ia$, with $a > 0$, and

$$\frac{1}{p-q}\left[(1-\lambda)\frac{zf^{(1+q)}(z_0)}{f^{(q)}(z_0)} + \lambda\left(1 + \frac{zf^{(2+q)}(z_0)}{f^{(1+q)}(z_0)}\right)\right]$$
$$= p(z_0)\left(1 + \frac{\lambda}{p-q}\frac{zp'(z_0)}{p^2(z_0)}\right) = a^\beta e^{i\frac{\pi\beta}{2}}\left[1 + \frac{\lambda k\beta}{(p-q)a^\beta}e^{i\frac{\pi(1-\beta)}{2}}\right],$$

with $k \geq \frac{1}{2}\left(a + \frac{1}{a}\right)$. It follows that $\frac{k\beta}{a^\beta} \geq \frac{\beta}{2}\left(a^{1-\beta} + a^{-1-\beta}\right)$, and we can easily check that the function $g : (0, +\infty) \to \mathbb{R}$ defined by $g(a) = \frac{1}{2}\left(a^{1-\beta} + a^{-1-\beta}\right)$ takes the minimum value at $a_* = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$ because

$$g'(a) = \frac{1}{2}\left(\frac{1-\beta}{a^\beta} - \frac{1+\beta}{a^{2+\beta}}\right) = \frac{1-\beta}{2a^{2+\beta}}\left(a^2 - \frac{1+\beta}{1-\beta}\right).$$

Therefore, we have

$$\arg\left\{\frac{1}{p-q}\left[(1-\lambda)\frac{zf^{(1+q)}(z_0)}{f^{(q)}(z_0)} + \lambda\left(1 + \frac{zf^{(2+q)}(z_0)}{f^{(1+q)}(z_0)}\right)\right]\right\}$$
$$= \arg p(z_0) + \arg\left[1 + \frac{\lambda}{p-q}\frac{zp'(z_0)}{p^2(z_0)}\right] = \frac{\pi}{2}\beta + \arg\left[1 + \frac{\lambda k\beta}{(p-q)}e^{i\frac{\pi(1-\beta)}{2}}\right]$$
$$\geq \frac{\pi}{2}\beta + \tan^{-1}\left\{\frac{\lambda\beta\sin\frac{\pi(1-\beta)}{2}}{(p-q)(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}} + \lambda\beta\cos\frac{\pi(1-\beta)}{2}}\right\} = \frac{\pi}{2}\delta,$$

which contradicts the assumption (2.36).

If $\arg p(z_0) = -\frac{\pi}{2}\beta$, by applying the same method, we have

$$\arg\left\{\frac{1}{p-q}\left[(1-\lambda)\frac{zf^{(1+q)}(z_0)}{f^{(q)}(z_0)} + \lambda\left(1 + \frac{zf^{(2+q)}(z_0)}{f^{(1+q)}(z_0)}\right)\right]\right\}$$
$$\leq -\frac{\pi}{2}\beta - \tan^{-1}\left\{\frac{\lambda\beta\sin\frac{\pi(1-\beta)}{2}}{(p-q)(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}} + \lambda\beta\cos\frac{\pi(1-\beta)}{2}}\right\} = -\frac{\pi}{2}\delta,$$

which contradicts the assumption (2.36).

Concluding, the above results implies that there is no point $z_0 \in \mathbb{D}$ such that

$$|\arg p(z_0)| = \left|\arg\left(\frac{z_0 f^{(1+q)}(z_0)}{(p-q)f^{(q)}(z_0)}\right)\right| = \frac{\pi}{2}\beta.$$

Putting $q = 0$ in Theorem 2.7 we get:

Corollary 2.17. *If $f \in \mathcal{A}(p)$ satisfies*

$$\left|\arg\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\right| < \frac{\pi}{2}\delta,\ z \in \mathbb{D},$$

with $0 < \delta \leq 1$, then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\beta,\ z \in \mathbb{D},$$

where β, with $0 < \beta < 1$, is the solution of the equation

$$\delta = \beta + \frac{2}{\pi}\tan^{-1}\left\{\frac{\lambda\beta\sin\frac{\pi(1-\beta)}{2}}{p(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}} + \lambda\beta\cos\frac{\pi(1-\beta)}{2}}\right\}.$$

Remark 2.12. (i) Putting $p = 1$ in Corollary 2.17 we obtain the result of Nunokawa et al. [27, Theorem 1 with $\gamma = 1$ and Theorem 3];

(ii) Putting $p = \lambda = 1$ in Corollary 2.17 we obtain the results of Nunokawa [23], and of Cho and Kim [13, Corollary 4].

3. Conclusions

This paper mainly focuses on defining a subclass of multivalent functions that is defined by using higher order derivatives and extends many previously and extensively studied classes of analytic functions, and that extends the *Bazilevič functions* of some order. We get sharp inclusion results using the *Briot–Bouquet* type of subordination together with the useful Lemma 1.2, and also a reverse inclusion in a subset of the unit disc. Also, using the previous results, we get an inclusion theorem between these classes with respect to the values of the parameters.

We gave some implications involving differential inequalities, where the novelties consist in the proofs that uses Lemma 1.5 and the subordination Lemma 1.7. These are connected with some differential expressions that belongs to a disc or a conic domain with the apex in origin, symmetric with respect to the real axe, and in particular it gives simple sufficient condition for starlikeness.

All our results extend and generalize some other previous ones, and many of them are sharp, in the sense that there are the best possible under the given assumptions. The complexity of many proofs could help some further studies to follow similarly methods to obtain convenient results in *Geometric Function Theory*, while to obtain the best results for those theorems that are not the best possible remain a challenge for those that work in this area.

References

[1] Aouf M. K., On a class of p-valent starlike functions of order α, International Journal of Mathematics and Mathematical Sciences, 10 (4) (1987), 733-744.

[2] Aouf M. K., A generalization of multivalent functions with negative coefficients, Journal of the Korean Mathematical Society, 25 (1) (1988), 53-66.

[3] Aouf M. K., On a class of p-valent close-to-convex functions of order β and type α, International Journal of Mathematics and Mathematical Sciences, 11 (2) (1988), 259-266.

[4] Aouf M. K., Certain subclasses of p-valent starlike functions defined by using a differential operator, Applied Mathematics and Computation, 206 (2008), 867-875.

[5] Aouf M. K., Some families of p-valent functions with negative coefficients, Acta Mathematica Universitatis Comenianae, 78 (1) (2009), 121-135.

[6] Aouf M. K., Bounded p-valent functions defined by using a differential operator, Journal of the Franklin Institute, 347 (2010), 431-437.

[7] Aouf M. K., On certain multivalent functions with negative coefficients defined by using a differential operator, Matematički Vesnik, 62 (1) (2010), 23-35.

[8] Aouf M. K., Some inclusion relationships associated with Dziok-Srivastava operator, Applied Mathematics and Computation, 216 (2010), 431-437.

[9] Aouf M. K., Bulboacă T., and El-Ashwah R. M., Subordination properties of multivalent functions involving an extended fractional differeintegral operator, Acta Mathematica Scientia (Series B) (English Edition), 34 (2) (2014), 367-379.

[10] Aouf M. K., Hossen H. M., and Srivastava H. M., Some families of multivalent functions, Computers & Mathematics with Applications, 39 (2000), 39-48.

[11] Bulboacă T., Differential Subordinations and Superordinations. Recent Results, House of Scientific Book Publ. Cluj-Napoca, 2005.

[12] Carathéodory C., Funktionentheorie, Vol. II., Birkhäuser Verlag, Basel, Switzerland, 1950.

[13] Cho N. E. and Kim Ji A., On a sufficient condition and an angular estimation for Φ-like functions, Taiwanese Journal of Mathematics, 2 (4) (1998), 397-403.

[14] Dinggong Y., On a criterion for multivalently starlikeness, Taiwanese Journal of Mathematics, 1 (2) (1997), 143-148.

[15] Goel R. M. and Sohi N. S., Multivalent functions with negative coefficients, Indian Journal of Pure and Applied Mathematics, 12 (7) (1981), 844-853.

[16] Guo D. and Liu M.-S., On certain subclass of Bazilevič functions, Journal of Inequalities in Pure and Applied Mathematics (JIPAM), 8 (1) (2007), Article ID: 12, 1-12.

[17] Liu M. S., On a subclass of p-valent close-to-convex functions of order β ond type α, Journal of Mathematical Study (JMS), 30 (1997), 102-104.

[18] Liu M. S., On certain subclass of analytic functions, Journal of South China Normal University, 4 (2002), 15-20.

[19] MacGregor T. H., The radius of convexity for starlike function of order, Proceedings of the American Mathematical Society, 14 (1963), 71-76.

[20] Miller S. S. and Mocanu P. T., Univalent solutions of Briot-Bouquet differential equations, Journal of Differential Equations, 56 (3) (1985), 297-309.

[21] Miller S. S. and Mocanu P. T., Differential Subordination. Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[22] Mocanu P. T., Some integral operators and starlike functions, Revue Roumaine de Mathématiques Pures et Appliquées, 31 (1986), 231-235.

[23] Nunokawa M., On properties of non-Carathéodory functions, Proceedings of the Japan Academy, Ser. A, Mathematical Sciences, 68 (6) (1992), 152-153.

[24] Nunokawa M., On the order of strongly starlikeness of strongly convex functions, Proceedings of the Japan Academy, Ser. A, Mathematical Sciences, 69 (1993), 234-237.

[25] Nunokawa M., On α-starlike functions, Bulletin of the Institute of Mathematics Academia Sinica New Series, 22 (4) (1994), 319-322.

[26] Nunokawa M., On a sufficient conditions for multivalently starlikeness, Tsukuba Journal of Mathematics, 18 (1) (1994), 131-134.

[27] Nunokawa M., Owa S., Hayami T., and Kuroki K., Some conditions for strongly starlikeness of certain analytic functions, International Mathematical Forum, 6 (25) (2011), 1199-1208.

[28] Owa S., On certain classes of p-valent functions with negative coefficients, Bulletin of the Belgian Mathematical Society - Simon Stevin, 59 (1985), 385-402.

[29] Owa S., Certain sufficient conditions for starlikeness and convexity of order α, Chinese Journal of Mathematics, 19 (1)(1991), 55-60.

[30] Patel J., On certain subclass of p-valently Bazilevič functions, Journal of Inequalities in Pure and Applied Mathematics (JIPAM), 6 (1) (2005), Article ID: 16, 1-13.

[31] Patel J. and Cho N. E., Some classes of analytic functions involving Noor integral operator, Journal of Mathematical Analysis and Applications, 312 (2005), 564-575.

[32] Singh R. and Singh S., Some sufficient conditions for univalence and starlikeness, Colloquium Mathematicum, 47 (1982), 309-314.

[33] Srivastava H. M., Patel J., and Mohapatra G. P., A certain class of p-valently analytic functions, Mathematical and Computer Modelling, 41 (2-3) (2005), 321-334.

[34] Wang Z.-G. and Jiang Y. P., Notes on certain subclass of p-valently Bazilevič functions, Journal of Inequalities in Pure and Applied Mathematics (JIPAM), 9 (3) (2008), Article ID: 70, 1-7.

[35] Whittaker E. T. and Watson G. N., A course on modern analysis: An introduction to the general theory of infinite processes and of analytic functions; With an account of the principal transcendental functions, Fourth edition (reprinted), Cambridge University Press, Cambridge, 1927.

[36] Wilken D. R. and Feng J., A remark on convex and starlike functions, Journal of the London Mathematical Society, 21 (1980), 287-290.