J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 11, No. 1 (2023), pp. 63-78

DOI: 10.56827/JRSMMS.2023.1101.5 **ISSN (Online): 2582-5461**

ISSN (Print): 2319-1023

β-b-REGULARITY IN FUZZY SETTING

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(**Received:** *Aug. 07, 2023* **Accepted:** *Dec. 02, 2023* **Published:** *Dec. 30, 2023)*

Abstract: This paper deals with a new type of fuzzy open-like set, viz., fuzzy β-b-open set, the class of which is strictly larger than that of fuzzy open set as well as fuzzy preopen set [11], fuzzy semiopen set [1], fuzzy α-open set [4] and fuzzy β-open set [6]. However, three different types of fuzzy continuous-like functions are introduced and studied and also the mutual relationships of these functions are established. Afterwards, two new types of separation axioms and a new type of compactness are introduced and studied. It is shown that in a fuzzy β-b-regular space, fuzzy β-b-open set coincides with fuzzy open set. In the last section some applications of the functions defined here are established.

Keywords and Phrases: Fuzzy β-b-open set, fuzzy regular open set, fuzzy β-b-r-continuous function, fuzzy β-b-continuity, fuzzy almost β-b-continuity, fuzzy extremally disconnected space.

2020 Mathematics Subject Classification: 54A40, 03E72.

1. Introduction

Fuzzy β-open set is introduced in [6]. Using this concept as a basic tool, here we introduce fuzzy β-b-open set. Fuzzy continuity is introduced in [5]. Here we introduce fuzzy β-b-r-continuous function, an independent concept of fuzzy continuity. Next two new types of functions are introduced here, viz., fuzzy β-b-continuity, and fuzzy almost β-b-continuity. It is shown that fuzzy continuity implies fuzzy

β-b-continuity and fuzzy almost β-b-continuity, but not conversely. Here we introduce fuzzy β-b-regular space in which fuzzy open set and fuzzy β-b-open set coincide. Also we introduce fuzzy β-b-compactness and fuzzy β-b-T_2-space. It is established that fuzzy β-b-compactness implies fuzzy compactness, but not conversely and fuzzy T_2-space is fuzzy β-b-T_2-space, but not conversely.

2. Preliminary

Throughout the paper, (X, τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [5]. In [15], Zadeh introduced fuzzy set as follows : A fuzzy set A in an fts X is a mapping from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$. The support [15] of a fuzzy set A, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [15] of a fuzzy set A in X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$ [15]. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [15] while AqB means A is quasi-coincident (q-coincident, for short) [12] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not q B$ respectively. For a fuzzy set A, clA and $intA$ stand for fuzzy closure and fuzzy interior of A in X [5]. $A \in I^X$ is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy preopen [11], fuzzy α-open [4], fuzzy β-open [6]) if $A = int(clA)$ (resp., $A \leq cl(intA)$, $A \leq int(clA)$, $A \leq int(cl(intA))$, $A \leq cl(int(clA))$). The complement of fuzzy regular open (resp., fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open) set is called fuzzy regular closed [1] (resp., fuzzy semiclosed [1], fuzzy preclosed [11], fuzzy α-closed [4], fuzzy β-closed [6]) set. The smallest fuzzy semiclosed (resp., fuzzy preclosed, fuzzy α-closed, fuzzy β-closed) set containing a fuzzy set A in X is called fuzzy semiclosure [1] (resp., fuzzy preclosure [11], fuzzy α-closure [4], fuzzy β-closure [6]) of A, denoted by $sclA$ (resp., $pclA$, αclA, βclA). $A \in I^X$ is fuzzy semiclosed (resp., fuzzy preclosed, fuzzy α-closed, fuzzy β-closed) if $A = sclA$ (resp., $A = pclA$, $A = \alpha clA$, $A = \beta clA$). The collection of all fuzzy regular open (resp., fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open) sets in X is denoted by $FRO(X)$ (rssp., $FSO(X)$, $FPO(X)$, $F\alpha O(X)$, $F\beta O(X)$) and the collection of all fuzzy regular closed (resp., fuzzy semiclosed, fuzzy preclosed, fuzzy α-closed, fuzzy β-closed) sets in X is denoted by $FRC(X)$ (rssp., $FSC(X)$, $FPC(X)$, $F\alpha C(X)$, $F\beta C(X)$). For a fuzzy open set A in X, $sclA = int(clA)$ [2].

3. Fuzzy β-b-Open Set : Some Properties

In this section fuzzy β-b-open set is introduced and studied. It is shown that the set of all fuzzy β-b-open sets in an fts X does not form a fuzzy topology.

First we recall some definitions from [3] for ready references.

Definition 3.1. [3] *Let (X, τ) be an fts and $A \in I^X$. A fuzzy point x_t in X is said to be fuzzy θ-semicluster point of A if $clUqA$ for all $U \in FSO(X)$ with x_tqU. The union of all fuzzy θ-semicluster points of A is called fuzzy θ-semiclosure of A and is denoted by θ-$sclA$.*
$A(\in I^X)$ is fuzzy θ-semiclosed if $A = \theta$-$sclA$.

Definition 3.2. [3] *Let (X, τ) be an fts and $A \in I^X$. Then r-kernel of A, denoted by r-KerA, is defined as follows :*

$$r\text{-Ker}A = \bigwedge\{U : U \in FRO(X), A \leq U\}.$$

Let us now introduce the following concept.

Definition 3.3. *A fuzzy set A in an fts (X, τ) is called fuzzy β-b-open if $A \leq cl(\beta int(clA))$.*
The complement of a fuzzy β-b-open set is called fuzzy β-b-closed.
The collection of all fuzzy β-b-open (resp., fuzzy β-b-closed) sets in an fts X is denoted by $F\beta bO(X)$ (resp., $F\beta bC(X)$).

Remark 3.4. *Union of any two fuzzy β-b-open sets is also so. But intersection of any two fuzzy β-b-open sets may not be so, as it seen from the following example.*

Example 3.5. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Consider two fuzzy sets B, C, defined by $B(a) = 0.4, B(b) = 0.6, C(a) = 0.6, C(b) = 0.4$. Then clearly $B, C \in F\beta bO(X)$. Let $D = B \bigwedge C$. Then $D(a) = D(b) = 0.4$. Now $cl(\beta int(clD)) = cl(\beta int(1_X \setminus A)) = cl0_X = 0_X \not\geq D \Rightarrow D \notin F\beta bO(X)$.

So we can conclude that the set of all fuzzy β-b-open sets in an fts X does not form a fuzzy topology.

Remark 3.6. *It is clear from definitions that fuzzy open set, fuzzy semiopen set, fuzzy preopen set, fuzzy α-open set, fuzzy β-open set imply fuzzy β-b-open set, but the reverse implications are not necessarily true follow from the following example.*

Example 3.7. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Consider a fuzzy set B defined by $B(a) = B(b) = 0.5$. Clearly $B \notin \tau$, $B \notin FPO(X)$, $B \notin F\alpha O(X)$. But clearly $B \in F\beta bO(X)$.
Next consider the fuzzy set C defined by $C(a) = C(b) = 0.6$. Then clearly $C \notin$

$FSO(X)$, but $C \in F\beta bO(X)$.
As $\tau \subseteq F\beta O(X)$, clearly fuzzy β-b-open set may not necessarily fuzzy β-open set.

Theorem 3.8. *Let (X, τ) be an fts. Then the union of any collection of fuzzy β-b-open sets in X is fuzzy β-b-open in X.*

Proof. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ be any collection of fuzzy β-b-open sets in X. Then for any $\alpha \in \Lambda$, $G_\alpha \leq cl(\beta int(clG_\alpha))$. Also, $G_\alpha \leq \bigvee_{\alpha\in\Lambda} G_\alpha$. Then $clG_\alpha \leq cl(\bigvee_{\alpha\in\Lambda} G_\alpha)$ implies that $G_\alpha \leq cl(\beta int(clG_\alpha)) \leq cl(\beta int(cl(\bigvee_{\alpha\in\Lambda} G_\alpha)))$, and this is true for all $\alpha \in \Lambda$. Taking union on both sides, $\bigvee_{\alpha\in\Lambda} G_\alpha \leq cl(\beta int(cl(\bigvee_{\alpha\in\Lambda} G_\alpha)))$. Hence $\bigvee_{\alpha\in\Lambda} G_\alpha$ is a fuzzy β-b-open in X.

Let us now introduce a new type of closure-like operator.

Definition 3.9. *Let (X, τ) be an fts and $A \in I^X$. Then fuzzy β-b-closure of A, denoted by $\beta bclA$, is defined by*

$$\beta bclA = \bigwedge\{U \in I^X : A \leq U, U \in F\beta bC(X)\}$$

and fuzzy β-b-interior of A, denoted by $\beta bintA$, is defined by

$$\beta bintA = \bigvee\{G : G \leq A, G \in F\beta bO(X)\}.$$

Note 3.10. By Remark 3.4, we can conclude that for any fuzzy set A in an fts X, $\beta bclA$ is fuzzy β-b-closed and $\beta bintA$ is fuzzy β-b-open. Again, if $A \in F\beta bC(X)$, then $A = \beta bclA$ and is $A \in F\beta bO(X)$, then $A = \beta bintA$.

Result 3.11. *Let (X, τ) be an fts. Then the following statements are true :*
(i) for any fuzzy point x_t in X and any $U \in I^X$, $x_t \in \beta bclU$ implies that for any $V \in F\beta bO(X)$ with x_tqV, VqU,
(ii) for any two fuzzy sets U, V where $V \in F\beta bO(X)$, $U \not{q} V \Rightarrow \beta bclU \not{q} V$.

Proof (i). Let $x_t \in \beta bclU$ and $V \in F\beta bO(X)$ with x_tqV. Then $x_t \notin 1_X \setminus V \in F\beta bC(X)$. Then $U \not\leq 1_X \setminus V$ implies that UqV.

(ii). If possible, let $\beta bclUqV$, but $U \not{q} V$. Then there exists $x \in X$ such that $(\beta bclU)(x) + V(x) > 1$ and so $V(x) + t > 1$ where $t = (\beta bclU)(x)$. Then $x_t \in \beta bclU$ where $x_tqV, V \in F\beta bO(X)$. By (i), VqU, a contradiction.

Result 3.12. *Let (X, τ) be an fts and $A \in I^X$. Then the following statements are true :*
(i) $\beta bcl(1_X \setminus A) = 1_X \setminus \beta bintA$,
(ii) $1_X \setminus \beta bclA = \beta bintA(1_X \setminus A)$.

Proof. (i) Let $x_t \in \beta bcl(1_X \setminus A)$. If possible, let $x_t \notin 1_X \setminus \beta int A$. Then $x_t q \beta int A \Rightarrow$ there exists $U \in F\beta bO(X)$ with $U \leq A$ such that $x_t q U$. Since $x_t \in \beta bcl(1_X \setminus A)$, $Uq(1_X \setminus A)$ (by Result 3.11 (i)) $\Rightarrow Aq(1_X \setminus A)$, a contradiction. Hence

$$\beta bcl(1_X \setminus A) \leq 1_X \setminus \beta int A ...(1)$$

Conversely, let $x_t \in 1_X \setminus \beta int A$. Then $1 - \beta int A(x) \geq t \Rightarrow x_t \not{q} \beta int A \Rightarrow x_t \not{q} U$ where $U \in F\beta bO(X)$ with $U \leq A$ ... (2)
Let $V \in F\beta bC(X)$ with $1_X \setminus A \leq V$. Then $1_X \setminus V \leq A$ where $1_X \setminus V \in F\beta bO(X)$. By (2), $x_t \not{q}(1_X \setminus V) \Rightarrow x_t \in V \Rightarrow x_t \in \beta bcl(1_X \setminus A)$. Hence $1_X \setminus \beta int A \leq \beta bcl(1_X \setminus A)$ ... (3).
Combining (1) and (3), we get the result.
(ii) Writing $1_X \setminus A$ for A in (i), we get the proof.

Let us now recall the following Lemma from [3] for ready references.

Lemma 3.13. [3] *Let (X, τ) be an fts and $A \in I^X$. Then the following statements hold.:*
(i) for any $A \in FRO(X)$, θ-$sclA = A$,
(ii) for any $A \in F\beta O(X), clA = \alpha clA$,
(iii) for any $A \in FSO(X)$, $clA = pclA$,
(iv) for any $A \in \tau, sclA = \theta$-$sclA$.

4. Fuzzy β-b-r-Continuous, Fuzzy β-b-Continuous and Fuzzy Almost β-b-Continuous Functions

In this section we first introduce fuzzy β-b-r-continuous function and characterize it in several ways. Afterwards, two new types of functions, viz., fuzzy β-b-continuous function and fuzzy almost β-b-continuous function are introduced. The mutual relationships of these three functions are established here .

Definition 4.1. *Let (X, τ) and (Y, τ_1) be two fts's. Then $f : X \to Y$ is called fuzzy β-b-r-continuous function if $f^{-1}(A) \in F\beta bC(X)$, for all $A \in FRO(Y)$.*

Theorem 4.2. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then the following statements are equivalent :*
(i) f is fuzzy β-b-r-continuous,
(ii) $f^{-1}(A) \in F\beta bO(X)$, for all $A \in FRC(Y)$,
(iii) $f(\beta bcl_\tau U) \leq r$-$ker(f(U))$, for all $U \in I^X$,
(iv) $\beta bcl_\tau(f^{-1}(A)) \leq f^{-1}(r$-$ker(A))$, for all $A \in I^Y$,
(v) $\beta bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta$-$scl_{\tau_1} R))$, for all $R \in \tau_1$.
(vi) $\beta bcl_\tau(f^{-1}(R)) \leq f^{-1}(scl_{\tau_1} R))$, for all $R \in \tau_1$.
(vii) $\beta bcl_\tau(f^{-1}(R)) \leq f^{-1}(int_{\tau_1}(cl_{\tau_1} R))$, for all $R \in \tau_1$.

(viii) $f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) \in F\beta bC(X)$, *for all* $A \in \tau_1$,
(ix) $f^{-1}(cl_{\tau_1}(int_{\tau_1}F)) \in F\beta bO(X)$, *for all* $F \in \tau_1^c$,
(x) $f^{-1}(cl_{\tau_1}U) \in F\beta bO(X)$, *for all* $U \in F\beta O(Y)$,
(xi) $f^{-1}(cl_{\tau_1}U) \in F\beta bO(X)$, *for all* $U \in FSO(Y)$,
(xii) $f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\beta bC(X)$, *for all* $U \in FPO(Y)$,
(xiii) $f^{-1}(\alpha cl_{\tau_1}U) \in F\beta bO(X)$, *for all* $U \in F\beta O(Y)$,
(xiv) $f^{-1}(pcl_{\tau_1}U) \in F\beta bO(X)$, *for all* $U \in FSO(Y)$.

Proof (i) ⇔ (ii). Obvious.

(ii) ⇒ (iii). Let $U \in I^X$ and suppose that y_t be a fuzzy point in Y with $y_t \notin r$-ker$(f(U))$. Then there exists $V \in FRO(Y)$ such that $f(U) \leq V$ and $y_t \notin V \Rightarrow V(y) < t$ and so $y_t q(1_Y \setminus V) \in FRC(Y)$ and $1_Y \setminus f(U) \geq 1_Y \setminus V$. So $f(U) \not{q}(1_Y \setminus V)$ implies that $U \not{q} f^{-1}(1_Y \setminus V)$. By (ii), $f^{-1}(1_Y \setminus V) = 1_X \setminus f^{-1}(V) \in F\beta bO(X)$. By Result 3.11(ii), $\beta bcl_\tau U \not{q}(1_X \setminus f^{-1}(V))$. Then $\beta bcl_\tau U \leq f^{-1}(V)$. So $f(\beta bcl_\tau U) \leq V$ implies that $1_Y \setminus f(\beta bcl_\tau U) \geq 1_Y \setminus V$. So $1 - f(\beta bcl_\tau U)(y) \geq 1 - V(y) > 1 - t$. Then $t > f(\beta bcl_\tau U)(y)$. Then $y_t \notin f(\beta bcl_\tau U)$. Therefore, $f(\beta bcl_\tau U) \leq r$-ker$(f(U))$.

(iii) ⇒ (iv). Let $A \in I^Y$. Then $f^{-1}(A) \in I^X$. By (iii), $f(\beta bcl_\tau f^{-1}(A)) \leq r$-ker$(f(f^{-1}(A))) \leq r$-ker$(A)$ implies that $\beta bcl_\tau(f^{-1}(A)) \leq f^{-1}(r\text{-ker}(A))$.

(iv) ⇒ (i). Let $A \in FRO(Y)$. By (iv), $\beta bcl_\tau(f^{-1}(A)) \leq f^{-1}(r\text{-ker}(A)) = f^{-1}(A)$. But $f^{-1}(A) \leq \beta bcl_\tau(f^{-1}(A))$ and so $f^{-1}(A) = \beta bcl_\tau(f^{-1}(A))$ and so $f^{-1}(A) \in F\beta bC(X)$. Hence f is fuzzy β-b-r-continuous function.

(v) ⇔ (vi). Follows from Lemma 3.13 (iv).

(vi) ⇔ (vii). Obvious.

(vii) ⇒ (i). Let $A \in FRO(Y)$. By (vii), $\beta bcl_\tau(f^{-1}(A)) \leq f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) = f^{-1}(A)$ implies that $f^{-1}(A) \in F\beta bC(X)$. Hence f is fuzzy β-b-r-continuous function.

(i) ⇒ (vii). Let $A \in \tau_1$. Then $int_{\tau_1}(cl_{\tau_1}A) \in FRO(Y)$. By (i), $f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) \in F\beta bC(X)$ and so $\beta bcl_\tau(f^{-1}(A)) \leq \beta bcl_\tau(f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) = f^{-1}(int_{\tau_1}(cl_{\tau_1}A))$.

(i)⇒ (viii). Let $A \in \tau_1$. Then $int_{\tau_1}(cl_{\tau_1}A) \in FRO(Y)$ and so by (i), $f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) \in F\beta bC(X)$.

(viii) ⇒ (i). Let $A \in FRO(Y)$. Then $A \in \tau_1$.
By (viii), $f^{-1}(A) = f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) \in F\beta bC(X)$.

(ii) ⇒ (ix). Let $F \in \tau_1^c$. Then $cl_{\tau_1}int_{\tau_1}F \in FRC(Y)$. By (ii), $f^{-1}(cl_{\tau_1}(int_{\tau_1}F)) \in F\beta bO(X)$.

(ix) ⇒ (ii). Let $F \in FRC(Y)$. By (ix), $f^{-1}(F) = f^{-1}(cl_{\tau_1}(int_{\tau_1}F)) \in F\beta bO(X)$.

(ii) ⇒ (x). Let $U \in F\beta O(Y)$. Then $U \leq cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U)) \leq cl_{\tau_1}U$ implies that $cl_{\tau_1}U \leq cl_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))) = cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U)) \leq cl_{\tau_1}(cl_{\tau_1}U) = cl_{\tau_1}U$. So $cl_{\tau_1}U = cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))$. Then $cl_{\tau_1}U \in FRC(Y)$ and so by (ii), $f^{-1}(cl_{\tau_1}U) \in F\beta bO(X)$.

(x) $\Rightarrow$ (xi). Since $FSO(Y) \subseteq F\beta O(Y)$, by (x), $f^{-1}(cl_{\tau_1}U) \in F\beta bO(X)$, for all $U \in FSO(Y)$.
(xi) $\Rightarrow$ (xii). Let $U \in FPO(Y)$. Then $U \leq int_{\tau_1}(cl_{\tau_1}U)$. We claim that $int_{\tau_1}(cl_{\tau_1}U) \in FRO(Y)$. Indeed, $int_{\tau_1}(cl_{\tau_1}U) \leq int_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))) \leq int_{\tau_1}(cl_{\tau_1}U)$ implies that $int_{\tau_1}(cl_{\tau_1}U) = int_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U)))$. So $1_Y \setminus int_{\tau_1}(cl_{\tau_1}U) \in FRC(Y)$. So $1_Y \setminus int_{\tau_1}(cl_{\tau_1}U) \in FSO(Y)$. By (xi), $f^{-1}(cl_{\tau_1}(1_Y \setminus int_{\tau_1}(cl_{\tau_1}U))) \in F\beta bO(X)$. Then $1_X \setminus f^{-1}(int_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))) = 1_X \setminus f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\beta bO(X)$. Hence $f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\beta bC(X)$.
(xii) $\Rightarrow$ (i). Let $U \in FRO(Y)$. Then $U \in FPO(Y)$. By (xii), $f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\beta bC(X)$. Hence $f^{-1}(U) = f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\beta bC(X)$. Then (i) follows.
(x) $\Leftrightarrow$ (xiii). The proof follows from Lemma 3.13(ii).
(xi) $\Leftrightarrow$ (xiv). The proof follow from Lemma 3.13(iii).

Theorem 4.3. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then the following statements hold :*
(i) for each fuzzy point x_t in X and each $A \in FSO(Y)$ with $f(x_t)qA$, there exists $U \in F\beta bO(X)$ with x_tqU, $f(U) \leq cl_{\tau_1}A$,
$\Leftrightarrow$ (ii) $f(\beta bcl_\tau P) \leq \theta\text{-}scl_{\tau_1}(f(P))$, for all $P \in I^X$,
$\Leftrightarrow$ (iii) for each fuzzy point x_t in X and each $A \in FSO(Y)$ with $f(x_t) \in A$, there exists $U \in F\beta bO(X)$ such that $x_t \in U$ and $f(U) \leq cl_{\tau_1}A$,
$\Leftrightarrow$ (iv) $f^{-1}(A) \leq \beta bint_\tau(f^{-1}(cl_{\tau_1}A))$, for all $A \in FSO(Y)$,
$\Leftrightarrow$ (v) $\beta bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R))$, for all $R \in I^Y$,
$\Rightarrow$ (vi) f is fuzzy β-b-r-continuous function.
Proof (i) $\Rightarrow$ (ii). Let $P \in I^X$ and x_t be any fuzzy point in X such that $x_t \in \beta bcl_\tau P$ and let $G \in FSO(Y)$ with $f(x_t)qG$. By (i), there exists $U \in F\beta bO(X)$ with x_tqU, $f(U) \leq cl_{\tau_1}G$. As $x_t \in \beta bcl_\tau P$, by Result 3.11(i), UqP and so $f(U)qf(P)$. Then $f(P)qcl_{\tau_1}G$. Then $f(x_t) \in \theta\text{-}scl_{\tau_1}(f(P))$. Hence $f(\beta bcl_\tau P) \leq \theta\text{-}scl_{\tau_1}(f(P))$.
(ii) $\Rightarrow$ (v). Let $R \in I^Y$. Then $f^{-1}(R) \in I^X$. By (ii), $f(\beta bcl_\tau(f^{-1}(R))) \leq \theta\text{-}scl_{\tau_1}(f(f^{-1}(R))) \leq \theta\text{-}scl_{\tau_1}R$ and so $\beta bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R)$.
(v) $\Rightarrow$ (i). Let x_t be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_t)qA$. Since, $cl_{\tau_1}A \not q (1_Y \setminus cl_{\tau_1}A)$, by definition $f(x_t) \notin \theta\text{-}scl_{\tau_1}(1_Y \setminus cl_{\tau_1}A)$ and so $x_t \notin f^{-1}(\theta\text{-}scl_{\tau_1}(1_Y \setminus cl_{\tau_1}A))$. By (v), $x_\alpha \notin \beta bcl_\tau(f^{-1}(1_Y \setminus cl_{\tau_1}A))$. So there exists $U \in F\beta bO(X)$ with x_tqU, $U \not q f^{-1}(1_Y \setminus cl_{\tau_1}A)$ implies that $f(U) \not q(1_Y \setminus cl_{\tau_1}A)$. Hence $f(U) \leq cl_{\tau_1}A$.
(i) $\Rightarrow$ (iv). Let $A \in FSO(Y)$ and x_t be any fuzzy point in X such that $x_tqf^{-1}(A)$. Then $f(x_t)qA$. By (i), there exists $U \in F\beta bO(X)$ such that x_tqU, $f(U) \leq cl_{\tau_1}A \Rightarrow x_tqU \leq f^{-1}(cl_{\tau_1}A)$ and so $x_tqU = \beta bint_\tau U \leq \beta bint_\tau(f^{-1}(cl_{\tau_1}A))$ implies that $x_tq\beta bint_\tau(f^{-1}(cl_{\tau_1}A))$ as $\beta bint_\tau(f^{-1}(cl_{\tau_1}A))$ is the union of all fuzzy β-b-open sets in X contained in $f^{-1}(cl_{\tau_1}A)$ and hence $f^{-1}(A) \leq \beta bint_\tau(f^{-1}(cl_{\tau_1}A))$.
(iv) $\Rightarrow$ (i). Let x_t be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_t)qA$. Then

$x_t q f^{-1}(A) \leq \beta bint_\tau(f^{-1}(cl_{\tau_1}A))$ (by (iv)) implies that there exists $U \in F\beta bO(X)$ with $x_t qU$, $U \leq f^{-1}(cl_{\tau_1}A)$. Hence $f(U) \leq cl_{\tau_1}A$.
(iii) ⇒ (iv). Let $A \in FSO(Y)$ and x_t be any fuzzy point in X such that $x_t \in f^{-1}(A)$. Then $f(x_t) \in A$. By (iii), there exists $U \in F\beta bO(X)$ with $x_t \in U$ and $f(U) \leq cl_{\tau_1}A$ implies that $U \leq f^{-1}(cl_{\tau_1}A)$ and so $x_t \in U = \beta bint_\tau U \leq \beta bint_\tau(f^{-1}(cl_{\tau_1}A))$. Hence $f^{-1}(A) \leq \beta bint_\tau(f^{-1}(cl_{\tau_1}A))$.
(iv) ⇒ (iii). Let x_t be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_t) \in A$. Then $x_t \in f^{-1}(A) \leq \beta bint_\tau(f^{-1}(cl_{\tau_1}A))$ (by (iv)) implies that there exists $U \in F\beta bO(X)$ with $x_t \in U$ and $U \leq f^{-1}(cl_{\tau_1}A)$. Hence $f(U) \leq cl_{\tau_1}A$.
(v) ⇒ (vi). Let $A \in FRO(Y)$. By (v), $\beta bcl_\tau(f^{-1}(A)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}A) = f^{-1}(A)$ (by Lemma 3.13 (i)) $\Rightarrow f^{-1}(A) \in F\beta bC(X) \Rightarrow f$ is fuzzy β-b-r-continuous function.

Theorem 4.4. *Let (X,τ) and (Y,τ_1) be two fts's and $f : X \to Y$ be a function satisfying $\beta bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R)$, for all $R \in I^Y$. Then the following statements hold :*
(i) $\beta bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R)$, for all $R \in FSO(Y)$,
(ii) $\beta bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R)$, for all $R \in FPO(Y)$,
(iii) $\beta bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R)$, for all $R \in F\beta O(Y)$.
Proof. Obvious.

Definition 4.5. *Let (X,τ) and (Y,τ_1) be two fts's and $f : X \to Y$ be a function. Then f is said to be fuzzy*
(i) β-b-continuous if $f^{-1}(A) \in F\beta bO(X)$, for all $A \in \tau_1$,
(ii) almost β-b-continuous if $f^{-1}(A) \in F\beta bO(X)$, for all $A \in FRO(Y)$.

Let us now recall the following definition from [5] for ready references.

Definition 4.6. [5] *Let (X,τ) and (Y,τ_1) be two fts's and $f : X \to Y$ be a function. Then f is said to be fuzzy continuous function if $f^{-1}(U) \in \tau$, for all $U \in \tau_1$.*

Remark 4.7. *It is clear from definitions that*
(i) fuzzy continuity ⇒ fuzzy β-b-continuity ⇒ fuzzy almost β-b-continuity, but reverse implications are not necessarily true, in general, follow from the next examples,
(ii) fuzzy β-b-r-continuity is an independent concept of fuzzy continuity, fuzzy β-b-continuity and fuzzy almost β-b-continuity, follow from the next examples.

Example 4.8. Fuzzy continuity, fuzzy β-b-continuity and fuzzy almost β-b-continuity $\not\Rightarrow$ fuzzy β-b-r-continuity
Let $X = \{a,b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = A(b) = 0.5, B(a) = 0.5, B(b) = 0.4$. Then (X,τ_1) and (X,τ_2) are fts's. Consider the iden-

tity function $i : (X, \tau_1) \to (X, \tau_2)$. Clearly i is fuzzy continuous and hence fuzzy β-b-continuous as well as fuzzy almost β-b-continuous function. Now $1_X \setminus B \in FRC(X, \tau_2)$. $i^{-1}(1_X \setminus B) = 1_X \setminus B$. Then $cl_{\tau_1}(\beta int_{\tau_1}(cl_{\tau_1}(1_X \setminus B))) = A \not\geq 1_X \setminus B \Rightarrow 1_X \setminus B \notin F\beta bO(X, \tau_1) \Rightarrow i$ is not fuzzy β-b-r-continuous function.

Example 4.9. Fuzzy β-b-r-continuity, fuzzy β-b-continuity and fuzzy almost β-b-continuity $\not\Rightarrow$ fuzzy continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since every fuzzy set in (X, τ_1) is fuzzy β-b-open in (X, τ_1), clearly i is fuzzy β-b-r-continuous, fuzzy β-b-continuous and fuzzy almost β-b-continuous function. Since $A \in \tau_2$, but $i^{-1}(A) = A \notin \tau_1, i$ is not fuzzy continuous function.

Example 4.10. Fuzzy β-b-r-continuity, fuzzy almost β-b-continuity $\not\Rightarrow$ fuzzy β-b-continuity
Lt $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, C\}$ where $A(a) = A(b) = 0.4, B(a) = B(b) = 0.5, C(a) = 0.5, C(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $C \in \tau_2$ and $i^{-1}(C) = C$. Then $cl_{\tau_1}(\beta int_{\tau_1}(cl_{\tau_1} C)) = B \not\geq C \Rightarrow C \notin F\beta bO(X, \tau_1) \Rightarrow i$ is not fuzzy β-b-continuous function. Since $0_X, 1_X \in FRO(X, \tau_2)$ only, so clearly i is fuzzy β-b-r-continuous and fuzzy almost β-b-continuous function.

Example 4.11. Fuzzy β-b-r-continuity $\not\Rightarrow$ fuzzy almost β-b-continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.3$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $B \in FRO(X, \tau_2)$, $i^{-1}(B) = B$. Then $int_{\tau_1}(\beta cl_{\tau_1}(int_{\tau_1} B)) = 0_X \leq B \Rightarrow B \in F\beta bC(X, \tau_1) \Rightarrow i$ is fuzzy β-b-r-continuous function. But $cl_{\tau_1}(\beta int_{\tau_1}(cl_{\tau_1} B)) = 0_X \not\geq B \Rightarrow B \notin F\beta bO(X, \tau_1) \Rightarrow i$ is not fuzzy almost β-b-continuous function.

Definition 4.12. [10] *An fts (X, τ) is said to be fuzzy extremally disconnected if the closure of every fuzzy open set in X is fuzzy open in X.*

Theorem 4.13. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. If (Y, τ_1) is fuzzy extremally disconnected space, then f is fuzzy β-b-r-continuous function if and only if f is fuzzy almost β-b-continuous function.*
Proof. First suppose that f is fuzzy β-b-r-continuous function and let $U \in FRO(Y)$. Then $U = int_{\tau_1}(cl_{\tau_1} U)$. As Y is fuzzy extremely disconnected, $cl_{\tau_1} U \in \tau_1$ and so $U = int_{\tau_1} cl_{\tau_1} U = cl_{\tau_1} U = cl_{\tau_1} int_{\tau_1} U$ implies that $U \in FRC(Y)$. By hypothesis, $f^{-1}(U) \in F\beta bO(X)$ and so f is fuzzy almost β-b-continuous function.

Conversely, let $U \in FRC(Y)$. As Y is fuzzy extremely disconnected, $U \in$

$FRO(Y)$. By hypothesis, $f^{-1}(U) \in F\beta bO(X)$. Hence f is fuzzy β-b-r-continuous function.

Remark 4.14. *Composition of two fuzzy β-b-r-continuous (resp., fuzzy β-b-continuous and fuzzy almost β-b-continuous) functions need not be so, as it seen from the following examples.*

Example 4.15. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$ where $A(a) = A(b) = 0.5, B(a) = 0.5, B(b) = 0.4$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$, $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are fuzzy β-b-r-continuous functions. Let $i_3 = i_2 \circ i_1$. Now $1_X \setminus B \in FRC(X, \tau_3)$, $i_3^{-1}(1_X \setminus B) = 1_X \setminus B$. Now $cl_{\tau_1}(\beta int_{\tau_1}(cl_{\tau_1}(1_X \setminus B))) = A \ngeq 1_X \setminus B \Rightarrow 1_X \setminus B \notin F\beta bO(X, \tau_1) \Rightarrow i_3$ is not fuzzy β-b-r-continuous function.

Example 4.16. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.3$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are fuzzy β-b-continuous and hence fuzzy almost β-b-continuous functions. Let $i_3 = i_2 \circ i_1$. Bow $B \in \tau_3$ as well as $B \in FRO(X, \tau_3)$. $i_3^{-1}(B) = B$. Now $cl_{\tau_1}(\beta int_{\tau_1}(cl_{\tau_1} B)) = 0_X \ngeq B \Rightarrow B \notin F\beta bO(X, \tau_1) \Rightarrow i_3$ is not fuzzy β-b-continuous and also fuzzy almost β-b-continuous functions.

5. Fuzzy β-b-Regular, Fuzzy β-b-compact and Fuzzy β-b-T_2-Spaces

In this section new types of separation axioms and compactness are introduced and studied. Then the mutual relationships of these spaces with the spaces defined in [5, 8] are established.

Definition 5.1. *An fts (X, τ) is called β-b-regular space if for each fuzzy point x_t in X and each fuzzy β-b-closed set F with $x_t \notin F$, there exist a fuzzy open set U and a fuzzy β-b-open set V in X such that $x_t qU$, $F \leq V$ and $U \not{q} V$.*

Theorem 5.2. *For an fts (X, τ), the following statements are equivalent:*
(a) X is fuzzy β-b-regular space,
(b) for each fuzzy point x_t in X and each fuzzy β-b-open set U in X with $x_t qU$, there exists a fuzzy open set V in X such that $x_t qV \leq \beta bclV \leq U$,
(c) for each fuzzy β-b-closed set F in X, $\bigwedge\{clV : F \leq V, V \in F\beta bO(X)\} = F$,
(d) for each fuzzy set G in X and each fuzzy β-b-open set U in X such that GqU, there exists a fuzzy open set V in X such that GqV and $\beta bclV \leq U$.
Proof (a)⇒(b). Let x_t be a fuzzy point in X and U, a fuzzy β-b-open set in X with $x_t qU$. Then $x_t \notin 1_X \setminus U \in F\beta bC(X)$. By (a), there exist a fuzzy open set V and a fuzzy β-b-open set W in X such that $x_t qV$, $1_X \setminus U \leq W$, $V \not{q} W$. Then

$x_tqV \leq 1_X \setminus W \leq U \Rightarrow x_tqV \leq \beta bclV \leq \beta bcl(1_X \setminus W) = 1_X \setminus W \leq U$.
(b)⇒(a). Let F be a fuzzy β-b-closed set in X and x_t be a fuzzy point in X with $x_t \notin F$. Then $x_tq(1_X \setminus F) \in F\beta bO(X)$. By (b), there exists a fuzzy open set V in X such that $x_tqV \leq \beta bclV \leq 1_X \setminus F$. Put $U = 1_X \setminus \beta bclV$. Then $U \in F\beta bO(X)$ and x_tqV, $F \leq U$ and $U \not{q} V$.
(b)⇒(c). Let F be fuzzy β-b-closed set in X. Then $F \leq \bigwedge\{clV : F \leq V, V \in F\beta bO(X)\}$.
Conversely, let $x_t \notin F \in F\beta bC(X)$. Then $F(x) < t \Rightarrow x_tq(1_X \setminus F)$ where $1_X \setminus F \in F\beta bO(X)$. By (b), there exists a fuzzy open set U in X such that $x_tqU \leq \beta bclU \leq 1_X \setminus F$. Put $V = 1_X \setminus \beta bclU$. Then $F \leq V$ and $U \not{q} V \Rightarrow x_t \notin clV \Rightarrow \bigwedge\{clV : F \leq V, V \in F\beta bO(X)\} \leq F$.
(c)⇒(b). Let V be any fuzzy β-b-open set in X and x_t be any fuzzy point in X with x_tqV. Then $V(x) + t > 1 \Rightarrow x_t \notin (1_X \setminus V)$ where $1_X \setminus V \in F\beta bC(X)$. By (c), there exists $G \in F\beta bO(X)$ such that $1_X \setminus V \leq G$ and $x_t \notin clG$. Then there exists a fuzzy open set U in X with x_tqU, $U \not{q} G \Rightarrow U \leq 1_X \setminus G \leq V$ $\Rightarrow x_tqU \leq \beta bclU \leq \beta bcl(1_X \setminus G) = 1_X \setminus G \leq V$.
(c)⇒(d). Let G be any fuzzy set in X and U be any fuzzy β-b-open set in X with GqU. Then there exists $x \in X$ such that $G(x) + U(x) > 1$. Let $G(x) = t$. Then $x_tqU \Rightarrow x_t \notin 1_X \setminus U$ where $1_X \setminus U \in F\beta bC(X)$. By (c), there exists $W \in F\beta bO(X)$ such that $1_X \setminus U \leq W$ and $x_t \notin clW \Rightarrow (clW)(x) < t \Rightarrow x_tq(1_X \setminus clW)$. Let $V = 1_X \setminus clW$. Then V is fuzzy open set in X and $V(x) + t > 1 \Rightarrow V(x) + G(x) > 1 \Rightarrow VqG$ and $\beta bclV = \beta bcl(1_X \setminus clW) \leq \beta bcl(1_X \setminus W) = 1_X \setminus W \leq U$.
(d)⇒(b). Obvious.

Note 5.3. It is clear from Theorem 5.2 that in a fuzzy β-b-regular space, every fuzzy β-b-closed set is fuzzy closed and hence every fuzzy β-b-open set is fuzzy open. As a result, in a fuzzy β-b-regular space, the collection of all fuzzy closed (resp., fuzzy open) sets and fuzzy β-b-closed (resp., fuzzy β-b-open) sets coincide.

Definition 5.4. *Let A be a fuzzy set in X. A collection $\mathcal{U}$ of fuzzy sets in X is called a fuzzy cover of A if $sup\{U(x) : U \in \mathcal{U}\} = 1$, for each $x \in suppA$* [7]. *In particular, if $A = 1_X$, we get the definition of fuzzy cover of X* [5].

Definition 5.5. *A fuzzy cover $\mathcal{U}$ of a fuzzy set A in X is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 \geq A$, i.e., $\mathcal{U}_0$ is also a fuzzy cover of A* [7]. *In particular, if $A = 1_X$, we get $\bigcup \mathcal{U}_0 = 1_X$* [5].

Definition 5.6. *A fuzzy set A in an fts (X, τ) is said to be fuzzy compact* [7] *if every fuzzy covering $\mathcal{U}$ of A by fuzzy open sets in X has a finite subcovering $\mathcal{U}_0$ of $\mathcal{U}$. In particular, if $A = 1_X$, we get the definition of fuzzy compact space* [5].

Definition 5.7. *An fts (X, τ) is said to be fuzzy s-closed* [13] *(resp. fuzzy nearly compact* [10]*) if every fuzzy covering of X by fuzzy regular closed (resp., fuzzy regular open) sets of X contains a finite subcovering.*

Let us now introduce the following concept.

Definition 5.8. *A fuzzy set A in an fts (X, τ) is called fuzzy β-b-compact set if every fuzzy covering of A by fuzzy β-b-open sets of X has a finite subcovering. In particular, if $A = 1_X$, we get the definition of fuzzy β-b-compact space.*

Remark 5.9. *It is clear from above discussion that fuzzy β-b-compact space is fuzzy compact. But the converse is not necessarily true follows from the next example.*

Example 5.10. Let $X = \{a\}$, $\tau = \{0_X, 1_X\}$. The clearly (X, τ) is a fuzzy compact space. Here every fuzzy set is fuzzy β-b-open set in X. Consider the fuzzy cover $\mathcal{U} = \{U_n(a) : n \in \mathbf{N}\}$ where $U_n(a) = \{\frac{n}{n+1} : n \in \mathbf{N}\}$. Then $\mathcal{U}$ is a fuzzy β-b-open cover of X. But it does not have any subcovering of X. Hence X is not fuzzy β-b-compact space.

Theorem 5.11. *Every fuzzy β-b-closed set A in a fuzzy β-b-compact space X is fuzzy β-b-compact.*

Proof. Let A be a fuzzy β-b-closed set in a fuzzy β-b-compact space X. Let $\mathcal{U}$ be a fuzzy covering of A by fuzzy β-b-open sets in X. Then $\mathcal{V} = \mathcal{U} \bigcup (1_X \setminus A)$ is a fuzzy β-b-open covering of X. By hypothesis, there exists a finite subcollection $\mathcal{V}_0$ of $\mathcal{V}$ which also covers X. If $\mathcal{V}_0$ contains $1_X \setminus A$, we omit it and get a finite subcovering of A. Consequently, A is fuzzy β-b-compact set.

Let us now recall the following definition from [8] for ready references.

Definition 5.12. [8] *Let (X, τ) be an fts. Then X is said to be fuzzy T_2-space if for each pair of distinct fuzzy points x_α, y_β : when $x \neq y$, there exist fuzzy open sets U_1, U_2, V_1, V_2 such that $x_\alpha \in U_1, y_\beta q V_1$ and $U_1 \not{q} V_1$ and $x_\alpha q U_2$, $y_\beta \in V_2$ and $U_2 \not{q} V_2$; when $x = y$, $\alpha < \beta$ (say), there exist fuzzy open sets U, V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not{q} V$.*

Now we introduce the following concept.

Definition 5.13. *Let (X, τ) be an fts. Then X is said to be fuzzy β-b-T_2-space if for each pair of distinct fuzzy points x_α, y_β : when $x \neq y$, there exist fuzzy β-b-open sets U_1, U_2, V_1, V_2 such that $x_\alpha \in U_1, y_\beta q V_1$ and $U_1 \not{q} V_1$ and $x_\alpha q U_2$, $y_\beta \in V_2$ and $U_2 \not{q} V_2$; when $x = y$, $\alpha < \beta$ (say), there exist fuzzy β-b-open sets U, V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not{q} V$.*

Let us now recall the following definition from [8] for ready references.

Definition 5.14. [8] *An fts (X, τ) is said to be fuzzy regular space if for any fuzzy* *point x_t in X and any fuzzy closed set F in X with $x_t \notin F$, there exist fuzzy open sets U, V in X such that $x_t q U, F \leq V$ and $U \not{q} V$.*

Remark 5.15. *It is clear from Note 5.3 that fuzzy β-b-regular space is fuzzy regular and fuzzy T_2-space is fuzzy β-b-T_2-space. But the reverse implications are not necessarily true, follow from the next example.*

Example 5.16. Consider Example 5.10. It is clear that (X, τ) is fuzzy regular and fuzzy β-b-T_2-space (as every fuzzy set is fuzzy β-b-open set as well as fuzzy β-b-closed set). Now consider the fuzzy point $a_{0.4}$ and a fuzzy set A defined by $A(a) = 0.3$. Then $a_{0.4} \notin A \in F\beta bC(X)$. But there do not exist any fuzzy open set U and a fuzzy β-b-open set V in X such that $a_{0.4} q U, A \leq V$ and $U \not{q} V$ (because 1_X is the only fuzzy open set in X with $a_{0.4} q 1_X$ and $1_X q V$ for all fuzzy set $V (\neq 0_X)$ in X). Hence X is not fuzzy β-b-regular space.
Consider two fuzzy points $a_{0.4}$ and $a_{0.5}$ in X. But there do not exist fuzzy open sets U, V in X such that $a_{0.4} \in U, a_{0.5} q V$ and $U \not{q} V$. So X is not fuzzy T_2-space.

6. Applications of Fuzzy β-b-r-Continuous, Fuzzy β-b-continuous and Fuzzy almost β-b-Continuous Functions

In this section the applications of the functions introduced in this paper are established.

First we recall the following definition from [14] for ready references.

Definition 6.1. [14] *A function $f : X \to Y$ is said to be fuzzy open function if $f(U)$ is a fuzzy open set in Y for every fuzzy open set U in X.*

Theorem 6.2. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be surjective, fuzzy β-b-r-continuous function. If X is fuzzy β-b-compact space, then Y is fuzzy s-closed space.*
Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy covering of Y by fuzzy regular closed sets of Y. As f is fuzzy β-b-r-continuous, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ covers X by fuzzy β-b-open sets of X. As X is fuzzy β-b-compact space, there exists a finite subset Λ_0 of Λ such that $1_X = \bigvee_{\alpha \in \Lambda_0} f^{-1}(U_\alpha)$ implies that $1_Y = f(\bigvee_{\alpha \in \Lambda_0} f^{-1}(U_\alpha)) = \bigvee_{\alpha \in \Lambda_0} f(f^{-1}(U_\alpha)) \leq \bigvee_{\alpha \in \Lambda_0} U_\alpha$. Hence Y is fuzzy s-closed space.

Theorem 6.3. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be fuzzy β-b-continuous function. If A is fuzzy β-b-compact set relative to X, then the image $f(A)$ is fuzzy compact relative to Y.*
Proof. Let A be fuzzy β-b-compact relative to X and $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a

fuzzy covering of $f(A)$ by fuzzy open sets of Y, i.e, $f(A) \leq \bigvee_{\alpha\in\Lambda} U_\alpha$ implies that $A \leq f^{-1}(\bigvee_{\alpha\in\Lambda} U_\alpha) = \bigvee_{\alpha\in\Lambda} f^{-1}(U_\alpha)$. So $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy covering of A by fuzzy β-b-open sets in X. As A is fuzzy β-b-compact set relative to X, there exists a finite subcollection $\mathcal{V}_0 = \{f^{-1}(U_{\alpha_i}) : 1 \leq i \leq n\}$ of $\mathcal{V}$ such that $A \leq \bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})$. Then $f(A) \leq f(\bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})) = \bigvee_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) \leq \bigvee_{i=1}^{n} U_{\alpha_i}$ implies that $\mathcal{U}_0 = \{U_{\alpha_i} : 1 \leq i \leq n\}$ is a finite subcovering of $f(A)$. Hence the proof.

Theorem 6.4. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be fuzzy almost β-b-continuous function. If A is fuzzy β-b-compact relative to X, then the image $f(A)$ is fuzzy nearly compact relative to Y.*
Proof. The proof is similar to that of Theorem 6.3.

Theorem 6.5. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be injective, fuzzy β-b-continuous function and Y is fuzzy T_2-space. Then X is fuzzy β-b-T_2-space.*
Proof. Let x_α and y_β be two distinct fuzzy points in X where $x \neq y$. Then as f is injective, $f(x_\alpha) \neq f(y_\beta)$. As Y is fuzzy T_2-space, there exist fuzzy open sets U_1, U_2, V_1, V_2 in Y such that $f(x_\alpha) \in U_1, f(y_\beta) q V_1$ and $U_1 \not{q} V_1$ and $f(x_\alpha) q U_2$, $f(y_\beta) \in V_2$ and $U_2 \not{q} V_2$. Then $x_\alpha \in f^{-1}(U_1), y_\beta q f^{-1}(V_1)$ and $f^{-1}(U_1) \not{q} f^{-1}(V_1)$. Indeed, $f^{-1}(U_1) q f^{-1}(V_1) \Rightarrow$ there exists $z \in X$ such that $[f^{-1}(U_1)](z) + [f^{-1}(V_1)](z) > 1 \Rightarrow U_1(f(z)) + V_1(f(z)) > 1 \Rightarrow U_1 q V_1$, a contradiction. Also $x_\alpha q f^{-1}(U_2), y_\beta \in f^{-1}(V_2)$ and $f^{-1}(U_2) \not{q} f^{-1}(V_2)$ where $f^{-1}(U_1), f^{-1}(V_1), f^{-1}(U_2), f^{-1}(V_2) \in F\beta\, bO(X, \tau_1)$. Similarly, when $x = y$, $\alpha < \beta$ (say), there exist $U_1, U_2 \in \tau_1$ such that $f(x_\alpha) \in U_1, f(y_\beta) q U_2$ and $U_1 \not{q} U_2$. Then $x_\alpha \in f^{-1}(U_1), y_\beta q f^{-1}(U_2)$ and $f^{-1}(U_1) \not{q} f^{-1}(U_2)$ where $f^{-1}(U_1), f^{-1}(U_2) \in F\beta bO(X, \tau_1)$. Hence X is fuzzy β-b-T_2-space.

Theorem 6.6. *If a bijective function $h : X \to Y$ is fuzzy β-b-continuous, fuzzy open function from a fuzzy β-b-regular space X onto an fts Y, then Y is fuzzy regular space.*
Proof. Let y_t be a fuzzy point in Y and F, a fuzzy closed set in Y with $y_t \notin F$. As h is bijective, there exists unique $x \in X$ such that $h(x) = y$. So $h(x_t) \notin F \Rightarrow x_t \notin h^{-1}(F) \in F\beta bC(X)$ as h is fuzzy β-b-continuous function. As X is fuzzy β-b-regular space, there exist a fuzzy open set U and a fuzzy β-b-open set V in X such that $x_t q U, h^{-1}(F) \leq V$ and $U \not{q} V$. Since X is fuzzy β-b-regular space, by Note 5.3, V is also fuzzy open set in X. As h is fuzzy open function, we have $h(x_\alpha) q h(U), F \leq h(V)$ and $h(U) \not{q} h(V)$ where $h(U), h(V)$ are fuzzy open sets in $Y \Rightarrow Y$ is fuzzy regular space.

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