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ON RAMANUJAN'S TAU-FUNCTION

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Abstract: We exhibit a recurrence relation for the Ramanujan's tau-function involving the sum of divisors function, whose solution gives a closed formula for $\tau(n)$ in terms of complete Bell polynomials. Besides, we show that it is possible to write $\tau(n)$ in terms of the compositions of n.

Keywords and Phrases: Z-transform, Sum of divisors function, Recurrence relations, Ramanujan's function $\tau(n)$, Complete Bell polynomials, Color partitions, Compositions.

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1. Introduction

We begin with the recurrence relation [1, 5, 6, 10, 20, 25, 30, 37]:

$$np_k(n) = -k\sum_{j=1}^{n}\sigma(j)\,p_k(n-j), \qquad n \geq 1, \tag{1}$$

where σ is the sum of divisors function [9, 13, 31, 32, 33] and $p_k(n)$ is the number of color partitions of n (See [20, 17, 18, 37]). The solution of (1) is given by

$$p_k(n) = \frac{1}{n!} B_n(-k \cdot 0!\sigma(1), -k \cdot 1!\sigma(2), -k \cdot 2!\sigma(3), \ldots, -k \cdot (n-1)!\sigma(n)), \quad (2)$$

in terms of the complete Bell polynomials [8, 14, 16, 19, 30, 38, 39]. In the Section 2, we employ (1) and (2) for the case $k = 24$ to obtain a recurrence relation verified by the Ramanujan's tau-function, and its corresponding closed expression via Bell polynomials. The Section 3 contains an observation on the polynomial structure of $p_k(n)$, and the Section 4 shows how the use of the Z-transform allows deducing the formula of Ramanujan for $\tau(p^n)$.

2. Ramanujan's function $\tau(n)$

We recall the connection

$$p_{24}(n) = \tau(n+1) \quad (3)$$

Then (1) gives the following recurrence relation for the Ramanujan's tau-function (See [28, 29]):

$$n\,\tau(n+1) = -24 \sum_{j=1}^{n} \sigma(j)\,\tau(n+1-j), \qquad n \geq 1, \quad (4)$$

which allows an easy recursive manner to calculate the values of $\tau(m)$:

$$1, -24, 252, -1472, 4830, -6048, \ldots,$$

that is, the sequence A000594 [34]. The property (4) obtained by Ramanujan is an alternative to the computationally efficient triangular recurrence formula:

$$(n-1)\,\tau(n) = \sum_{m=1}^{\lfloor b_n \rfloor} (-1)^{m+1}(2m+1)(n-1-\tfrac{9}{2}m(m+1))\tau(n-\tfrac{1}{2}m(m+1)), \quad (5)$$

where $b_n = \frac{1}{2}(\sqrt{8n+1}-1)$ (See [28, 15, 21, 35]).

From (2) and (3), the following closed expression for the Ramanujan's tau-function in terms of the complete Bell polynomials (See [30, 8, 14, 16, 19, 30, 38, 39]) is immediate:

$$\tau(n+1) = \frac{1}{n!} B_n(-24\cdot 0!\sigma(1), -24\cdot 1!\sigma(2), -24\cdot 2!\sigma(3), \ldots, -24\cdot(n-1)!\sigma(n)), \quad n \geq 0, \quad (6)$$

which also allows reproduce the sequence of integers A000594. We can consider (6) as another alternative to several expressions in the literature, for instance, for the expression

$$\tau(n) = n^4\sigma(n) - 24\sum_{k=1}^{n-1} k^2(35k^2 - 52kn + 18n^2)\sigma(k)\sigma(n-k), \qquad n \geq 1, \tag{7}$$

(See [24, 36]) or for the closed relations give in [4, 35]:

$$\tau(n) = 8000((\sigma_3 * \sigma_3) * \sigma_3)(n) - 147(\sigma_5 * \sigma_5)(n), \quad \sigma_r(m) = \sum_{d|m} d^r, \quad m \geq 1, \tag{8}$$

$$= \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sum_{k=1}^{n-1}\sigma_5(k)\sigma_5(n-k), \ \sigma_3(0) = \frac{1}{240}, \ \sigma_5(0) = -\frac{1}{504}, \tag{9}$$

where $*$ denotes Cauchy convolution [31].

3. Polynomial structure of $p_k(n)$

From (1) or (2) it is evident that $p_k(n)$ is a polynomial in k of degree n [7]:

$$p_k(n) = a(n,n)k^n + a(n,n-1)k^{n-1} + \cdots + a(n,2)k^2 + a(n,1)k, \tag{10}$$

which it is equivalent to the recent result of Alegri [2]:

$$p_k(n) = \sum_{j=1}^{n} \frac{(-k)^j}{j!} \sum_{(\omega_1,\ldots,\omega_j)\in C_(n)} \frac{\sigma(\omega_1)\cdots\sigma(\omega_j)}{\omega_1\cdots\omega_j} \tag{11}$$

where C_n is the set of compositions of n. Therefore, from (3), (10) and (11), we have

$$\tau(n+1) = \sum_{j=1}^{n} a(n,j)(24)^j, \ \ a(n,j) = \frac{(-1)^j}{j!} \sum_{(\omega_1,\ldots,\omega_j)\in C_(n)} \frac{\sigma(\omega_1)\cdots\sigma(\omega_j)}{\omega_1\cdots\omega_j}, \ \ n, j \geq 1, \tag{12}$$

which gives an another way to determine the sequence A000594 [34].

4. An expression of Ramanujan involving $\tau(p^n)$

We have the following relation of Ramanujan [4, 22, 23, 28, 29]:

$$\tau(p^{n+2}) = \tau(p)\,\tau(p^{n+1}) - p^{11}\,\tau(p^n), \qquad n \geq 0, \tag{13}$$

where p is a prime number, which gives the recurrence relation:

$$f_{n+2} = \tau(p) f_{n+1} - p^{11} f_n, \qquad f_n := \tau(p^n), \tag{14}$$

with the initial conditions $f_0 = \tau(1) = 1$ and $f_1 = \tau(p)$.

The Z-transform [12, 26] of (14) generates the expression:

$$F(z) = \frac{z^2}{z^2 - \tau(p) z + p^{11}} = \frac{z^2}{(z - \frac{\tau(p)}{2})^2 + A^2}, \qquad A^2 := p^{11} - \frac{(\tau(p))^2}{4}, \tag{15}$$

where $A^2 \geq 0$ due to the property (See [9, 22, 28, 29]):

$$|\tau(p)| \leq 2 p^{\frac{11}{2}}. \tag{16}$$

Then from (15), we have

$$F(z) = \frac{z}{z-a} \cdot \frac{z}{z-\bar{a}}, \qquad a = \frac{\tau(p)}{2} - iA, \qquad a\bar{a} = p^{11}, \tag{17}$$

and hence $\{f_n\}$ is the Cauchy convolution of the sequences $\{g_n\} = \{\bar{a}^n\}$ and $\{h_n\} = \{a^n\}$:

$$f_n = \sum_{m=0}^{n} h_m g_{n-m} = \bar{a}^n \sum_{m=0}^{n} \left(\frac{a^2}{p^{11}} \right)^m \tag{18}$$

and after several manipulations and simplifications the following expression results:

$$\tau(p^n) = \frac{1}{2}(a^n + \bar{a}^n) - \frac{\tau(p)}{2A} \frac{1}{2i}(a^n - \bar{a}^n). \tag{19}$$

Finally, if we define θ_p such that

$$\sin\theta_p = \frac{A}{p^{\frac{11}{2}}}, \quad \cos\theta_p = \frac{\tau(p)}{2\, p^{\frac{11}{2}}},$$

it follows that

$$a = p^{\frac{11}{2}} e^{-i\theta_p}. \tag{20}$$

Then (19) takes the form obtained by Ramanujan [28] (See also [29]):

$$\tau(p^n) = \frac{\sin(n+1)\theta_p}{\sin\theta_p} p^{\frac{11n}{2}}. \tag{21}$$

Remark 4.1. *The property (13) can be obtained if in the relation (see* [4, 23]*)*

$$\sum_{d|(m,N)} d^{11}\tau(\tfrac{mN}{d^2}) = \tau(m)\tau(N), \tag{22}$$

we use $m = p^{n+1}$ and $N = p$.

Remark 4.2. *From the expression (19), it is easy to deduce that*

$$\tau(p^n) = \left(\frac{\tau(p)}{2}\right)^n \sum_{j=0}^{\lfloor n/2\rfloor} \binom{n+1}{2j+1} \left(1 - \frac{4p^{11}}{(\tau(p))^2}\right)^j, \tag{23}$$

but if we use the identity

$$\sum_{r=m}^{\lfloor n/2\rfloor} \binom{n+1}{2r+1}\binom{r}{m} = 2^{n-2m}\binom{n-m}{n-2m}, \qquad 0 \leq m \leq \lfloor n/2\rfloor, \tag{24}$$

then (23) will be reduced to the form obtained by Gandhi [11]:

$$\tau(p^n) = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \binom{n-k}{n-2k} (\tau(p))^{n-2k} p^{11k}. \tag{25}$$

Remark 4.3. *From (7) the following congruences of Ramanujan* [27] *are immediate:*

$$\tau(jn) \equiv 0 \pmod{j}, \qquad j = 2, 3, 5, 7. \tag{26}$$

Remark 4.4. *It is very easy to show that the coefficients $a(n, j)$ defined in (12) verify the recurrence relations:*

$$n\, a(n+j, j) = \sum_{m=1}^{n} \frac{m(j+1) - n}{m+1} \sigma(m+1) a(n+j-m, j), \qquad j, n \geq 1, \tag{27}$$

$$n\, a(n, r) = -\sum_{l=r-1}^{n-1} \sigma(n-l) a(l, r-1), \qquad n \geq 1, \quad 1 \leq r \leq n. \tag{28}$$

Thus, we have

$$\begin{aligned} &a(n,1) = -\frac{\sigma(n)}{n}, \quad a(n,n) = \frac{(-1)^n}{n!}, \ldots, \\ &a(n+1,n) = \frac{3(-1)^n)}{2\cdot(n-1)!}, \quad a(n+2,n) = \frac{(-1)^n(27n+5)}{24\cdot(n-1)!}, \ldots \qquad n \geq 1. \end{aligned} \tag{29}$$

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