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ON SOME EXTENDED SPECIAL FUNCTIONS WITH THE BI-FOX-WRIGHT FUNCTION KERNEL

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Abstract: In the present article, our main goal is to study the new extended beta, Gauss and confluent hypergeometric functions defined by using the two Fox-Wright functions as their kernel. Some applications of the new extended beta function to distribution theory are also discussed.

Keywords and Phrases: Gamma function, Beta function, Fox-Wright function, Mellin transform.

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1. Introduction

In 1994, Chaudhry and Zubair [10] proposed the first extension of the gamma function that considered exponential kernel of the form $\exp\left(-\wp t^{-1}\right)$ as follows:

$$\Gamma_{\wp}(\Omega) = \int_0^{\infty} t^{\Omega-1} \exp\left(-t - \frac{\wp}{t}\right) dt, \tag{1}$$

where $Re(\wp) > 0$ and $Re(\Omega) > 0$.
Using the same concept in (1), Chaudhry et al., [9] studied the following extended Euler beta function:

$$B_{\wp}(\Omega, \mho) = \int_0^1 t^{\Omega-1}(1-t)^{\mho-1} \exp\left(\frac{-\wp}{t(1-t)}\right) dt, \tag{2}$$

where $Re(\wp) > 0$ and $\min\{Re(\Omega), Re(\mho)\} > 0$.
They also (Chaudhry et al., [9]) studied the properties of the extended Euler beta function in (2) such as functional relations, recurrence relations, summation formulas, Mellin transform, application to statistics and connection with other special functions such as error, MacDonald, and Whittaker functions.
Then in 2004, Chaudhry et al., [11] used the extended Euler beta function in (2) to establish the following extended Gauss and confluent hypergeometric functions:

$$F_{\wp}(\upsilon, \phi; \varphi; z) = \frac{1}{B(\phi, \varphi-\phi)} \int_0^1 \frac{t^{\phi-1}(1-t)^{\varphi-\phi-1}}{(1-tz)^{\upsilon}} \exp\left(\frac{-\wp}{t(1-t)}\right) dt,$$

where $Re(\wp) > 0$, $Re(\varphi) > Re(\phi) > 0$ and $|arg(1-z)| < \pi$; and

$$\Phi_{\wp}(\phi; \varphi; z) = \frac{1}{B(\phi, \varphi-\phi)} \int_0^1 t^{\phi-1}(1-t)^{\varphi-\phi-1} \exp\left(tz - \frac{\wp}{t(1-t)}\right) dt,$$

where $Re(\wp) > 0$ and $Re(\varphi) > Re(\phi) > 0$.
For various extension o beta function and their application, see for example Abubakar [1, 2], Abubakar and Patel [3], Ata and Kiymaz [4, 5], Kulip et al., [15], Mubeen et al., [16], Singhal and Mittal [20, 21].
Motivated by the above mentioned work, the following generalization of beta function is defined:

$$\begin{aligned} {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega, \mho) &= {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c} (r_k, R_k)_{1,h} & (\tau_i, T_i)_{1,\varepsilon} & \\ & & \Omega, \mho \\ (d_\iota, D_\iota)_{1,g} & (\ell_j, L_j)_{1,\zeta} & \end{array}\right] \\ &= \int_0^1 (1-t)^{\mho-1} t^{\Omega-1} {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right) {}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) dt, \end{aligned} \tag{3}$$

where $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$ and $\min\{Re(\Omega), Re(\mho)\} > 0$, and ${}_h\Psi_g(.)$ is the Fox-Wright function [14] defined by

$$ {}_h\Psi_g(z) = {}_h\Psi_g \left[\begin{array}{c|c} (r_k, R_k)_{1,h} & \\ & z \\ (d_\iota, D_\iota)_{1,g} & \end{array} \right] = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{h} \Gamma(r_k + nR_k)}{\prod_{\iota}^{g} \Gamma(d_\iota + nD_\iota)} \frac{z^n}{n!} \tag{4} $$

Remark 1. *The following symmetry properties is also true*

$$ {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega, \mho) = {}^{\Psi}B_{\Im,\wp}^{\Upsilon,\Lambda}(\mho, \Omega) $$

2. Properties of the Extended Beta Function

Properties such as functional relation and summation formulas are studied in this section

Theorem 2. (Functional relation) *The following formula holds:*

$$ {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega, \mho) = {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega + 1, \mho) + {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega, \mho + 1) \tag{5} $$

Proof. By direct calculation

$$ \begin{aligned} {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega, \mho) &= \int_0^1 (1-t)^{\mho} t^{\Omega} \left\{(1-t)^{-1} t^{-1}\right\} {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right) {}_h\Psi_g\left(\frac{-\wp}{t^{\Lambda}}\right) dt \\ &= \int_0^1 (1-t)^{\mho} t^{\Omega} \left\{(1-t)^{-1} + t^{-1}\right\} {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\wp}{(1-t)^{\Lambda}}\right) {}_h\Psi_g\left(\frac{-\Im}{t^{\Upsilon}}\right) dt \\ &= {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega + 1, \mho) + {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega, \mho + 1) \end{aligned} $$

This complete the proof of Theorem 2.

Theorem 3. *The following summation formula is valid:*

$$ {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega, 1 - \mho) = \sum_{n=0}^{\infty} \frac{(\mho)_n}{n!}\, {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega + n, 1) \qquad (Re(\mho) < 1) \tag{6} $$

Proof. Using the generalized binomial theorem

$$ \sum_{n=0}^{\infty} (\Delta)_n \frac{(tz)^n}{n!} = (1 - tz)^{-\Delta} \quad (|t| < 1), \tag{7} $$

in new extended beta function (3) and interchanging the order of integral and summation, we obtained

$$ {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega, 1 - \mho) = \sum_{n=0}^{\infty} \frac{(\mho)_n}{n!} \int_0^1 t^{\Omega+n-1} {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right) {}_h\Psi_g\left(\frac{-\wp}{t^{\Lambda}}\right) dt \tag{8} $$

Using (3) and (8), (6) follows.

Theorem 4. *The following second summation formula holds.*

$$ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho)=\sum_{n=0}^{\infty}\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega+n,\mho+1) \tag{9}$$

Proof. Equation (9) follow from first summation formula in (8), binomial theorem in (7) and the extended beta function in (3). The follow result also follows from (3), (5), (8) and (9).

Corollary 5.

$$ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho)=\sum_{n=0}^{m}\binom{m}{n}\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega+n,\mho+m-n)\qquad(m\in\mathbb{N}) \tag{10}$$

3. Integral Representations of the New Extended Beta Function

In this section, integral representations of the new extended beta function are investigated.

Theorem 6. *The following integral representations hold true.*

$$\begin{aligned}
&{}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho)\\
&=2\int_0^1(1-u^2)^{\mho-1}u^{2\Omega-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-u^2)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{u^{2\Lambda}}\right)du &(11)\\
&=2\int_0^{\frac{\pi}{2}}\cos^{2\mho-1}(\vartheta)\sin^{2\Omega-1}(\vartheta)\ {}_{\varepsilon}\Psi_{\zeta}\left(-\Im\sec^{2\Upsilon}(\vartheta)\right){}_{h}\Psi_{g}\left(-\wp\csc^{2\Lambda}(\vartheta)\right)d\vartheta &(12)\\
&=\int_0^{\infty}(1-u)^{1-\mho}u^{\Omega-1}{}_{\varepsilon}\Psi_{\zeta}\left(-\Im(1+u)^{\Upsilon}\right){}_{h}\Psi_{g}\left(-\wp\left(u^{-1}+1\right)^{\Lambda}\right)du &(13)\\
&=2^{1-(\Omega+\mho)}\int_{-1}^{1}(1-u)^{\mho-1}(u+1)^{\Omega-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-2^{\Upsilon}\Im}{(1-u)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-2^{\Lambda}\wp}{(u+1)^{\Lambda}}\right)du &(14)\\
&=(q_2-q_1)^{1-(\Omega+\mho)}\\
&\times\int_{q_1}^{q_2}(q_2-u)^{\mho-1}(u-q_1)^{\Omega-1}{}_{\varepsilon}\Psi_{\zeta}\left(-\frac{\Im(q_2-q_1)^{\Upsilon}}{(q_2-u)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(-\frac{\wp(q_2-q_1)^{\Lambda}}{(u-q_1)^{\Lambda}}\right)du &(15)
\end{aligned}$$

Proof. Setting $t=u^2$, $t=\sin^2(\vartheta)$, $t=u(1+u)^{-1}$, $t=(u-q_1)(q_2-q_1)^{-1}$, and $t=2^{-1}(1+u)$ in (3) and change of variables, we obtained the desired results in (11)-(15). This complete the proof of Theorem 6.

4. The Mellin Transform of the New Extended Beta Function

The Mellin transform and inverse Mellin transform of the new extended beta function is given in the following:

Theorem 7. *The following Mellin transform formula hold.*

$$\mathfrak{M}\left\{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho)\right\}(r,s) = {}^{\Psi}\Gamma(r){}^{\Psi}\Gamma(s)B(\Omega+\Lambda r,\mho+\Upsilon s), \tag{16}$$

where $Re(r) > 0$, $Re(s) > 0$, $Re(\Omega + \Lambda r) > 0$ *and* $Re(\mho + \Upsilon s) > 0$.

Proof. On using direct substituting of (3) into left-hand side of (16), gives

$$\begin{aligned}\mathfrak{M}\left\{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho)\right\}(r,s) &= \int_0^\infty\int_0^\infty \wp^{r-1}\Im^{s-1} \\ &\times\left\{\int_0^1 (1-t)^{\mho-1}t^{\Omega-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)dt\right\}d\wp d\Im\end{aligned} \tag{17}$$

Interchanging the order of integrations in (17), we have

$$\begin{aligned}\mathfrak{M}\left\{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho)\right\}(r,s) &= \int_0^1 (1-t)^{\mho-1}t^{\Omega-1} \\ &\times\left\{\int_0^\infty \Im^{s-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right)d\Im\right\}\left\{\int_0^\infty \wp^{r-1}{}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)d\wp\right\}dt\end{aligned} \tag{18}$$

Setting $\Im = v(1-t)^{\Upsilon}$ and $\wp = ut^{\Lambda}$ to the inner integrals of (18), gives

$$\begin{aligned}\mathfrak{M}\left\{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho)\right\}(r,s) &= \int_0^1 (1-t)^{\mho+s\Upsilon-1}t^{\Omega+r\Lambda-1} \\ &\times\left\{\int_0^\infty v^{s-1}{}_{\varepsilon}\Psi_{\zeta}(-v)\,dv\right\}\left\{\int_0^\infty u^{r-1}{}_{h}\Psi_{g}(-u)\,du\right\}dt\end{aligned} \tag{19}$$

Simplifying (19), the desired result in (16) is obtained.

Theorem 8. *The following inverse Mellin transform hold true.*

$$\begin{aligned}&{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho) \\ &= \frac{1}{(2\pi i)^2}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} {}^{\Psi}\Gamma(r){}^{\Psi}\Gamma(s)B(\Omega+\Lambda r,\mho+\Upsilon s)\Lambda^{-r}\Upsilon^{-s}drds,\end{aligned} \tag{20}$$

where $\sigma_1 > 0$, $\sigma_2 > 0$, $Re(r) > 0$, $Re(s) > 0$, $Re(\Omega+r\Lambda) > 0$ *and* $Re(\mho+s\Upsilon) > 0$.

Proof. On taking inversion Mellin of Theorem 7, we obtain required result in (20).

5. Differential Formulas of the New Extended Beta Function

This section discussed differential formulas of the new generalized beta function

Theorem 9. *The following differential formula holds.*

$$\frac{\partial^n}{\partial\wp^n}\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho) = (-1)^n\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c} (r_k+nR_k,R_k)_{1,h} & (\tau_i,T_i)_{1,\varepsilon} & \Omega-n\Lambda,\mho \\ (d_\iota+nD_\iota,D_\iota)_{1,g} & (\ell_j,L_j)_{1,\zeta} & \end{array}\right], \tag{21}$$

where $Re(\Omega) > n\Lambda$ *and* $Re(\mho) > 0$.

Proof. Using the principal of mathematical induction, we have

$$\frac{\partial}{\partial\wp}\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho) = \frac{\partial}{\partial\wp}\left\{\int_0^1 (1-t)^{\mho-1}t^{\Omega-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)dt\right\} \tag{22}$$

On simplification of (22), we obtain

$$\frac{\partial}{\partial\wp}\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho) = (-1)\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c} (r_k+R_k,R_k)_{1,h} & (\tau_i,T_i)_{1,\varepsilon} & \Omega-\Lambda,\mho \\ (d_\iota+D_\iota,D_\iota)_{1,g} & (\ell_j,L_j)_{1,\zeta} & \end{array}\right] \tag{23}$$

Assuming m^{th} order derivative is holds, then

$$\frac{\partial^m}{\partial\wp^m}\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho) = (-1)^m\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c} (r_k+mR_k,R_k)_{1,h} & (\tau_i,T_i)_{1,\varepsilon} & \Omega-m\Lambda,\mho \\ (d_\iota+mD_\iota,D_\iota)_{1,g} & (\ell_j,L_j)_{1,\zeta} & \end{array}\right] \tag{24}$$

The $(m+1)^{th}$ order derivative is as follows

$$\frac{\partial^{m+1}}{\partial\wp^{m+1}}\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho) = \frac{\partial}{\partial\wp}\left\{\frac{\partial^m}{\partial\wp^m}\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho)\right\} \tag{25}$$

Applying (23) and (24) to (25) and some simplification, yields

$$\begin{aligned}&\frac{\partial^{m+1}}{\partial\wp^{m+1}}\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho)\\ &= (-1)^{m+1}\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c} (r_k+(m+1)R_k,R_k)_{1,h} & (\tau_i,T_i)_{1,\varepsilon} & \Omega-(m+1)\Lambda,\mho \\ (d_\iota+(m+1)D_\iota,D_\iota)_{1,g} & (\ell_j,L_j)_{1,\zeta} & \end{array}\right]\end{aligned}$$

This complete the proof of Theorem 9.

Theorem 10. *The following differential formula is also true.*

$$\frac{\partial^w}{\partial \Im^w}\ {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho) = (-1)^w\ {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c} (r_k,R_k)_{1,h} & (\tau_i+wT_i,T_i)_{1,\varepsilon} & \\ & & \Omega,\mho-w\Upsilon \\ (d_\iota,D_\iota)_{1,g} & (\ell_j+wL_j,L_j)_{1,\zeta} & \end{array}\right], \tag{26}$$

where $Re(\Omega)>0$, $Re(\mho)>w\Upsilon$.

Proof. Using (3) and the procedure in (21), (26) follows.

Theorem 11. *The following differential formula is also true.*

$$\begin{aligned}\frac{\partial^{n+w}}{\partial \wp^n \Im^w}\ {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho) &= (-1)^{n+w} \\ \times\ {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}&\left[\begin{array}{c|c|c} (r_k+wR_k,R_k)_{1,h} & (\tau_i+nT_i,T_i)_{1,\varepsilon} & \\ & & \Omega-n\Lambda,\mho-w\Upsilon \\ (d_\iota+wD_\iota,D_\iota)_{1,g} & (\ell_j+nL_j,L_j)_{1,\zeta} & \end{array}\right],\end{aligned} \tag{27}$$

where $Re(\Omega)>n\Lambda$, *and* $Re(\mho)>w\Upsilon$.

Proof. Equation (27) follows from (21) and (26).

Theorem 12. *The following partial differential formula is valid.*

$$\frac{\partial^n}{\partial\Omega^n}{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho) = \int_0^1 (1-t)^{\mho-1}t^{\Omega-1}\log^n(t)\ {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)dt \tag{28}$$

Proof. Using the principal of mathematical induction, yields

$$\begin{aligned}\frac{\partial}{\partial\Omega}{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho) &= \frac{\partial}{\partial\Omega}\left\{\int_0^1 (1-t)^{\mho-1}t^{\Omega-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)dt\right\} \\ &= \int_0^1 (1-t)^{\mho-1}t^{\Omega-1}\log(t)\ {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)dt\end{aligned} \tag{29}$$

Assuming m^{th} order derivative is hold, then

$$\frac{\partial^m}{\partial\Omega^m}{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho) = \int_0^1 (1-t)^{\mho-1}t^{\Omega-1}\log^m(t){}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)dt \tag{30}$$

The $(m+1)^{th}$ order derivative is as follows

$$\frac{\partial^{m+1}}{\partial\Omega^{m+1}}{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho) = \frac{\partial}{\partial\Omega}\left\{\frac{\partial^m}{\partial\Omega^m}{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho)\right\} \tag{31}$$

Applying (29) and (30) in (31) and simplifying, gives

$$\frac{\partial^{m+1}}{\partial\Omega^{m+1}} {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho) = \int_0^1 (1-t)^{\mho-1} t^{\Omega-1} \log^{m+1}(t)\ {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right) {}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) dt$$

This complete the proof of Theorem 12.

Theorem 13. *The following partial differential formula is valid.*

$$\frac{\partial^{w}}{\partial\mho^{w}} {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho) = \int_0^1 (1-t)^{\mho-1} \log^{w}(1-t) t^{\Omega-1}\ {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right) {}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) dt \tag{32}$$

Proof. Considering (3) and (29), (32) follows.

Theorem 14. *The following partial differential formula is valid.*

$$\frac{\partial^{n+w}}{\partial\Omega^{n}\partial\mho^{w}} {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho) = \int_0^1 (1-t)^{\mho-1} \log^{w}(1-t) t^{\Omega-1} \log^{n}(t)\ {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right) {}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) dt. \tag{33}$$

Proof. Equation (33) follows from (29) and (32).

6. Extended Beta Function Recurrence Relation

The new extended beta function recurrence relation is investigated in this section

Theorem 15. *The following formula holds.*

$$\begin{aligned} &\Omega\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho+1) + \mho\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho) \\ &= \Lambda\wp\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c} (\tau_i+T_i,T_i)_{1,\varepsilon} & (r_k,R_k)_{1,h} & \\ & & \Omega-\Lambda,\mho+1 \\ (\ell_j+L_j,L_j)_{1,\zeta} & (d_\iota,D_\iota)_{1,g} & \end{array}\right] \\ &- \Upsilon\Im\ {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c} (\tau_i,T_i)_{1,\varepsilon} & (r_k+R_k,R_k)_{1,h} & \\ & & \Omega+1,\mho-\Upsilon \\ (\ell_j,L_j)_{1,\zeta} & (d_\iota+D_\iota,D_\iota)_{1,g} & \end{array}\right], \end{aligned} \tag{34}$$

where $Re(\Omega) > \Lambda$ *and* $Re(\mho) > \Upsilon$.

Proof. Setting

$${}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Omega,\mho) = \mathfrak{M}\left\{{}^{\Psi}h^{\Lambda,\Upsilon}_{\wp,\Im}(t;y);\Omega\right\},$$

where

$$ {}^{\Psi}h_{\wp,\Im}^{\Lambda,\Upsilon}(t;y) = H(1-t)(1-t)^{\mho-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right). \tag{35} $$

Differentiating (35) with respect to t, we have

$$ \begin{aligned} \frac{\partial}{\partial t}{}^{\Psi}h_{\wp,\Im}^{\Lambda,\Upsilon}(t;y) &= -\delta(1-t)(1-t)^{\mho-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) - \Upsilon\Im H(1-t) \\ &\times(\mho-1)(1-t)^{\mho-\Upsilon-2}\left\{\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{\varepsilon}\Gamma(r_k+(n+1)R_k)}{\prod_{j=1}^{\zeta}\Gamma(d_\iota+(n+1)D_\iota)}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right)\right\}{}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) \\ &+\Lambda\wp H(1-t)\frac{(1-t)^{\mho-1}}{t^{\Lambda+1}}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right)\left\{\sum_{m=0}^{\infty}\frac{\prod_{i=1}^{h}\Gamma(\tau_i+(m+1)T_i)}{\prod_{j=1}^{g}\Gamma(\ell_j+(m+1)L_j)}\left(\frac{-\wp}{t^{\Lambda}}\right)\right\} \\ &-H(1-t)(\mho-1)(1-t)^{\mho-2}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right). \end{aligned} \tag{36} $$

Using the fact that in [5], $\delta(1-t)=\delta(t-1)=0$, and applying (3) in (36), gives

$$ \begin{aligned} &(\Omega-1)^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega-1,\mho+1)+(\mho-1)^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho-1) \\ &=\Lambda\wp^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c} (r_k+R_k,R_k)_{1,h} & (\tau_i,T_i)_{1,\varepsilon} & \\ & & \Omega-\Lambda-1,\mho \\ (d_\iota+D_\iota,D_\iota)_{1,g} & (\ell_j,L_j)_{1,\zeta} & \end{array}\right] \\ &-\Upsilon\Im^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c} (r_k,R_k)_{1,h} & (\tau_i+T_i,T_i)_{1,\varepsilon} & \\ & & \Omega,\mho-\Upsilon-1 \\ (d_\iota,D_\iota)_{1,g} & (\ell_j+L_j,L_j)_{1,\zeta} & \end{array}\right], \end{aligned} \tag{37} $$

on setting $\Omega\to\Omega+1$ and $\mho\to\mho+1$ in (37) the needful result in (34) is obtained.

7. Applications of the Extended Beta unction to Statistics

The new extended beta function is applied to distribution theory by studying the following extended beta distribution:

$$ f(t)=\begin{cases}\frac{1}{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta,\Theta)}(1-t)^{\Theta-1}t^{\Delta-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) & (0<t<1) \\ 0 & \text{elsewhere,}\end{cases} \tag{38} $$

where $\wp,\Im\geq 0$ and $-\infty<\Delta,\Theta<\infty$.
The moment of the new extended beta distribution in (38) is

$$ E\left(X^{n}\right)=\frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta+n,\Theta)}{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta,\Theta)}, $$

where $\wp, \Im \geq 0$, $-\infty < \Delta, \Theta < \infty$ and $n \in \mathbb{N}$.
The variance of the new extended beta function in (38) is given as

$$\delta^2 = E\left(X^2\right) - \{E(X)\} = \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta,\Theta)\,{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta+2,\Theta) - \left\{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta+1,\Theta)\right\}^2}{\left\{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta,\Theta)\right\}^2}.$$

The Coefficient of Variance (C.V) is defined as

$$C.V = \sqrt{\frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta,\Theta)\,{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta+2,\Theta)}{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta+1,\Theta)} - 1}.$$

The Moment Generating Function (m.g.t) is represented by

$$M_X(t) = \sum_{n=0}^{\infty} \frac{E\left(X^n\right)t^n}{n!} = \sum_{n=0}^{\infty} \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta+n,\Theta)t^n}{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta,\Theta)n!}.$$

The characteristic function is denoted by

$$M_X\left(\exp(it)\right) = \sum_{n=0}^{\infty} \frac{E\left(X^n\right)(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta+n,\Theta)(it)^n}{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta,\Theta)n!}.$$

The cumulative (probability) distribution function of the new extended beta function in (38) is expressed by

$$F(z) = F(X < z) = \int_0^z f(t)dt = \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon;z}(\Delta,\Theta)}{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Delta,\Theta)},$$

where ${}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon;z}(\Delta,\Theta)$ denoted the lower incomplete new extended beta function defined by

$$\begin{aligned}{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon;z}(\Delta,\Theta) &= {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon;z}\left[\begin{array}{c|c|c} (r_k,R_k)_{1,h} & (\tau_i,T_i)_{1,\varepsilon} & \\ & & \Delta,\Theta \\ (d_\iota,D_\iota)_{1,g} & (\ell_j,L_j)_{1,\zeta} & \end{array}\right] \\ &= \int_0^z (1-t)^{\Theta-1}t^{\Delta-1}\,{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)dt,\end{aligned}$$

where $\wp, \Im \geq 0$ and $-\infty < \Delta, \Theta < \infty$.
The reliability function of the new extended beta function in (38) is illustrated by the following equation

$$R(z) = f(X > z) = 1 - F(z) = \int_z^1 f(t)dt = \frac{{}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im;z}(\Delta,\Theta)}{{}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\Delta,\Theta)},$$

for ${}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im;z}(\Delta,\Theta)$ denoted the upper incomplete new extended beta function given by

$$\begin{aligned} {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im;z}(\Delta,\Theta) &= {}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im;z}\left[\begin{array}{c|c|c} (r_k,R_k)_{1,h} & (\tau_i,T_i)_{1,\varepsilon} & \\ & & \Delta,\Theta \\ (d_\iota,D_\iota)_{1,g} & (\ell_j,L_j)_{1,\zeta} & \end{array}\right] \\ &= \int_z^1 (1-t)^{\Theta-1} t^{\Delta-1} {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right) {}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) dt, \end{aligned}$$

for $\wp, \Im \geq 0$ and $-\infty < \Delta, \Theta < \infty$.

8. The New Extended Hypergeometric Function

The new extended Gauss and confluent hypergeometric functions are defined in this section

$$\begin{aligned} {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi) &= {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c} (r_k,R_k)_{1,h} & (\tau_i,T_i)_{1,\varepsilon} & \\ & & \upsilon,\phi;\varphi;z \\ (d_\iota,D_\iota)_{1,g} & (\ell_j,L_j)_{1,\zeta} & \end{array}\right] \\ &= \sum_{n=0}^{\infty} (\upsilon)_n \frac{{}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\phi+n,\varphi-\phi)}{B(\phi,\varphi-\phi)} \frac{z^n}{n!}, \end{aligned} \tag{39}$$

where $Re(\varphi) > Re(\phi) > 0$, $|z| < 1$, and

$$\begin{aligned} {}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi;\varphi) &= {}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c} (r_k,R_k)_{1,h} & (\tau_i,T_i)_{1,\varepsilon} & \\ & & \phi;\varphi;z \\ (d_\iota,D_\iota)_{1,g} & (\ell_j,L_j)_{1,\zeta} & \end{array}\right] \\ &= \sum_{n=0}^{\infty} \frac{{}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\phi+n,\varphi-\phi)}{B(\phi,\varphi-\phi)} \frac{z^n}{n!}, \end{aligned} \tag{40}$$

where $Re(\varphi) > Re(\phi) > 0$.

9. Integral Representation of Extended Hypergeometric Function

Theorem 16. *The following integral formula is valid*

$$ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;z) = \frac{1}{B(\phi,\varphi-\phi)}\int_0^1 \frac{(1-t)^{\varphi-\phi-1}t^{\varphi-1}}{(1-tz)^{\upsilon}}\, {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right) {}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) dt, \tag{41} $$

where $Re(\varphi) > Re(\phi) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Omega), Re(\mho)\} > 0$, *and* $|arg(1-z)| < \pi$.

Proof. Considering (39), we obtained

$$ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;z) = \frac{1}{B(\phi,\varphi-\phi)}\sum_{n=0}^{\infty}(\upsilon)_n \left\{\int_0^1 (1-t)^{\varphi-\phi-1}t^{\phi+n-1}\, {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right) {}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) dt\right\}\frac{z^n}{n!} \tag{42} $$

$$ = \frac{1}{B(\phi,\varphi-\phi)}\int_0^1 (1-t)^{\varphi-\phi-1}t^{\phi-1}\left\{\sum_{n=0}^{\infty}(\upsilon)_n\frac{(tz)^n}{n!}\right\} {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right) {}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) dt \tag{43} $$

Applying (7) to (42) the required result in (41) is obtained.

Theorem 17. *For the extended Gauss hypergeometric function, the following integral formulas holds*

$$ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;z) = \frac{1}{B(\phi,\varphi-\phi)}\int_0^{\infty} \frac{(1+t)^{\upsilon-\varphi}t^{\phi-1}}{\{1+(1-z)t\}^{\upsilon}}\, {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right) {}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) dt, \tag{44} $$

and

$$ \begin{aligned} {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;z) =& \frac{2}{B(\phi,\varphi-\phi)}\int_0^{\frac{\pi}{2}} \cos^{(2\varphi-2\phi-1)}(\vartheta)\sin^{(2\phi-1)}(\vartheta)(1-z\sin^2(\vartheta))^{-\upsilon} \\ & \times {}_{\varepsilon}\Psi_{\zeta}\left(-\Im\sec^{2\Upsilon}(\vartheta)\right) {}_{h}\Psi_{g}\left(-\wp\csc^{2\Lambda}(\vartheta)\right) d\vartheta \end{aligned} \tag{45} $$

Proof. The proof of (44) and (45) follows by directly by putting $u = t(1-t)^{-1}$, $t = \sin^2(\vartheta)$ in (41), respectively.

Corollary 18.

$$ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z) = \frac{1}{B(\phi,\varphi-\phi)}\int_0^1 (1-t)^{\varphi-\phi-1}t^{\varphi-1}\exp(tz)\,{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)dt, \quad (46) $$

where $Re(\varphi) > Re(\phi) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Omega), Re(\mho)\} > 0$ *and* $|arg(1-z)| < \pi$.

$$ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z) = \frac{1}{B(\phi,\varphi-\phi)}\int_0^{\infty} (1+t)^{-\varphi}t^{\phi-1}\exp\left(\frac{tz}{1+t}\right) \times {}_{\varepsilon}\Psi_{\zeta}\left(-\Im(1+t)^{\Upsilon}\right){}_{h}\Psi_{g}\left(-\wp\left(1+t^{-1}\right)\right)^{\Lambda}dt, \quad (47) $$

and

$$ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z) = \frac{2}{B(\phi,\varphi-\phi)}\int_0^{\frac{\pi}{2}} \cos^{(2\varphi-2\phi-1)}(\vartheta)\sin^{2\phi-1}(\vartheta)\exp\left(z\sin^2(\vartheta)\right) \times {}_{\varepsilon}\Psi_{\zeta}\left(-\Im\sec^{2\Upsilon}(\vartheta)\right){}_{h}\Psi_{g}\left(-\wp\csc^{2\Lambda}\right)d\vartheta $$

Remark 19. *Putting* $t \to 1-t$ *in (46) the following formula is obtained*

$$ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z) = \frac{\exp(z)}{B(\phi,\varphi-\phi)}\int_0^1 t^{\varphi-\phi-1}(1-t)^{\varphi-1}\exp(-tz)\,{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{t^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{(1-t)^{\Lambda}}\right)dt \quad (48) $$

10. The Mellin Transform of the New Extended Hypergeometric Function

The Mellin transform and inverse Mellin transform of the new extended hypergeometric functions is discussed in this section:

Theorem 20. *The following Mellin transform formula hold.*

$$ \mathfrak{M}\left\{{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)\right\}(r,s) = \frac{{}^{\Psi}\Gamma(r)\,{}^{\Psi}\Gamma(s)B(\phi+\Lambda r,\varphi-\phi+\Upsilon s)}{B(\phi,\varphi-\phi)}\,{}_2F_1(\upsilon,\phi+\Lambda r;\varphi+\Lambda r+\Upsilon s;z), \quad (49) $$

where $Re(r) > 0$, $Re(s) > 0$, $Re(\phi+\Lambda r) > 0$ *and* $Re(\varphi-\phi+\Upsilon s) > 0$.

Proof. Using direct substituting

$$\mathfrak{M}\left\{{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)\right\}(r,s)=\int_0^\infty\int_0^\infty d\wp d\Im\wp^{r-1}\Im^{s-1}$$
$$\times\left\{\frac{1}{B(\phi,\varphi-\phi)}\int_0^1(1-t)^{\varphi-\phi-1}t^{\phi-1}(1-tz)^{-\upsilon}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)dt\right\} \tag{50}$$

Interchanging the order of integrations in (50), we have

$$\mathfrak{M}\left\{{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)\right\}(r,s)=\frac{1}{B(\phi,\varphi-\phi)}\int_0^1(1-t)^{\varphi-\phi-1}t^{\phi-1}(1-tz)^{-\upsilon}$$
$$\times\left\{\int_0^\infty\Im^{s-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right)d\Im\right\}\left\{\int_0^\infty\wp^{r-1}{}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)d\wp\right\}dt. \tag{51}$$

Setting $\Im=\upsilon(1-t)^{\Upsilon}$ and $\wp=ut^{\Lambda}$ to the inner integrals of (51), gives

$$\mathfrak{M}\left\{{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)\right\}(r,s)=\frac{1}{B(\phi,\varphi-\phi)}\int_0^1(1-t)^{\varphi-\phi+s\Upsilon-1}t^{\phi+r\Lambda-1}(1-tz)^{-\upsilon}$$
$$\times\left\{\int_0^\infty\upsilon^{s-1}{}_{\varepsilon}\Psi_{\zeta}(-\upsilon)\,d\upsilon\right\}\left\{\int_0^\infty u^{r-1}{}_{h}\Psi_{g}(-u)\,du\right\}dt. \tag{52}$$

Simplifying (52), the desired result in (49) is obtained.

Theorem 21. *The following inverse Mellin transform hold true.*

$${}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)=\frac{1}{(2\pi i)^2}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty}\frac{{}^{\Psi}\Gamma(r){}^{\Psi}\Gamma(s)B(\phi+\Lambda r,\varphi-\phi+\Upsilon s)}{B(\phi,\varphi-\phi)}$$
$$\times{}_2F_1(\upsilon,\phi+\Lambda r,\varphi+\Lambda r+\Upsilon s;z)\Lambda^{-r}\Upsilon^{-s}drds, \tag{53}$$

where $\sigma_1>0$ and $\sigma_2>0$.

Proof. On taking Mellin inversion of Theorem 20, we obtain required result in (53).

Corollary 22.

$$\mathbb{M}\left\{{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z)\right\}(r,s)$$
$$=\frac{{}^{\Psi}\Gamma(r){}^{\Psi}\Gamma(s)B(\phi+\Lambda r,\varphi-\phi+\Upsilon s)}{B(\phi,\varphi-\phi)}\Phi(\phi+\Lambda r,\varphi+\Lambda r+\Upsilon s;z),$$

where $Re(r) > 0$, $Re(s) > 0$, $Re(\phi + \Lambda r) > 0$ and $Re(\varphi - \phi + \Upsilon s) > 0$.
Corollary 23.

$$^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z) = \frac{1}{(2\pi i)^2}\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty}\int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty}\frac{^{\Psi}\Gamma(r)^{\Psi}\Gamma(s)B(\phi+\Lambda r, \varphi - \phi + \Upsilon s)}{B(\phi, \varphi - \phi)}$$
$$\times\Phi(\phi+\Lambda r, \varphi + \Lambda r + \Upsilon s; z)\Lambda^{-r}\Upsilon^{-s}drds,$$

where $\sigma_1 > 0$ *and* $\sigma_2 > 0$.

11. The New Extended Hypergeometric Function Differential and Difference Formulas

Differential and difference formulas for the new extended Gauss and confluent hypergeometric functions is presented in this section

Theorem 24. *Gauss hypergeometric function, the following differential formula holds*

$$\frac{d^n}{dz^n}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi,\varphi;z) = \frac{(\upsilon)_n(\phi)_n}{(\varphi)_n}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon+n,\phi+n;\varphi+n;z) \qquad (n \in \mathbb{N}_0) \quad (54)$$

Proof. Using mathematical induction, for $n = 0$ equation (54) obviously follows and for $n = 1$, we have

$$\frac{d}{dz}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi,\varphi;z) = \sum_{n=1}^{\infty}(\upsilon)_n\frac{^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+n,\varphi-\phi)}{B(\phi,\varphi-\phi)}\frac{z^{n-1}}{(n-1)!}$$
$$= \frac{(\upsilon)(\phi)}{(\varphi)}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon+1,\phi+1;\varphi+1;z) \qquad (55)$$

Assuming m^{th}-order derivative is holds, that is

$$\frac{d^m}{dz^m}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi,\varphi;z) = \frac{(\upsilon)_m(\phi)_m}{(\varphi)_m}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon+m,\phi+m;\varphi+m;z), \qquad (56)$$

and so the $(m+1)^{th}$-order derivative is

$$\frac{d^{m+1}}{dz^{m+1}}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi,\varphi;z) = \frac{d}{dz}\left\{\frac{d^m}{dz^m}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi,\varphi;z)\right\} \qquad (57)$$

Using (55) and (56) in (57) gives

$$\frac{d^{m+1}}{dz^{m+1}}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi,\varphi;z) = \frac{(\upsilon)_{m+1}(\phi)_{m+1}}{(\varphi)_{m+1}}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon+m+1,\phi+m+1;\varphi+m+1;z)$$

This complete the proof of Theorem 24.

Theorem 25. *For the new extended confluent hypergeometric function, the following differential formula holds*

$$\frac{d^n}{dz^n}{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z)=\frac{(\phi)_n}{(\varphi)_n}{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+n;\varphi+n;z) \qquad (n\in\mathbb{N}_0) \tag{58}$$

Proof. This follows from Theorem 24.
Theorem 26.

$$\Delta_\upsilon{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)=\frac{\phi z}{\varphi}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon+1,\phi+1;\varphi+1;z), \tag{59}$$

and

$$\upsilon\Delta_\upsilon{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)=z\frac{d}{dz}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z) \tag{60}$$

Proof. From (39) we can obtain

$$\begin{aligned}&\Delta_\upsilon{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)={}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon+1,\phi;\varphi;z)-{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)\\&=\frac{z}{B(\phi,\varphi-\phi)}\int_0^1\frac{(1-t)^{\varphi-\phi-1}t^{\varphi-1}}{(1-tz)^{\upsilon-1}}{}_{\varepsilon}\Psi_\zeta\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_h\Psi_g\left(\frac{-\wp}{t^{\Lambda}}\right)dt\end{aligned} \tag{61}$$

Also, from (41) setting $\upsilon\to\upsilon+1$, $\phi\to\phi+1$ and $\varphi\to\varphi+1$, gives

$$\begin{aligned}&{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon+1,\phi+1;\varphi+1;z)\\&=\frac{1}{B(\phi+1,\varphi-\phi)}\int_0^1\frac{(1-t)^{\varphi-\phi-1}t^{\varphi}}{(1-tz)^{\upsilon+1}}{}_{\varepsilon}\Psi_\zeta\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_h\Psi_g\left(\frac{-\wp}{t^{\Lambda}}\right)dt,\end{aligned} \tag{62}$$

putting (62) into (61) and using (55) the desired result in (59) is obtained. Using the differential formula for $n=1$ in (55) the required result in (60) is received.

12. The New Extended Hypergeometric Function Recurrence Relations

The Gauss and confluent hypergeometric function recurrence relations is study in this section

Theorem 27. *The following formula holds.*

$$\begin{aligned}
&(\phi-1)B(\phi-1,\varphi-\phi+1)^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi-1,\varphi-1;z)\\
&=(\varphi-\phi-1)B(\phi,\varphi-\phi-1)^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi-1;z)\\
&-\upsilon zB(\phi,\varphi-\phi)^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon+1,\phi,\varphi;z)+\Upsilon\Im B(\phi,\varphi-\phi-\Upsilon-1)\\
&\times{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c}(r_k,R_k)_{1,h} & (\tau_i+T_i,T_i)_{1,\varepsilon} & \\ & & \upsilon,\phi-\Lambda-1;\varphi-\Upsilon-1;z\\ (d_\iota,D_\iota)_{1,g} & (\ell_j+L_j,L_j)_{1,\zeta} & \end{array}\right]\\
&-\Lambda\wp B(\phi,\varphi-\Lambda-1)\\
&\times{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c}(\tau_i+T_i,T_i)_{1,\varepsilon} & (r_k,R_k)_{1,h} & \\ & & \upsilon,\phi-\Lambda-1;\varphi-\Lambda-1;z\\ (\ell_j+L_j,L_j)_{1,\zeta} & (d_\iota,D_\iota)_{1,g} & \end{array}\right],
\end{aligned}\tag{63}$$

where $Re(\phi-\Lambda)>1$, $Re(\varphi-\upsilon)>\Upsilon+1$.

Proof. Setting

$$B(\phi,\varphi-\phi)^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;z)=\mathfrak{M}\left\{{}^{\Psi}h^{\Lambda,\Upsilon;\upsilon,\phi,\varphi}_{\wp,\Im}(t;z);\Omega\right\},$$

where

$${}^{\Psi}h^{\Lambda,\Upsilon}_{\wp,\Im}(t;y)=H(1-t)(1-t)^{\varphi-\phi-1}(1-tz)^{-\upsilon}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)\tag{64}$$

Differentiating (64) with respect to t, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}{}^{\Psi}h^{\Lambda,\Upsilon;\upsilon,\phi,\varphi}_{\wp,\Im}(t;z)=-\delta(1-t)^{\varphi-\phi-1}(1-tz)^{-\upsilon}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)\\
&-(\varphi-\phi-1)H(1-t)(1-t)^{\varphi-\phi-2}(1-tz)^{-\upsilon}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)\\
&-\Upsilon\Im H(1-t)(1-tz)^{-\upsilon}(1-t)^{\varphi-\phi-\Upsilon-2}\\
&\times\left\{\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{\varepsilon}\Gamma(r_k+(n+1)R_k)}{\prod_{j=1}^{\zeta}\Gamma(d_\iota+(n+1)D_\iota)}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right)\right\}{}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)+\Lambda\wp H(1-t)t^{-\Lambda-1}\\
&\times(1-t)^{\varphi-1}(1-tz)^{-\upsilon}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right)\left\{\sum_{m=0}^{\infty}\frac{\prod_{i=1}^{h}\Gamma(\tau_i+(m+1)T_i)}{\prod_{j=1}^{g}\Gamma(\ell_j+(m+1)L_j)}\left(\frac{-\wp}{t^{\Lambda}}\right)\right\}\\
&+\upsilon zH(1-t)(1-t)^{\varphi-\phi-1}(1-tz)^{-\upsilon-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)
\end{aligned}\tag{65}$$

Using the fact that in [5], $\delta(1-t)=\delta(t-1)=0$ and using (39) in (63), gives

$$\begin{aligned}
&-(\phi-1)B(\phi-1,\varphi-\phi+1)\,{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi-1,\varphi-1;z)=-(\varphi-\phi-1)\\
&\times B(\phi,\varphi-\phi-1)\,{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi-1;z)+azB(\phi,\varphi-\phi)\,{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon+1,\phi,\varphi;z)\\
&-{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c}(r_k,R_k)_{1,h} & (\tau_i+T_i,T_i)_{1,\varepsilon} & \\ & & \upsilon,\phi-\Upsilon-1;\varphi-\Upsilon-1;z\\ (d_\iota,D_\iota)_{1,g} & (\ell_j+L_j,L_j)_{1,\zeta} & \end{array}\right]\\
&\times \Upsilon\Im B(\phi,\varphi-\phi-\Upsilon-1)+\Lambda\wp B(\phi,\varphi-\phi-\Lambda-1)\\
&\times {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c}(\tau_i+T_i,T_i)_{1,\varepsilon} & (r_k,R_k)_{1,h} & \\ & & \upsilon,\phi-\Lambda-1;\varphi-\Lambda-1;z\\ (\ell_j+L_j,L_j)_{1,\zeta} & (d_\iota,D_\iota)_{1,g} & \end{array}\right] \qquad (66)
\end{aligned}$$

On algebraic simplifications of (6) the required result in (63) is obtained.

Theorem 28. *The following formula holds.*

$$\begin{aligned}
&(\phi-1)B(\phi-1,\varphi-\phi+1)\,{}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi-1,\varphi-1;z)=(\varphi-\phi-1)B(\phi,\varphi-\phi-1)\\
&\times {}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi;\varphi-1;z)-zB(\phi,\varphi-\phi)\,{}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi,\varphi-1;z)+\Upsilon\Im B(\phi,\varphi-\phi-\Upsilon-1)\\
&\times {}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c}(r_k,R_k)_{1,h} & (\tau_i+T_i,T_i)_{1,\varepsilon} & \\ & & \phi;\varphi-\Upsilon-1;z\\ (d_\iota,D_\iota)_{1,g} & (\ell_j+L_j,L_j)_{1,\zeta} & \end{array}\right]\\
&-{}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}\left[\begin{array}{c|c|c}(\tau_i+T_i,T_i)_{1,\varepsilon} & (r_k,R_k)_{1,h} & \\ & & \phi-\Lambda-1;\varphi-\Lambda-1;z\\ (\ell_j+L_j,L_j)_{1,\zeta} & (d_\iota,D_\iota)_{1,g} & \end{array}\right]\\
&\times \Lambda\wp B(\phi-\Lambda-1,\varphi-\Lambda-1) \qquad (67)
\end{aligned}$$

where $Re(\phi-\Lambda)>1,\ Re(\varphi-\upsilon)>\Upsilon+1.$

Proof. Setting

$$B(\phi,\varphi-\phi)\,{}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi;\varphi;z)=\mathfrak{M}\left\{{}^{\Psi}h^{\Lambda,\Upsilon;\phi,\varphi}_{\wp,\Im}(t;z);\Omega\right\},$$

where

$$ {}^{\Psi}h^{\Lambda,\Upsilon;\phi,\varphi}_{\wp,\Im}(t;y)=H(1-t)(1-t)^{\varphi-\phi-1}\exp(tz)\,{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right) \qquad (68)$$

On differentiating (68) with respect t and following similar argument as in Theorem 27, we get the desired result in (67).

13. The New Extended Hypergeometric Function Summation Formulas

This section contained functional relation and summation formulas for the new extended Gauss and confluent hypergeometric functions

Theorem 29. *The following functional relations holds*

$$\varphi\,{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)=\phi\,{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi+1;\varphi+1;z)+(\varphi-\phi)\,{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi+1;z),\quad(69)$$

and

$$\varphi\,{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z)=\phi\,{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+1;\varphi+1;z)+(\varphi-\phi)\,{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi+1;z)\quad(70)$$

Proof. Equations (69) and (70) are obtained from the functional relation of new extended beta function in (5).

Theorem 30.

$${}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)=\sum_{n=0}^{\infty}\frac{(\phi-\varphi+1)_n B(\phi+n,1\phi)}{B(\phi,\varphi-\phi)n!}\,{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi+n;\varphi+n+1;z),\quad(71)$$

and

$${}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z)=\sum_{n=0}^{\infty}\frac{(\phi-\varphi+1)_n B(\phi+n,1\phi)}{B(\phi,\varphi-\phi)n!}\,{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+n;\varphi+n+1;z)\quad(72)$$

Proof. Equations (71) and (72) follows directly from the new extended beta function summation formula in (6).

Theorem 31.

$${}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)=(\varphi-\phi)\sum_{n=0}^{\infty}\frac{(\phi)_n}{(\varphi)_{n+1}}\,{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi+n;\varphi+n+1;z),\quad(73)$$

and

$${}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z)=(\varphi-\phi)\sum_{n=0}^{\infty}\frac{(\phi)_n}{(\varphi)_{n+1}}\,{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+n;\varphi+n+1;z)\quad(74)$$

Proof. Equations (73) and (74) follows directly from the new extended beta function summation formula in (6).

Theorem 32. *The following finite summation formula is also true*

$${}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z)=\sum_{n=0}^{m}\binom{m}{n}\frac{B(\phi+n,\varphi-\phi-n+m)}{B(\phi,\varphi-\phi)}\,{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi+n;\varphi+m;z),\quad(75)$$

and

$$ {}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi;\varphi;z) = \sum_{n=0}^{m} \binom{m}{n} \frac{B(\phi+n,\varphi-\phi-n+m)}{B(\phi,\varphi-\phi)} {}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi+n;\varphi+m;z) \quad (76) $$

Proof. Equations (75) and (76) follows directly from the new extended beta function summation formula in (10).

14. The Hypergeometric Function Transformation Formulas

The Gauss and confluent hypergeometric functions related transformation formulas are studied in this section

Theorem 33. *For the new extended Gauss hypergeometric function, the following transformation formula holds*

$$ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi,\varphi;z) = (1-z)^{-\upsilon}\, {}^{\Psi}F^{\Upsilon,\Lambda}_{\Im,\wp}\left(\upsilon,\varphi-\phi;\varphi;\frac{z}{z-1}\right) \quad (|arg(1-z)|<\pi) \quad (77) $$

Proof. Considering the elementary identity

$$ [1-z(1-t)]^{-\upsilon} = (1-z)^{-\upsilon}\left(1+\frac{z}{1-z}t\right)^{-\upsilon} \quad (78) $$

Substituting t by $1-t$ in (78) and using the integral formula for the new extended Gauss hypergeometric function in (41), we get

$$ \begin{aligned} &{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;z) \\ &= \frac{1}{B(\phi,\varphi-\phi)}\int_0^1 \frac{t^{\varphi-\phi-1}(1-t)^{\varphi-1}}{[1-(1-t)z]^{\upsilon}}\, {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{t^{\Upsilon}}\right) {}_{h}\Psi_{g}\left(\frac{-\wp}{(1-t)^{\Lambda}}\right) dt \\ &= \frac{(1-z)^{-\upsilon}}{B(\phi,\varphi-\phi)}\int_0^1 \frac{t^{\varphi-\phi-1}(1-t)^{\varphi-1}}{\left[1+\frac{z}{1-z}t\right]^{\upsilon}}\, {}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{t^{\Upsilon}}\right) {}_{h}\Psi_{g}\left(\frac{-\wp}{(1-t)^{\Lambda}}\right) dt \end{aligned} \quad (79) $$

Simplifying (79) the needful result in (77) is obtained
Putting $z \longrightarrow 1-\frac{1}{z}$ and $z \longrightarrow -\frac{z}{1+z}$ in (77) the following corollary are obtained:

Corollary 34.

$$ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left(\upsilon,\phi,\varphi;1-\frac{1}{z}\right) = z^{\upsilon}\, {}^{\Psi}F^{\Upsilon,\Lambda}_{\Im,\wp}(\upsilon,\varphi-\phi;\varphi;1-z) \quad (|arg(z)|<\pi), \quad (80) $$

and

$$ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left(\upsilon,\phi;\varphi;\frac{z}{1+z}\right) = (1+z)^{\upsilon}\, {}^{\Psi}F^{\Upsilon,\Lambda}_{\Im,\wp}(\upsilon,\varphi-\phi;\varphi;z) \quad (|arg(1+z)|<\pi) \quad (81) $$

Corollary 35. *The following formula is also true*

$$ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z) = \exp(z)\ {}^{\Psi}\Phi_{\Im,\wp}^{\Upsilon,\Lambda}(\varphi-\phi;\varphi;-z) \tag{82} $$

Theorem 36. *The following transformation formula is also true*

$$ \int_0^\infty z^{\upsilon-1}\,{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;-z) = \Gamma(\upsilon)\ {}^{\Psi}F_{\Im,\wp}^{\Upsilon,\Lambda}(\upsilon,\phi;\varphi;1) \tag{83} $$

Proof. Putting $z \longrightarrow -z$ into (82), multiplying by $z^{\upsilon-1}$ and integrating with respect to z, limit form 0 to ∞, we have

$$ \begin{aligned} \int_0^\infty z^{\upsilon-1}\,{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;-z) &= \int_0^\infty z^{\upsilon-1}\exp(z)\left\{\sum_{n=0}^{\infty}\frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\varphi-\phi+n,\phi)}{B(\varphi-\phi,\phi)}\frac{z^n}{n!}\right\}dz \\ &= \sum_{n=0}^{\infty}\frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\varphi-\phi+n,\phi)}{B(\varphi-\phi,\phi)n!}\int_0^\infty z^{\upsilon+n-1}\exp(z)dz \\ &= \Gamma(\upsilon)\sum_{n=0}^{\infty}(\upsilon)_n\frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\varphi-\phi+n,\phi)}{B(\varphi-\phi,\phi)}\frac{(1)^n}{n!} \end{aligned} \tag{84} $$

Applying (39) to (84) the required result in (83) is obtained.

15. Conclusion

New generalized beta, Gauss and confluent hypergeometric functions with two Wright functions as their regularizer are studied. Many known beta, Gauss and confluent hypergeometric functions can expressed in term of the new generalized beta, Gauss and confluent hypergeometric functions, for example
Sahim et al., [19]

$$ B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho) = {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c} (1,0)_{1,1} & (1,0)_{1,1} & \Omega,\mho \\ (1,0)_{1,1} & (1,0)_{1,1} & \end{array}\right] $$

$$ F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z) = {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c} (1,0)_{1,1} & (1,0)_{1,1} & \upsilon,\phi;\varphi;z \\ (1,0)_{1,1} & (1,0)_{1,1} & \end{array}\right], $$

and

$$ \Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;z) = {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c} (1,0)_{1,1} & (1,0)_{1,1} & \phi;\varphi;z \\ (1,0)_{1,1} & (1,0)_{1,1} & \end{array}\right], $$

which will also reduces to the work of Choi et al., [12] if $\Lambda = \Upsilon = 1$, Chaudhry and Zubair [10], Chaudhry et al., [9, 11], if we set $\Lambda = \Upsilon = 1$ and $\wp = \Im$ and classical beta, Gauss and confluent hypergeometric functions [14] if we put $\wp = \Im = 0$.
Khan et al., [15]

$$B_{\wp,\Im}^{\mu;\Lambda,\Upsilon}(\Omega,\mho) = {}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c} (1,1)_{1,1} & (1,1)_{1,1} & \\ & & \Omega,\mho \\ (\mu,1)_{1,1} & (\mu,1)_{1,1} & \end{array}\right]$$

$$F_{\wp,\Im}^{\mu;\Lambda,\Upsilon}(\upsilon,\phi;\varphi;z) = {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c} (\mu,1)_{1,1} & (\mu,1)_{1,1} & \\ & & \upsilon,\phi;\varphi;z \\ (\mu,1)_{1,1} & (\mu,1)_{1,1} & \end{array}\right],$$

and

$$\Phi_{\wp,\Im}^{\mu;\Lambda,\Upsilon}(\phi;\varphi;z) = {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c} (\mu,1)_{1,1} & (\mu,1)_{1,1} & \\ & & \phi;\varphi;z \\ (\mu,1)_{1,1} & (\mu,1)_{1,1} & \end{array}\right],$$

which will also diminishes to the work of Rahman et al., in [17] if $\Lambda = \Upsilon = 1$.
Barahman [8] and Atash et al., [6]

$$B_{\wp,\Im}^{\mu,\xi,\omega}(\Omega,\mho) = \frac{1}{\Gamma^2(\omega)}\,{}^{\Psi}B_{\wp,\Im}^{1,1}\left[\begin{array}{c|c|c} (\omega,1)_{1,1} & (\omega,1)_{1,1} & \\ & & \Omega,\mho \\ (\mu,\xi)_{1,1} & (\mu,\xi)_{1,1} & \end{array}\right]$$

$$F_{\wp,\Im}^{\mu,\xi,\omega}(\upsilon,\phi;\varphi;z) = \frac{1}{\Gamma^2(\omega)}\,{}^{\Psi}F_{\wp,\Im}^{1,1}\left[\begin{array}{c|c|c} (\omega,1)_{1,1} & (\mu,1)_{1,1} & \\ & & \upsilon,\phi;\varphi;z \\ (\mu,\xi)_{1,1} & (\mu,\xi)_{1,1} & \end{array}\right],$$

and

$$\Phi_{\wp,\Im}^{\mu,\xi,\omega}(\phi;\varphi;z) = \frac{1}{\Gamma^2(\omega)}\,{}^{\Psi}\Phi_{\wp,\Im}^{1,1}\left[\begin{array}{c|c|c} (\omega,1)_{1,1} & (\omega,1)_{1,1} & \\ & & \phi;\varphi;z \\ (\mu,\xi)_{1,1} & (\mu,\xi)_{1,1} & \end{array}\right],$$

which will also diminishes to the work of Atash et al., in [7] and Atash et al., [6] if $\omega = 1$.
Rahman et al., [18]

$$B_{\wp,\Im}^{\mu,\xi}(\Omega,\mho) = \frac{\Gamma^2(\xi)}{\Gamma^2(\mu)}\,{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c} (\mu,1)_{1,1} & (\mu,1)_{1,1} & \\ & & \Omega,\mho \\ (\xi,1)_{1,1} & (\xi,1)_{1,1} & \end{array}\right]$$

$$F_{\wp,\Im}^{\mu,\xi}(\upsilon,\phi;\varphi;z)=\frac{\Gamma^2(\xi)}{\Gamma^2(\mu)}\,{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c}(\mu,1)_{1,1} & (\mu,1)_{1,1} & \\ & & \upsilon,\phi;\varphi;z\\ (\xi,1)_{1,1} & (\xi,1)_{1,1} & \end{array}\right],$$

and

$$\Phi_{\wp,\Im}^{\mu,\xi}(\phi;\varphi;z)=\frac{\Gamma^2(\xi)}{\Gamma^2(\mu)}\,{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c}(\mu,1)_{1,1} & (\mu,1)_{1,1} & \\ & & \phi;\varphi;z\\ (\xi,1)_{1,1} & (\xi,1)_{1,1} & \end{array}\right],$$

and of course all other properties like functional relations, summation formulas and applications to beta distribution, and fractional calculus will follow.

Special functions of mathematical physics such as the Mittag-Leffler, Bessel and Wright functions can be represented in the form of our new generalized beta function and their properties and applications can also be studied. More importantly, the new generalized special functions investigated in this work will have potential applications in various fields of science and engineering.

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