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SOME INEQUALITIES RELATING TO GENERALIZED RELATIVE ORDER (α, β) AND GENERALIZED RELATIVE TYPE (α, β) OF MEROMORPHIC FUNCTIONS WITH RESPECT TO ENTIRE FUNCTION

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Abstract: In this paper, we intend to find out some inequalities of generalized relative order (α, β), generalized relative type (α, β) and generalized relative weak type (α, β) of a meromorphic function f with respect to an entire function g when generalized relative order (γ, β), generalized relative type (γ, β) and generalized relative weak type (γ, β) of f and generalized relative order (γ, α), generalized relative type (γ, α) and generalized relative weak type (γ, α) of g with respect another entire function h are given, where α, β, γ are slowly increasing functions.

Keywords and Phrases: Entire function, meromorphic function, growth, generalized relative order (α, β), generalized relative type (α, β).

2020 Mathematics Subject Classification: 30D35, 30D30, 30D20.

1. Introduction, Definitions and Notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [7, 9, 12]. We also use the standard notations and definitions of the theory of entire functions which are available in [11] and therefore we do not explain those in details. Let f be an entire function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f (see [7, p.4]), playing the same role as $M_f(r)$.

However, the Nevanlinna's characteristic function of a meromorphic function f is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r,a)(\overline{N}_f(r,a))$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows:

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a)\log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t} dt + \overline{n}_f(0,a)\log r\right),$$

in addition we represent by $n_f(r,a)(\overline{n}_f(r,a))$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

$$\text{where} \quad \log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0.$$

Also we may employ $m(r, \frac{1}{f-a})$ by $m_f(r,a)$.

If f is entire, then the Nevanlinna's characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

Moreover, if f is non-constant entire then $T_f(r)$ is also strictly increasing and continuous functions of r. Therefore its inverse $T_f^{-1} : (T_f(0), \infty) \to (0,\infty)$ exists and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$.

Now let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1$, if $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1 + o(1))\alpha(\exp(x))$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$.

Clearly, $L_2 \subset L_1$, as both the classes L_1 and L_2 are the subset of the class L with $\log^{[2]}(x)$ belongs to the classes L_1 and L_2 but $\log x$ belongs to only L_1 class.

Considering the above, Sheremeta [10] introduced the concept of generalized order (α, β) of an entire function. For details about generalized order (α, β) one may see [10]. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α, β) in some different direction. For the purpose of further applications, Biswas et al. [5] rewrote the definition of the generalized order (α, β) of entire and meromorphic functions in the following way after giving a minor modification to the original definition (e.g. see, [10]) which considerably extend the definition of φ-order of entire and meromorphic function introduced by Chyzhykov et al. [6].

Definition 1. [5] *Let* $\alpha \in L_2$ *and* $\beta \in L_1$. *The generalized order* (α, β) *and generalized lower order* (α, β) *of a meromorphic function* f *denoted by* $\rho_{(\alpha,\beta)}[f]$ *and* $\lambda_{(\alpha,\beta)}[f]$ *respectively are defined as:*

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}.$$

If f *is an entire function, then*

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to\infty} \frac{\alpha(M_f(r))}{\beta(r)}.$$

where $\alpha,\ \beta \in L_1$.

Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.[7]\}$, for an entire function f, one may easily verify that

$$\begin{matrix} \rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(M_f(r))}{\beta(r)} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\exp(T_f(r)))}{\beta(r)},$$

when $\alpha \in L_2$ and $\beta \in L_1$.

The function f is said to be of regular generalized (α, β) growth when generalized order (α, β) and generalized lower order (α, β) of f are the same. Functions which are not of regular generalized (α, β) growth are said to be of irregular generalized (α, β) growth.

Mainly the growth investigation of entire and meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire or meromorphic function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2, 8]) will come. Now in order to make some progresses in the study of relative order, Biswas et al.[5] have introduced the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of a meromorphic function with respect to another entire function in the following way:

Definition 2. [5] *Let $\alpha,\ \beta \in L_1$. The generalized relative order (α, β) and generalized relative lower order (α, β) of a meromorphic function f with respect to an entire function g denoted by $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively are defined as:*

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}.$$

Further if generalized relative order (α, β) and the generalized relative lower order (α, β) of a meromorphic function f with respect to an entire function g are the same, then f is called a function of regular generalized relative (α, β) growth with respect to g. Otherwise, f is said to be irregular generalized relative (α, β) growth.with respect to g.

Now in order to refine the above growth scale, Biswas et al.[5] have introduced the definitions of other growth indicators, such as generalized relative type (α, β) and generalized relative lower type (α, β) of meromorphic function with respect to an entire function which are as follows:

Definition 3. [5] *Let $\alpha,\ \beta \in L_1$. The generalized relative type (α, β) denoted by $\sigma_{(\alpha,\beta)}[f]_g$ and generalized relative lower type_(α, β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ of a meromorphic function f with respect to an entire function g having non-zero finite generalized relative order (α, β) are defined as:*

$$\sigma_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}.$$

Analogously, to determine the relative growth of a meromorphic function f having same non zero finite generalized relative lower order (α, β) with respect to an entire function g, Biswas et al.[5] have introduced the definitions of generalized relative upper weak type (α, β) and generalized relative weak type (α, β) of f with respect to g in the following way:

Definition 4. [5] *Let $\alpha,\ \beta \in L_1$. The generalized relative upper weak type (α, β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$ and generalized relative weak type (α, β) denoted by $\tau_{(\alpha,\beta)}[f]_g$ of a meromorphic function f with respect to an entire function g having non-zero finite generalized relative lower order (α, β) are defined as:*

$$\overline{\tau}_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}} \text{ and } \tau_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}.$$

However the main aim of this paper is to investigate some growth properties of entire and meromorphic functions using generalized relative order (α, β) and generalized relative type (α, β), of a meromorphic function with respect to an entire function, which improves and extends some earlier results (see, e.g., [3,4]). Henceforth we assume that $\alpha,\ \beta, \gamma \in L_1$, all the growth indicators are non-zero finite.

2. Main Results

In this section, we present the main results of the paper.

Theorem 1. *Let f is a meromorphic function and g, h are entire functions such that $0 < \lambda_{(\gamma,\beta)}[f]_h \le \rho_{(\gamma,\beta)}[f]_h < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g]_h \le \rho_{(\gamma,\alpha)}[g]_h < \infty$. Then*

$$\frac{\lambda_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h} \le \lambda_{(\alpha,\beta)}[f]_g \le \min\left\{\frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}, \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}\right\}$$
$$\le \max\left\{\frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}, \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}\right\} \le \rho_{(\alpha,\beta)}[f]_g \le \frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}.$$

Proof. From the definitions of $\rho_{(\gamma,\beta)}[f]_h$ and $\lambda_{(\gamma,\beta)}[f]_h$ we have for all sufficiently large values of r that

$$T_f(r) \le T_h(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]_h + \varepsilon)\beta(r))), \tag{1}$$
$$T_f(r) \ge T_h(\gamma^{-1}((\lambda_{(\gamma,\beta)}[f]_h - \varepsilon)\beta(r))) \tag{2}$$

and also for a sequence of values of r tending to infinity we get that

$$T_f(r) \ge T_h(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]_h - \varepsilon)\beta(r))), \tag{3}$$
$$T_f(r) \le T_h(\gamma^{-1}((\lambda_{(\gamma,\beta)}[f]_h + \varepsilon)\beta(r))). \tag{4}$$

Further from the definitions of $\rho_{(\gamma,\alpha)}[g]_h$ and $\lambda_{(\gamma,\alpha)}[g]_h$ it follows for all sufficiently large values of r that

$$T_g(r) \le T_h(\gamma^{-1}((\rho_{(\gamma,\alpha)}[g]_h + \varepsilon)\alpha(r)))$$
$$i.e.,\ T_h(r) \ge T_g\Big(\alpha^{-1}\Big(\frac{\gamma(r)}{\rho_{(\gamma,\alpha)}[g]_h + \varepsilon}\Big)\Big) \tag{5}$$

$$\text{and } T_h(r) \leq T_g\Big(\alpha^{-1}\Big(\frac{\gamma(r)}{\lambda_{(\gamma,\alpha)}[g]_h - \varepsilon}\Big)\Big). \tag{6}$$

Also from the definitions of $\rho_{(\gamma,\alpha)}[g]_h$ and $\lambda_{(\gamma,\alpha)}[g]_h$, we get for a sequence of values of r tending to infinity we obtain that

$$T_h(r) \leq T_g\Big(\alpha^{-1}\Big(\frac{\gamma(r)}{(\rho_{(\gamma,\alpha)}[g]_h - \varepsilon)}\Big)\Big) \tag{7}$$

$$\text{and } T_h(r) \geq T_g\Big(\alpha^{-1}\Big(\frac{\gamma(r)}{\lambda_{(\gamma,\alpha)}[g]_h + \varepsilon}\Big)\Big). \tag{8}$$

Now from (3) and in view of (5), for a sequence of values of r tending to infinity we get that

$$\alpha(T_g^{-1}(T_f(r))) \geq \alpha(T_g^{-1}(T_h(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]_h - \varepsilon)\beta(r))))$$

$$\alpha(T_g^{-1}(T_f(r))) \geq \alpha\Big(T_g^{-1}\Big(T_g\Big(\alpha^{-1}\Big(\frac{(\rho_{(\gamma,\beta)}[f]_h - \varepsilon)\beta(r)}{\rho_{(\gamma,\alpha)}[g]_h + \varepsilon}\Big)\Big)\Big)\Big).$$

$$i.e.,\ \alpha(T_g^{-1}(T_f(r))) \geq \frac{(\rho_{(\gamma,\beta)}[f]_h - \varepsilon)\beta(r)}{(\rho_{(\gamma,\alpha)}[g]_h + \varepsilon)}$$

$$i.e.,\ \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)} \geq \frac{\rho_{(\gamma,\beta)}[f]_h - \varepsilon}{\rho_{(\gamma,\alpha)}[g]_h + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, it follows that

$$\rho_{(\alpha,\beta)}[f]_g \geq \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}. \tag{9}$$

Analogously from (2) and in view of (8) it follows that

$$\rho_{(\alpha,\beta)}[f]_g \geq \frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}. \tag{10}$$

Again from (2) and in view of (5) we obtain that

$$\lambda_{(\alpha,\beta)}[f]_g \geq \frac{\lambda_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}. \tag{11}$$

Now in view of (6) we have from (1) for all sufficiently large values of r that

$$\alpha(T_g^{-1}(T_f(r))) \leq \alpha(T_g^{-1}(T_h(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]_h + \varepsilon)\beta(r)))))$$

$$\alpha(T_g^{-1}(T_f(r))) \leq \alpha\Big(T_g^{-1}\Big(T_g\Big(\alpha^{-1}\Big(\frac{\rho_{(\gamma,\beta)}[f]_h+\varepsilon)\beta(r)}{\lambda_{(\gamma,\alpha)}[g]_h-\varepsilon}\Big)\Big)\Big)\Big)$$

$$i.e.,\ \alpha(T_g^{-1}(T_f(r))) \leq \frac{(\rho_{(\gamma,\beta)}[f]_h+\varepsilon)\beta(r)}{(\lambda_{(\gamma,\alpha)}[g]_h-\varepsilon)}$$

$$i.e.,\ \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)} \leq \frac{\rho_{(\gamma,\beta)}[f]_h+\varepsilon}{\lambda_{(\gamma,\alpha)}[g]_h-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\rho_{(\alpha,\beta)}[f]_g \leq \frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}. \tag{12}$$

Similarly in view of (7), we get from (1) that

$$\lambda_{(\alpha,\beta)}[f]_g \leq \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}. \tag{13}$$

Again from (4) and in view of (6) it follows that

$$\lambda_{(\alpha,\beta)}[f]_g \leq \frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}. \tag{14}$$

The theorem follows from (9), (10), (11), (12), (13) and (14).

Remark 1. *From the conclusion of the above result, one may write* $\rho_{(\alpha,\beta)}[f]_g = \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}$ *and* $\lambda_{(\alpha,\beta)}[f]_g = \frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}$ *when* $\lambda_{(\gamma,\alpha)}[g]_h = \rho_{(\gamma,\alpha)}[g]_h$. *Similarly* $\rho_{(\alpha,\beta)}[f]_g = \frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}$ *and* $\lambda_{(\alpha,\beta)}[f]_g = \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}$ *when* $\lambda_{(\gamma,\beta)}[f]_h = \rho_{(\gamma,\beta)}[f]_h$.

Theorem 2. *Let f is a meromorphic function and g, h are entire functions such that* $0 < \rho_{(\gamma,\beta)}[f]_h < \infty$ *and* $0 < \lambda_{(\gamma,\alpha)}[g]_h \leq \rho_{(\gamma,\alpha)}[g]_h < \infty$. *Then*

$$\max\Big\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\Big\} \leq \sigma_{(\alpha,\beta)}[f]_g \leq \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}.$$

Proof. Let us consider that $\varepsilon(>0)$ is arbitrary number. Now from the definitions of $\sigma_{(\gamma,\beta)}[f]_h$ and $\overline{\sigma}_{(\gamma,\beta)}[f]_h$, we have for all sufficiently large values of r that

$$T_f(r) \leq T_h(\gamma^{-1}(\log[(\sigma_{(\gamma,\beta)}[f]_h+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}])), \tag{15}$$

$$T_f(r) \geq T_h(\gamma^{-1}(\log[(\overline{\sigma}_{(\gamma,\beta)}[f]_h-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}])) \tag{16}$$

and also for a sequence of values of r tending to infinity, we get that

$$T_f(r) \geq T_h(\gamma^{-1}(\log[(\sigma_{(\gamma,\beta)}[f]_h - \varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}])), \quad (17)$$
$$T_f(r) \leq T_h(\gamma^{-1}(\log[(\overline{\sigma}_{(\gamma,\beta)}[f]_h + \varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}])). \quad (18)$$

Similarly from the definitions of $\sigma_{(\gamma,\alpha)}[g]_h$ and $\overline{\sigma}_{(\gamma,\alpha)}[g]_h$, it follows for all sufficiently large values of r that

$$T_g(r) \leq T_h(\gamma^{-1}(\log[(\sigma_{(\gamma,\alpha)}[g]_h + \varepsilon)(\exp(\alpha(r)))^{\rho_{(\gamma,\alpha)}[g]_h}]))$$
$$i.e., T_h(r) \geq T_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\sigma_{(\gamma,\alpha)}[g]_h + \varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\Big)\Big) \quad (19)$$

$$\text{and } T_h(r) \leq T_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\overline{\sigma}_{(\gamma,\alpha)}[g]_h - \varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\Big)\Big). \quad (20)$$

Also for a sequence of values of r tending to infinity, we obtain that

$$T_h(r) \leq T_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\sigma_{(\gamma,\alpha)}[g]_h - \varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\Big)\Big) \quad (21)$$

$$\text{and } T_h(r) \geq T_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\overline{\sigma}_{(\gamma,\alpha)}[g]_h + \varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\Big)\Big). \quad (22)$$

Further from the definitions of $\overline{\tau}_{(\gamma,\beta)}[f]_h$ and $\tau_{(\gamma,\beta)}[f]_h$, we have for all sufficiently large values of r that

$$T_f(r) \leq T_h(\gamma^{-1}(\log((\overline{\tau}_{(\gamma,\beta)}[f]_h + \varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]_h}))), \quad (23)$$
$$T_f(r) \geq T_h(\gamma^{-1}(\log((\tau_{(\gamma,\beta)}[f]_h - \varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]_h}))) \quad (24)$$

and also for a sequence of values of r tending to infinity, we get that

$$T_f(r) \geq T_h(\gamma^{-1}(\log((\overline{\tau}_{(\gamma,\beta)}[f]_h - \varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]_h}))), \quad (25)$$
$$T_f(r) \leq T_h(\gamma^{-1}(\log((\tau_{(\gamma,\beta)}[f]_h + \varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]_h}))). \quad (26)$$

Similarly from the definitions of $\overline{\tau}_{(\gamma,\alpha)}[g]_h$ and $\tau_{(\gamma,\alpha)}[g]_h$, it follows for all sufficiently large values of r that

$$T_h(r) \geq T_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\overline{\tau}_{(\gamma,\alpha)}[g]_h + \varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\Big)\Big) \quad (27)$$

$$\text{and } T_h(r) \leq T_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\tau_{(\gamma,\alpha)}[g]_h - \varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\Big)\Big). \quad (28)$$

Also for a sequence of values of r tending to infinity, we obtain that

$$T_h(r) \le T_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\overline{\tau}_{(\gamma,\alpha)}[g]_h-\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\Big)\Big) \tag{29}$$

$$\text{and}\ \ T_h(r) \ge T_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\tau_{(\gamma,\alpha)}[g]_h+\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\Big)\Big). \tag{30}$$

Now from (17) and in view of (27), we get for a sequence of values of r tending to infinity that

$$\exp(\alpha(T_g^{-1}(T_f(r)))) \ge \exp(\alpha(T_g^{-1}(T_h(\gamma^{-1}(\log((\sigma_{(\gamma,\beta)}[f]_h-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}))))))$$

$$\exp(\alpha(T_g^{-1}(T_f(r)))) \ge \Big(\frac{(\sigma_{(\gamma,\beta)}[f]_h-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}}{(\overline{\tau}_{(\gamma,\alpha)}[g]_h+\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}$$

$$i.e.,\ \ \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}}} \ge \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h-\varepsilon}{\overline{\tau}_{(\gamma,\alpha)}[g]_h+\varepsilon}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}.$$

Since in view of Theorem 1, $\frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h} \ge \rho_{(\alpha,\beta)}[f]_g$, and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}} \ge \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h-\varepsilon}{\overline{\tau}_{(\gamma,\alpha)}[g]_h+\varepsilon}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}$$

$$i.e.,\ \ \sigma_{(\alpha,\beta)}[f]_g \ge \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}. \tag{31}$$

Analogously from (16) and (30), we get that

$$\sigma_{(\alpha,\beta)}[f]_g \ge \Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \tag{32}$$

as in view of Theorem 1, it follows that $\frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h} \ge \rho_{(\alpha,\beta)}[f]_g$.

Again in view of (20), we have from (15) for all sufficiently large values of r that

$$\exp(\alpha(T_g^{-1}(T_f(r)))) \le \Big(\frac{(\sigma_{(\gamma,\beta)}[f]_h+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}}{(\overline{\sigma}_{(\gamma,\alpha)}[g]_h-\varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}}} \leq \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h + \varepsilon}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h - \varepsilon}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}.$$

Since in view of Theorem 1, it follows that $\frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h} \leq \rho_{(\alpha,\beta)}[f]_g$ and $\varepsilon(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}}} \leq \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h + \varepsilon}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h - \varepsilon}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \sigma_{(\alpha,\beta)}[f]_g \leq \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}. \tag{33}$$

Thus the theorem follows from (31), (32) and (33).

The conclusion of the following theorem can be carried out from (20) and (23); (23) and (28) respectively after applying the same technique of Theorem 2 and with the help of Theorem 1. Therefore its proof is omitted.

Theorem 3. *Let f is a meromorphic function and g, h are entire functions such that $0 < \lambda_{(\gamma,\beta)}[f]_h < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g]_h \leq \rho_{(\gamma,\alpha)}[g]_h < \infty$. Then*

$$\sigma_{(\alpha,\beta)}[f]_g \leq \min\Big\{\Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\Big\}.$$

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carry out the following theorem from pairwise inequalities numbers (24) and (27) ; (21) and (23); (20) and (26) respectively and therefore its proofs is omitted:

Theorem 4. *Let f is a meromorphic function and g, h are entire functions such that $0 < \lambda_{(\gamma,\beta)}[f]_h \leq \rho_{(\gamma,\beta)}[f]_h < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g]_h \leq \rho_{(\gamma,\alpha)}[g]_h < \infty$. Then*

$$\Big(\frac{\tau_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}} \leq \tau_{(\alpha,\beta)}[f]_g \leq \min\Big\{\Big(\frac{\tau_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\Big\}.$$

Theorem 5. *Let f is a meromorphic function and g, h are entire functions such that $0 < \rho_{(\gamma,\beta)}[f]_h < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g]_h \leq \rho_{(\gamma,\alpha)}[g]_h < \infty$. Then*

$$\tau_{(\alpha,\beta)}[f]_g \geq \max\Big\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\Big\}.$$

With the help of Theorem 1, the conclusion of the above theorem can be carried out from (16), (19) and (16), (27) respectively after applying the same technique of Theorem 2 and therefore its proof is omitted.

Theorem 6. *Let f is a meromorphic function and g, h are entire functions such that $0 < \rho_{(\gamma,\beta)}[f]_h < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g]_h \le \rho_{(\gamma,\alpha)}[g]_h < \infty$. Then*

$$\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}} \le \overline{\sigma}_{(\alpha,\beta)}[f]_g \le \min\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right\}.$$

Proof. From (16) and in view of (27), we get for all sufficiently large values of r that

$$\exp(\alpha(T_g^{-1}(T_f(r)))) \ge \exp(\alpha(T_g^{-1}(T_h(\gamma^{-1}(\log((\overline{\sigma}_{(\gamma,\beta)}[f]_h - \varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}))))))$$

$$i.e., \exp(\alpha(T_g^{-1}(T_f(r)))) \ge \left(\frac{(\overline{\sigma}_{(\gamma,\beta)}[f]_h - \varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}}{(\overline{\tau}_{(\gamma,\alpha)}[g]_h + \varepsilon)}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}}} \ge \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h - \varepsilon}{\overline{\tau}_{(\gamma,\alpha)}[g]_h + \varepsilon}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}.$$

Since in view of Theorem 1, $\frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h} \ge \rho_{(\alpha,\beta)}[f]_g$, and $\varepsilon(> 0)$ is arbitrary, we get from above that

$$\liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}} \ge \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h - \varepsilon}{\overline{\tau}_{(\gamma,\alpha)}[g]_h + \varepsilon}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \overline{\sigma}_{(\alpha,\beta)}[f]_g \ge \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}. \tag{34}$$

Further in view of (21), we get from (15) for a sequence of values of r tending to infinity that

$$\exp(\alpha(T_g^{-1}(T_f(r)))) \le \left(\frac{(\sigma_{(\gamma,\beta)}[f]_h + \varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}}{(\sigma_{(\gamma,\alpha)}[g]_h - \varepsilon)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}}} \le \left(\frac{\sigma_{(\gamma,\beta)}[f]_h + \varepsilon}{\sigma_{(\gamma,\alpha)}[g]_h - \varepsilon}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}.$$

Again as in view of Theorem 1, $\frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h} \le \rho_{(\alpha,\beta)}[f]_g$ and $\varepsilon(> 0)$ is arbitrary, therefore we get from above that

$$\liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}} \le \left(\frac{\sigma_{(\gamma,\beta)}[f]_h + \varepsilon}{\sigma_{(\gamma,\alpha)}[g]_h - \varepsilon}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \overline{\sigma}_{(\alpha,\beta)}[f]_g \leq \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}. \tag{35}$$

Similarly from (18) and (20), we get that

$$i.e., \quad \overline{\sigma}_{(\alpha,\beta)}[f]_g \leq \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \tag{36}$$

as in view of Theorem 1 it follows that $\frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h} \leq \rho_{(\alpha,\beta)}[f]_g$.

Thus the theorem follows from (34), (35) and (36).

Theorem 7. *Let f is a meromorphic function and g, h are entire functions such that $0 < \lambda_{(\gamma,\beta)}[f]_h < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g]_h \leq \rho_{(\gamma,\alpha)}[g]_h < \infty$. Then*

$$\overline{\sigma}_{(\alpha,\beta)}[f]_g \leq \min\left\{\left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}},\right.$$
$$\left.\left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right\}.$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (20) and (26); (21) and (23); (26) and (28); (23) and (29) respectively after applying the same technique of Theorem 6 and with the help of Theorem 1. Therefore its proof is omitted.

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carry out the following theorem from pairwise inequalities numbered (25) and (27); (24) and (30); (20) and (23) respectively and therefore its proof is omitted:

Theorem 8. *Let f is a meromorphic function and g, h are entire functions such that $0 < \lambda_{(\gamma,\beta)}[f]_h < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g]_h \leq \rho_{(\gamma,\alpha)}[g]_h < \infty$. Then*

$$\max\left\{\left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\right\} \leq \overline{\tau}_{(\alpha,\beta)}[f]_g \leq \left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}.$$

Theorem 9. *Let f is a meromorphic function and g, h are entire functions such that $0 < \lambda_{(\gamma,\beta)}[f]_h \leq \rho_{(\gamma,\beta)}[f]_h < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g]_h \leq \rho_{(\gamma,\alpha)}[g]_h < \infty$. Then*

$$\overline{\tau}_{(\alpha,\beta)}[f]_g \geq \max\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}},\right.$$
$$\left.\left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\right\}.$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (17) and (19); (16) and (22); (17) and (27); (16) and (30) respectively after applying the same technique of Theorem 6 and with the help of Theorem 1. Therefore its proof is omitted.

Theorem 10. *Let f is a meromorphic function and g,h are entire functions such that $0<\rho_{(\gamma,\beta)}[f]_h<\infty$ and $0<\rho_{(\gamma,\alpha)}[g]_h(=\lambda_{(\gamma,\alpha)}[g]_h)<\infty$. Then*

$$\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}} \leq \overline{\sigma}_{(\alpha,\beta)}[f]_g$$
$$\leq \min\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right\}$$
$$\leq \max\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right\}$$
$$\leq \sigma_{(\alpha,\beta)}[f]_g \leq \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}.$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (16) and (19); (18) and (20); (15) and (21); (16) and (22); (17) and (19); (15) and (20) respectively after applying the same technique of Theorem 6 and with the help of Theorem 1. Therefore its proof is omitted.

Remark 2. *In Theorem 10, if we replace the conditions "$0<\rho_{(\gamma,\beta)}[f]_h<\infty$ and $0<\rho_{(\gamma,\alpha)}[g]_h(=\lambda_{(\gamma,\alpha)}[g]_h)<\infty$" by "$0<\rho_{(\gamma,\beta)}[f]_h(=\lambda_{(\gamma,\beta)}[f]_h)<\infty$ and $0<\rho_{(\gamma,\alpha)}[g]_h<\infty$" respectively, then Theorem 10 remains valid with $\tau_{(\alpha,\beta)}[f]_g$ and $\overline{\tau}_{(\alpha,\beta)}[f]_g$ replaced by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ and $\sigma_{(\alpha,\beta)}[f]_g$ respectively.*

Theorem 11. *Let f is a meromorphic function and g,h are entire functions such that $0<\rho_{(\gamma,\beta)}[f]_h(=\lambda_{(\gamma,\beta)}[f]_h)<\infty$ and $0<\lambda_{(\gamma,\alpha)}[g]_h<\infty$. Then*

$$\left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}} \leq \overline{\sigma}_{(\alpha,\beta)}[f]_g$$
$$\leq \min\left\{\left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\right\}$$
$$\leq \max\left\{\left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\right\}$$
$$\leq \sigma_{(\alpha,\beta)}[f]_g \leq \left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}.$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (24) and (27); (26) and (28); (23) and (29); (24) and (30); (25) and (27); (23) and (28) respectively after applying the same technique of Theorem 6 and with the help of Theorem 1. Therefore its proof is omitted.

Remark 3. *In Theorem 11, if we replace the conditions "*$0 < \rho_{(\gamma,\beta)}[f]_h$ $(= \lambda_{(\gamma,\beta)}[f]_h) < \infty$ *and* $0 < \lambda_{(\gamma,\alpha)}[g]_h < \infty$*" by "*$0 < \lambda_{(\gamma,\beta)}[f]_h < \infty$ *and* $0 < \rho_{(\gamma,\alpha)}[g]_h (= \lambda_{(\gamma,\alpha)}[g]_h) < \infty$*" respectively, then Theorem 11 remains valid with* $\tau_{(\alpha,\beta)}[f]_g$ *and* $\overline{\tau}_{(\alpha,\beta)}[f]_g$ *replaced by* $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ *and* $\sigma_{(\alpha,\beta)}[f]_g$ *respectively.*

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