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# Transformation of Whittaker function of three variables

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**Abstract:** The main object of the present paper is to obtain transformation of Whittaker functions of three variables into Srivastava's triple hypergeometric series $F^{(3)}$. A number of known and new transformations for Kamp$e'$ de F$e'$riet function, Appell function $F_2$ are also obtained as special cases.

**Keywords:** Whittaker function, Modified Bessel function, and Laplace transform.
**2000 AMS Subject Classifications:** 33C15, 33C20, 33C65.

## 1. Introduction and Definition

Whittaker functions $M_{\lambda,\mu}(x)$ and $W_{\lambda,\mu}(x)$ are solutions of the Whittaker equation

$$W'' + \left(\frac{\frac{1}{4}-\mu^2}{x^2} + \frac{\lambda}{x} - \frac{1}{4}\right) w = 0, \tag{1.1}$$

the function $W_{\lambda,\mu}(x)$ satisfies the equation

$$W_{\lambda.\mu}(x) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\lambda-\mu)}\, M_{\lambda,\mu}(x) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}-\lambda+\mu)}\, M_{\lambda,-\mu}(x). \tag{1.2}$$

The pair of functions $M_{\lambda,\mu}(x)$, $M_{\lambda,-\mu}(x)$ and $W_{\lambda,\mu}(x)$ and $W_{\lambda,-\mu}(x)$ are linearly independent solutions of equation (1.1). Whittaker function $M_{\lambda,\mu}(x)$, was introduced by Whittaker [11] (see also Whittaker and Watson [12]) in terms of Confluent hypergeometric function ${}_1F_1$ (or Kummer's function).

$$M_{\lambda,\mu}(x) = x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}x}\ {}_1F_1\left(\mu-\lambda+\frac{1}{2}, 2\mu+1; x\right), \tag{1.3}$$

$$W_{\lambda,\mu}(x) = x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}x}\ \Psi\left(\mu-\lambda+\frac{1}{2}, 2\mu+1; x\right). \tag{1.4}$$

Further generalization of Whittaker function $M_{\lambda,\mu}(x)$ was introduced by Humbert [6; p. 63(15)] in the following form

$$M_{\lambda,\ \mu_1,......\mu_n}(x_1,.....x_n) = x_1^{\mu_1+\frac{1}{2}}.........x_n^{\mu_n+\frac{1}{2}} \exp\left[-\frac{1}{2}(x_1+......+x_n)\right]$$

$$.\Psi_2^n\left[\mu_1+.......+\mu_n-\lambda+\frac{n}{2};2\mu_1+1......2\mu_n+1;x_1.....x_n\right], \tag{1.5}$$

where $\Psi_2^{(n)}$ denotes Humbert's confluent hypergeometric function of $n$- variable, [6; p. 62(11)]

$$\Psi_2^{(n)}[a;b_1......,b_n;x_1.......,x_n] = \sum_{m_1....,m_n=0}^{\infty} \frac{(a)_{m_1+......+m_n}}{(b_1)_{m_1}.....(b_n)_{mn}} \frac{x_1^{m_1}.......x_n^{mn}}{m_1!........m_n!} \tag{1.6}$$

$$(\max\{|\ x_1\ |,\ldots,|\ x_n\ |\}<\infty).$$

Relation with other functions with the degenerate hypergeometric function:

$$M_{k,m}(z) \ = \ z^{m+1/2}e^{-z/2}\phi\left(m-k+\frac{1}{2};2m+1;z\right), \tag{1.7}$$

also with the modified Bessel functions and the MacDonald function

$$M_{0,m}(z) \ = \ 2^{2m}\Gamma(m+1)\sqrt{z}\ I_m\left(\frac{z}{2}\right). \tag{1.8}$$

$$W_{0,m}(z) \ = \ \sqrt{\frac{2}{\pi}}K_m\left(\frac{z}{2}\right). \tag{1.9}$$

**2. Integral Transformation**

Here we can establish an integral transformation of Whittaker functions in the following form

$$\int_0^\infty t^{\lambda-1}e^{-(z+p/2)t}W_{k,\mu}(pt)M_{k_1,m_1-\frac{1}{2}}(x_1t)M_{k_2,m_2-\frac{1}{2}}(x_2t)dt$$

$$= \frac{p^{\mu+\frac{1}{2}}x_1^{m_1}x_2^{m_2}\Gamma(a+\mu)\Gamma(a-\mu)}{\Gamma(a-k+\frac{1}{2})\ \delta^{a+\mu}}$$

$$F^{(3)}\left[\begin{array}{l} a+\mu :: a-\mu;\_\_\ ;\_\_\ ;m_1-k_1,m_2-k_2;\mu-k+\frac{1}{2}; \\ a-k+\frac{1}{2} :: \_\_\ ;\_\_\ ;\_\_\ ;2m_1,2m_2;\_\_; \end{array} \quad \frac{x_1}{\delta},\frac{x_2}{\delta},\frac{\delta-p}{\delta}\right]. \tag{2.1}$$

where $a=\lambda+m_1+m_2+\frac{1}{2},\ Re(a+\mu)>0,\ Re(z+p+\frac{x_1}{2}+\frac{x_2}{2})>0,\ \delta=(z+p+\frac{x_1}{2}+\frac{x_2}{2})$, and $F^{(3)}[x,y,z]$ is a general triple hypergeometric series of Srivastava's [9; p. 69(39)] defined as

$$F^{(3)}\left[\begin{array}{ll}(a)::(b);(b');(b'');(e);(e');\ (e'')\ ; & \\ & x,y,z \\ (d)::(g);(g');(g'');(h);(h');(h'')\ ; & \end{array}\right]$$

$$=\sum_{m,n,p=0}^{\infty}\frac{[(a)]_{m+n+p}[(b)]_{m+n}[(b')]_{n+p}[(b'')]_{p+m}[(e)]_m[(e')]_n[(e'')]_p}{[(d)]_{m+n+p}[(g)]_{m+n}[(g')]_{n+p}[(g'')]_{p+m}[(h)]_m[(h')]_n[(h'')]_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \quad (2.2)$$

where (a) is the sequence of A parameters $a_1$,$a_2$,.....$a_A$ and

$$[(a)]_n=\prod_{j=1}^{A}(a_j)_m=\frac{\Gamma(a_j+n)}{\Gamma(a_j)}. \quad (2.3)$$

Equation (2.1) can be obtain by expanding $M_{k,m}(x)$ in series and using integral transform [3; p.216(16)], then we arrive at the main integral (2.1).

**3. Special cases**

Some special cases of the main result (2.1) are given below:

**1.** On setting $k=\mu+\frac{1}{2}$ in (2.1), and using a relation [3; p. 432]

$$\int_0^{\infty} t^{\lambda+\mu+\frac{1}{2}-1}e^{-(z+p)t}\ M_{k_1,m_1-\frac{1}{2}}(x_1t)\ M_{k_2,m_2-\frac{1}{2}}\ (x_2t)dt$$

$$=\frac{p^{\mu+\frac{1}{2}}\ x_1^{m_1}x_2^{m_2}\Gamma(a+\mu)}{\delta^{a+\mu}}$$

$$F_{0:1:1}^{1:1:1}\left[\begin{array}{ll}a+\mu::m_1-k_1,m_2-k_2; & \\ & \frac{x_1}{\delta},\frac{x_2}{\delta} \\ \_\_::2m_1,2m_2; & \end{array}\right] \quad (3.1)$$

where $F_{E:G:H}^{A:B:D}$ is Kampe$'$ de Fe$'$riet function defined by [10; p.63(16)]

$$F_{E:G;H}^{A:B;D}\left[\begin{array}{ll}(a_A):(b_B);(d_D)\ ; & \\ & x,y \\ (e_E):(g_G);(h_H)\ ; & \end{array}\right]=\sum_{m,n=0}^{\infty}\frac{[(a_A)]_{m+n}[(b_B)]_m[(d_D)]_n}{[(e_E)]_{m+n}[(g_G)]_m[(h_H)]_n}\frac{x^m}{m!}\frac{y^n}{n!} \quad (3.2)$$

In view of a known result of Srivastava and Karlsson [8; p. 270 (2)]

$$F_{0:1:1}^{1:1:1}=F_2, \quad (3.3)$$

therefore equation (3.1) becomes

$$\int_0^\infty t^{\lambda+\mu+\frac{1}{2}-1} e^{-(z+p)t}\ M_{k_1,m_1-\frac{1}{2}}(x_1t)\ M_{k_2,m_2-\frac{1}{2}}\ (x_2t)dt$$

$$= \frac{p^{\mu+\frac{1}{2}}\ x_1^{m_1} x_2^{m_2} \Gamma(a+\mu)}{\delta^{a+\mu}}$$

$$F_2\left[[a+\mu, m_1-k_1, m_2-k_2; 2m_1, 2m_2; \frac{x_1}{\delta}, \frac{x_2}{\delta}]\right]. \tag{3.4}$$

where $F_2$ is Appell's function defined as

$$F_2[a,b,b';c,c';x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} b_m (b')_n}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{3.5}$$

$$|\ x\ | + |\ y\ | < 1$$

**2.** On setting $k_1 = k_2 = 0$ in (2.1), we get

$$\int_0^\infty t^{\lambda-1} e^{-(z+\frac{p}{2})t}\ W_{k,\mu}(pt)\ M_{0,m_1-\frac{1}{2}}\ (x_1t)\ M_{0,m_2-\frac{1}{2}}\ (x_2t)\ dt$$

$$= \frac{p^{\mu+\frac{1}{2}}\ x_1^{m_1} x_2^{m_2}\ \Gamma(a+\mu)\ \Gamma(a-\mu)}{\Gamma(a-k+\frac{1}{2})\ \delta^{a+\mu}}$$

$$F^{(3)}\left[\begin{array}{l} a+\mu, a-\mu : \_\_; \_\_;\ m_1, m_2;\ \mu-k+\frac{1}{2}; \\ a-k+\frac{1}{2} :: \_\_; \_\_; \_\_;\ 2m_1, 2m_2; \_\_; \end{array}\right.$$

$$\left. \frac{x_1}{\delta}, \frac{x_2}{\delta}, \frac{\delta-p}{\delta} \right]. \tag{3.6}$$

**3.** On setting $k_1 = -m_1$ in (2.1), we get

$$\int_0^\infty t^{\lambda+m_1-1}\ e^{-(z+\frac{p}{2}-\frac{x_1}{2})t}\ W_{k,\mu}(pt) M_{k_2,m_2-\frac{1}{2}}\ (x_2t)\ dt$$

$$= \frac{p^{\mu+\frac{1}{2}} x_2^{m_2} \Gamma(a+\mu)\ \Gamma(a-\mu)}{\Gamma(a-k+\frac{1}{2})\ \sigma^{a+\mu}}$$

$$F^{1:2:1}_{1:1:0}\left[\begin{array}{l} a+\mu :: a-\mu;\ m_2-k_2; \mu-k+\frac{1}{2}\ ; \\ \\ a-k+\frac{1}{2} ::\ 2m_2\ ; \_\_; \end{array} \begin{array}{c} \frac{x_2}{\sigma}, \frac{\sigma-p}{\sigma} \end{array}\right]. \tag{3.7}$$

where $a = \lambda + m_1 + m_2 + \frac{1}{2}$, $Re(a+\mu) > 0$, $Re(z+p-\frac{x_1}{2}+\frac{x_2}{2}) > 0$, $\sigma = (z+p-\frac{x_1}{2}+\frac{x_2}{2})$.

**4.** On setting $k_1 = k_2 = 0, m_1 = \mu+\frac{1}{2}, m_2 = \nu+\frac{1}{2}$ and $x_1 = 2\alpha, x_2 = 2\beta$ in (2.1) and using a relation [3 p. 432], the integral reduces to

$$\int_0^\infty t^\lambda e^{-(z+\frac{p}{2})t}\ W_{k,\mu}(pt)\ I_\mu(\alpha t) I_\nu(\beta t) dt$$

$$= \frac{p^{\mu+\frac{1}{2}}\ \alpha^\mu \beta^\nu \Gamma(\gamma+2\mu+\nu+\frac{3}{2})\ \Gamma(\gamma+\nu+\frac{3}{2})}{2^{\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)\ \Gamma(\gamma+\mu+\nu+2-k)\ \delta^{\lambda+2\mu+\nu+\frac{3}{2}}}$$

$$F^{(3)}\left[\begin{array}{l} \gamma+2\mu+\nu+\frac{3}{2} :: \gamma+\nu+\frac{3}{2}; \_\_; \_\_; \mu+\frac{1}{2}, \nu+\frac{1}{2}; \mu-k+\frac{1}{2}; \\ \\ \gamma+\mu+\nu+2-k :: \_\_; \_\_; \_\_; 2\mu+1; 2\nu+1; \end{array} \begin{array}{c} \frac{2\alpha}{\delta}, \frac{2\beta}{\delta}, \frac{\delta-p}{\delta} \end{array}\right]. \tag{3.8}$$

**5.** On setting $k_1 = 0, m_1 = -m$ and $k_2 = 1, m_2 = \frac{1}{2}$ in (2.1), and using the relations [4; p. 579, 580(18),(2)], then the integral reduces to

$$\int_0^\infty t^{\lambda+1-1} e^{-(z+\frac{p}{2})t} W_{k,\mu}(pt)[\frac{m!}{(2m)!\sqrt{\pi}} K_{m+\frac{1}{2}}(\frac{x_1 t}{2})].[(1-x_2 t) I_0(\frac{x_2 t}{2}) + x_2 t I_1(\frac{x_2 t}{2})] dt$$

$$= \frac{p^{\mu+\frac{1}{2}} x_1^{(-m-\frac{1}{2})}\ \Gamma(a+\mu)\Gamma(a-\mu)}{\Gamma(a-k+\frac{1}{2})\ \delta^{a+\mu}}$$

$$F^{(3)}\left[\begin{array}{l} a+\mu :: a-\mu\ ; \_\_\ ; \_\_\ ; -m, \frac{-1}{2};\ \mu-k+\frac{1}{2}; \\ \\ a-k+\frac{1}{2} :: \_\_\ ; \_\_\ ; \_\_\ ; -2m, 1;\ \_\_; \end{array} \begin{array}{c} \frac{x_1}{\delta}, \frac{x_2}{\delta}, \frac{\delta-p}{\delta} \end{array}\right]. \tag{3.9}$$

where $a = \lambda - m + 1$, $Re(a+\mu) > 0, Re(z+p+\frac{x_1}{2}+\frac{x_2}{2})$, $\delta = (z+p+\frac{x_1}{2}+\frac{x_2}{2})$.

## 4. Conclusion:

In the present investigation, we have established an integral transformation of Whittaker function of three variables into Srivastava's triple hypergeometric series $F^{(3)}$. A number of known and new transformations for Kamp$e'$ de F$e'$riet function, Appell function $F_2$ are also obtained as special cases. The results appears in this paper may be found useful in some aria of mathematical physics and engineering.

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