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A NEW CLASS OF TAN-G FAMILY OF DISTRIBUTIONS WITH PROPERTIES AND APPLICATIONS

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Abstract: This research article is dedicated to the exploration of a novel family of distributions that is based on a tangent transformation. The new class of distributions, which we have named the new class tan-G (NCT-G) family, has been developed using the ratio of the cumulative distribution function (CDF) G(x) and 1+G(x) of a baseline distribution. We provide an overview of the general properties of this family of distributions. To demonstrate the applicability of the NCT-G family, we have utilized the inverse Weibull distribution as a baseline and introduced a new member of the suggested family. This new distribution exhibits a reverse-J, increasing, or inverted bathtub-shaped hazard function. We have also explored some statistical properties of this distribution, as well as its associated parameters estimated through maximum likelihood estimation (MLE). To evaluate the accuracy of the estimation procedure, we have conducted a Monte Carlo simulation. We found that even for small sample sizes, biases and mean square errors decrease as the sample size increases. Additionally, we have applied the NCT-IW distribution to two real data sets. By using model selection criteria and goodness of fit test statistics, we empirically proved that our suggested model outperforms other existing models, most of which have a greater number of parameters.

Keywords and Phrases: Inverse Weibull distribution, Tan-G class, Mean residual life, Lorenz curve, Entropy.

2020 Mathematics Subject Classification: 62F15, 65C05.

1. Introduction

The study of real-world events often involves the use of statistical distributions, with both the practical application and theoretical aspects of these distributions being extensively explored. Over time, a variety of distribution families have been developed to account for a range of real-world phenomena and this development of distribution theory is an ongoing process. While many probability distributions outlined in the literature contain numerous parameters, thereby enhancing the model's adaptability, some experts argue that obtaining accurate parameter estimates using numerical resources can be difficult (Grimmett & Stirzaker, 2020). To better model actual data, it may be more effective to create models with a limited number of model parameters that offer a noticeable degree of flexibility. To this end, a team of scientists has turned to trigonometric functions in the pursuit of novel distributions. Trigonometric models have become increasingly popular among scholars in recent years because of their versatility and the ability to be mathematically understood. Al-Mofleh (2018) introduced a family of distributions with the tangent function and discussed its four members. Another group of distributions, which also utilize the tangent function, has been defined by (Nanga et al., 2022). Souza et al. (2021) proposed a novel class of trigonometric distribution with an increasing failure rate or bathtub-shaped function, known as the Tan-G Class of distribution. This class of distribution is considered to be one of the various trigonometric G-families and has base parameters ($\omega > 0$). The PDF and CDF for the Tan-G Class of distribution are

$$F(x;\omega) = \int_0^{\frac{\pi}{4}K(x;\omega)} sec^2(t)dt = \tan\left[\frac{\pi}{4}K(x;\omega)\right]; x \in \Re. \tag{1.1}$$

$$f(x;\omega) = \frac{\pi}{4}k(x;\omega)sec^2\left[\frac{\pi}{4}K(x;\omega)\right]; x \in \Re. \tag{1.2}$$

where $K(x;\omega)$ is CDF of any base distribution. Souza et al. (2019) employed a comparable approach to propose the Sin-G family of distributions and identified the Sin-Inverse Weibull distribution as a member of the Sin-G class. Additionally, they introduced a novel Cos-G class with an increasing failure rate function or a bathtub-shaped failure rate function and specifically examined the Cos-W distribution as a member. Both the Sin-G and Cos-G classes of distributions CDFs are

$$F(x;\omega) = \int_0^{\frac{\pi}{2}K(x;\omega)} \cos(t)dt = \sin\left[\frac{\pi}{2}K(x;\omega)\right]; x \in \Re.$$

$$F(x;\omega) = -\int_{0}^{\frac{\pi}{2}K(x;\omega)} sin(t)dt = 1 - \cos\left[\frac{\pi}{2}K(x;\omega)\right]; x \in \Re.$$

Another new sin-G family was created by (Mahmood et al., 2019), who also studied the sin-inverse Weibull model in specific. The PDF and CDF of the novel sin-G family of distributions are

$$F(x;\omega) = \int_{0}^{\frac{\pi}{4}K(x;\omega)(K(x;\omega)+1)} cos(t)dt = \sin\left[\frac{\pi}{4}K(x;\omega)(K(x;\omega)+1)\right]; x \in \Re.$$

$$f(x;\omega) = \frac{\pi}{4}k(x;\omega)[2K(x;\omega)+1]cos\left[\frac{\pi}{4}K(x;\omega)(K(x;\omega)+1)\right]; x \in \Re.$$

In addition, the sine Kumaraswamy-G family has been defined by (Chesneau & Jamal, 2020), which introduces two extra parameters to the family. Another family of distributions related to the trigonometric function is the exponentiated sine-G family, which has been introduced by (Muhammad et al., 2021). They have also analyzed a specific model known as the exponentiated sine-Weibull distribution. Similar works also can be found in (Kyurkchiev et al., 2021a) and (Kyurkchiev et al., 2021b) Chaudhary et al. (2021) introduced another probability model related to trigonometric functions, called the Arctan generalized exponential distribution. Various modifications of this class of distributions have been proposed and studied by a number of researchers. In Kyurkchiev et al. (2021c), some general classes of trigonometric CDF and 'saturation' in the Hausdorff sense for some special cases of the families are studied. Isa et al. (2022) have utilized the sine-G family of distribution to develop a new two-parameter model, known as the sine Burr XII distribution. Through our observations, we have discovered that simple functions possess a trigonometric distribution and can be formalized with ease, as detailed in (see Souza et al., 2019a). Moreover, we found that the sine transformation can significantly enhance the flexibility of G(x) without requiring additional parameters, as emphasized by (Chesneau and Jamal, 2020). These desirable qualities led us to explore the tangent metamorphosis family. In our study, we introduced a novel family of trigonometric models utilizing the Tangent function, which we have dubbed the NCT-G family of distributions. To provide a clear and concise overview of our research, the remaining parts of this paper are ordered as follows: Section 2 introduces the methodology of model development and key functions of the distribution family. In Section 3, we present some general properties of the NCT-G family and its estimation procedure is presented in Section 4. Moving forward to Section 5, we introduce a specific member of the NCT-G family, providing a thorough study and application of this model. Lastly, we conclude our findings in Section 6.

2. The NCT-G family of Distribution (NCT-G FD)

In this study, the NCT-G FD is proposed using the T-X approach introduced by (Alzaatreh et al., 2013). To do so, let a baseline CDF $G(x;\xi)$, where $\xi > 0$ is a vector of related parameters, then the ratio between $G(x;\xi)$ and $1+G(x;\xi)$ can be considered as a distribution function of the new family of distributions, for further information (see Marshal and Olkin, 2007). Here $\frac{G(x;\xi)}{1+G(x;\xi)} \to 0$ as $G(x;\xi) \to 0$ and $\frac{G(x;\xi)}{1+G(x;\xi)} \to \frac{1}{2}$ as $G(x;\xi) \to 1$, hence the CDF $F(x;\xi)$ of the NCT-G family of distributions can be defined as

$$F(x,\xi) = \tan\left[\frac{\pi}{2}\frac{G(x,\xi)}{1+G(x,\xi)}\right]; x \in \Re, \xi > 0. \tag{2.1}$$

Differentiating the CDF defined in Equation (2.1), the PDF is $f(x;\xi)$ of NCT-G FD is presented as

$$f(x,\xi) = \frac{\pi}{2}\sec^2\left[\frac{\pi}{2}\frac{G(x,\xi)}{1+G(x,\xi)}\right]\frac{g(x,\xi)}{(1+G(x,\xi))^2}; x \in \Re. \tag{2.2}$$

Reliability function

The Reliability function of NCT-G FD is given as

$$R(x,\xi) = 1 - \tan\left[\frac{\pi}{2}\frac{G(x,\xi)}{1+G(x,\xi)}\right]; x \in \Re.$$

Hazard Function

The Hazard function of NCT-G FD is given as

$$H(x,\xi) = \frac{\pi}{2}\sec^2\left[\frac{\pi}{2}\frac{G(x,\xi)}{1+G(x,\xi)}\right]\frac{g(x,\xi)}{(1+G(x,\xi))^2}\left[1 - \tan\left(\frac{\pi}{2}\frac{G(x,\xi)}{1+G(x,\xi)}\right)\right]^{-1}; x \in \Re$$

The Quantile Function (QF)

The QF is useful in statistical analysis and modeling, as it provides a way to estimate percentiles and other summary statistics of a probability distribution. Suppose $Q(p)$ is the smallest value of X for which the probability that X is less than or equal to that value is at least p. The QF $Q(p;\xi)$ of CDF $F(x;\xi)$ of NCT-G FD can be obtained as

$$Q(p;\xi) = G^{-1}\left[\frac{2\tan^{-1}p}{\pi - 2\tan^{-1}p}\right]; p \in (0,1). \tag{2.3}$$

The random deviate function of NCT-G FD can be generated by

$$x = G^{-1}\left[\frac{2\tan^{-1}u}{\pi - 2\tan^{-1}u}\right]; u \in (0,1)$$

Using Equation (2.3) we can calculate the median, upper and lower quartile, quartile deviation (QD), coefficient of QD, skewness, and kurtosis are presented in Table 1.

Table 1: Various measures based on quantiles of NCT-G FD.

Median	$G^{-1}\left[\frac{2\tan^{-1}(0.5)}{\pi-2\tan^{-1}(0.5)}\right]$
Lower Quartile	$G^{-1}\left[\frac{2\tan^{-1}(0.25)}{\pi-2\tan^{-1}(0.25)}\right]$
Upper Quartile	$G^{-1}\left[\frac{2\tan^{-1}(0.75)}{\pi-2\tan^{-1}(0.75)}\right]$
QD	$\frac{1}{2}\left[G^{-1}\left(\frac{2\tan^{-1}(0.75)}{\pi-2\tan^{-1}(0.75)}\right) - G^{-1}\left(\frac{2\tan^{-1}(0.25)}{\pi-2\tan^{-1}(0.25)}\right)\right]$
Coefficient of QD	$\frac{\left[G^{-1}\left(\frac{2\tan^{-1}(0.75)}{\pi-2\tan^{-1}(0.75)}\right)-G^{-1}\left(\frac{2\tan^{-1}(0.25)}{\pi-2\tan^{-1}(0.25)}\right)\right]}{\left[G^{-1}\left(\frac{2\tan^{-1}(0.75)}{\pi-2\tan^{-1}(0.75)}\right)+G^{-1}\left(\frac{2\tan^{-1}(0.25)}{\pi-2\tan^{-1}(0.25)}\right)\right]}$
Skewness (Kenney and Keeping 1962)	$\frac{Q\left(\frac{3}{4};\xi\right)-2Q\left(\frac{1}{2};\xi\right)+Q\left(\frac{1}{4};\xi\right)}{Q\left(\frac{3}{4};\xi\right)-Q\left(\frac{1}{4};\xi\right)}$
Kurtosis (Moors, 1988)	$\frac{Q\left(\frac{7}{8};\xi\right)-Q\left(\frac{5}{8};\xi\right)-Q\left(\frac{1}{8};\xi\right)+Q\left(\frac{3}{8};\xi\right)}{Q\left(\frac{3}{4};\xi\right)-Q\left(\frac{1}{4};\xi\right)}$

3. General Properties of NCT-G FD

3.1. Linear form of NCT-G family of Distribution

One can derive useful linear expansions using exponentiated distributions. Specifically, the exponentiated-G (Exp-G) distribution with power parameter $z > 0$ has a CDF and PDF as

$$G_z(x;\varphi) = [G(x;\varphi)]^z; x \in \Re. \tag{3.1}$$

$$g_z(x;\varphi) = zg(x;\varphi)\left[G(x;\varphi)\right]^{(z-1)}, x \in \Re. \tag{3.2}$$

Exponentiated distributions have well-known properties for a wide range of baseline CDF $G(x;\varphi)$, for more information (see Nadarajah and Gupta, 2007; Lemonte et al., 2013). We can express the CDF of the NCT-G family of distributions as a linear form using the following series expansions

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!}x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots; -\infty < x < \infty.$$

where B_{2n} is so-called $2k^{th}$ Bernoulli number.

$$(1+x)^b = \sum_{n=0}^{\infty} \binom{b}{n} x^n = 1 + \frac{b}{1!}x + \frac{b(b-1)}{2!}x^2 + \frac{b(b-1)(b-2)}{3!}x^3 + \cdots; |x| < 1.$$

The NCT-G family of distribution is

$$F(x,\xi) = \sum_{i=1}^{\infty} \left(\frac{\pi}{2}\right)^{2i-1} \frac{B_{2i}(-4)^i(1-4^i)}{(2i)!} \left(G(x,\xi)\right)^{2i-1} \left(1+G(x,\xi)\right)^{2i-1}. \tag{3.3}$$

Further expanding Equation (3.3) using generalized binomial series expansion, the expression for $F(x;\xi)$ becomes

$$F(x,\xi) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \left(\frac{\pi}{2}\right)^{2i-1} \frac{B_{2i}(-4)^i(1-4^i)}{(2i)!} \binom{2i-1}{j} \left(G(x,\xi)\right)^{2i+j-1}. \tag{3.4}$$

Using Equation (3.4) the PDF of the NCT-G family of distribution can be presented in the form of

$$f(x,\xi) = g(x,\xi) \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \Omega_{ij} \left(G(x,\xi)\right)^{2i+j-2}. \tag{3.5}$$

where $\Omega_{ij} = \left(\frac{\pi}{2}\right)^{2i-1} \frac{B_{2i}(-4)^i(1-4^i)(2i+j-1)}{(2i)!} \binom{2i-1}{j}$.

3.2. Moments

The r^{th} order moment (μ_r') for the NCT-G family of distribution is

$$\begin{aligned}\mu_r' &= \int_{-\infty}^{\infty} x^r f(x)dx \\ &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \Omega_{ij} \int_{-\infty}^{\infty} x^r g(x,\xi) \left(G(x,\xi)\right)^{2i+j-2} dx.\end{aligned}$$

Further moments can also be calculated using quantile function for more detail (see Balakrishnan and Cohen, 1991) as Let $G(x;\xi) = p \Rightarrow g(x;\xi)dx = dp; 0 \leqslant p \leqslant 1$

$$\mu_r' = E(X^r) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \Omega_{ij} \int_0^1 p^{2i+j-2} Q_G^r(p)dp, 0 < p < 1.$$

where $G(x;\xi) = p$ and $Q_G(p)$ is the QF.

3.3. Moment Generating Function (MGF)

The MGF ($M_X(t)$) for the NCT-G FD is

$$\begin{aligned} M_X(t) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu_r^{'} \\ &= \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{ij} \frac{t^k}{k!} \int_{-\infty}^{\infty} x^k g(x,\xi) \left(G(x,\xi)\right)^{2i+j-2} dx. \end{aligned}$$

Also using the QF we can calculate the MGF as, Let $G(x;\xi) = p \Rightarrow g(x;\xi)dx = dp; 0 \leqslant p \leqslant 1$

$$M_X(t) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{ij} \frac{t^k}{k!} \int_{0}^{1} p^{2i+j-2} Q_G^k(p) dp, \quad 0 < p < 1.$$

where $G(x;\xi) = p$ and $Q_G(p)$ is the QF of the baseline distribution.

3.4. Incomplete Moments

The Incomplete moment of the random variable X is defined as $M_r(y) = \int_0^y x^r f(x) dx$. Therefore incomplete moment for NCT-G FD is given by

$$M_r(y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \int_{-\infty}^{y} \Omega_{ij} x^r g(x;\xi) \left(G(x,\xi)\right)^{2i+j-2} dx.$$

Alternately, $M_r(y)$ may be presented in terms of QF as

$$M_r(y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Omega_{ij} \int_{0}^{G(y)} p^{2i+j-2} Q_G^r(p) dp; 0 < p < 1.$$

3.5. Mean Residual Life

The Mean residual life of the random variable T is defined as

$$\bar{M}(x) = \frac{1}{F(x)} \left[\mu - \int_{-\infty}^{x} t f(t) dt \right] - x.$$

Therefore Mean residual life for NCT-G FD is given by

$$\bar{M}(x) = \frac{1}{F(x)} \left[\mu - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Omega_{ij} \int_{-\infty}^{x} t g(t;\xi) \left\{ G(t;\xi) \right\}^{2i+j-2} dt \right] - x.$$

Alternately, $\bar{M}(x)$ may be expressed in term of quantile function as

$$\bar{M}(x) = \frac{1}{F(x)} \left[\mu - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Omega_{ij} \int_{0}^{G(y)} p^{2i+j-2} Q_G(p) dp \right] - x,$$

where $\Omega_{ij} = \left(\frac{\pi}{2}\right)^{2i-1} \frac{B_{2i}(-4)^i(1-4^i)(2i+j-1)}{(2i)!} \binom{2i-1}{j}$.

3.6. Inequality Measure

Lorenz and Bonferroni curves are utilized in the analysis of inequality measures such as income and poverty across various disciplines, including demography, social science, etc.

- Lorenz Curve: Lorenz curve is defined as $L_{F(y)} = \frac{1}{\mu} \int_{-\infty}^{y} x f(x) dx$, hence Lorenz curve for NCT-G family of distribution is given by

$$L_{F(y)} = \frac{1}{\mu} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \Omega_{ij} \int_{-\infty}^{y} x g(x,\xi) \left(G(x,\xi) \right)^{2i+j-2} dx. \tag{3.6}$$

  Alternatively, it can be expressed in terms of QF as

$$L_{F(y)} = \frac{1}{\mu} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \Omega_{ij} \int_{-\infty}^{G(y)} p^{2i+j-2} Q_G(p) dp.$$

- Bonferroni Curve: The Bonferroni curve is another measure of inequality given by $B_{F(y)} = \frac{L_{F(y)}}{F(y)}$. From Equation (3.6), the Bonferroni curve for the NCT-G family of distribution is obtained as

$$B_{F(y)} = \frac{1}{\mu F(y)} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \Omega_{ij} \int_{-\infty}^{y} x g(x;\xi) \left(G(x,\xi) \right)^{2i+j-2} dx.$$

3.7. Entropy

Entropy is a metric used to quantify the level of variation or uncertainty present in a random variable, and its utility extends across a broad range of fields including engineering, econometrics, probability theory, and social and medical sciences.

i) Rényi's Entropy

It was introduced by (Rényi, 1961), and is one such measure of entropy that can be employed to estimate the extent of variability in uncertainty. This quantity can be computed using a formula that captures the degree of uncertainty and variation in the system under consideration. Additionally, entropy has been utilized in other domains, such as financial mathematics, to assess the degree of uncertainty in market variables.

$$R_\rho(X) = \frac{1}{1-\rho} \log \int_{-\infty}^{\infty} \{f(x)\}^\rho \, dx,$$

where $\rho > 0$ and $\rho \neq 1$. Applying Taylor's series expansion $[f(x,\xi)]^\rho$ can be presented in the form

$$[f(x,\xi)]^\rho = \left(\frac{\pi}{2}\right)^\rho (g(x;\xi))^\rho \sec^{2\rho}\left[\frac{\pi}{2}\frac{G(x,\xi)}{1+G(x,\xi)}\right](1+G(x,\xi))^{-2\rho}\,; x \in \Re. \quad (3.7)$$

Let us suppose the function $W(s) = \sec^{2\rho}\left[\frac{\pi}{2}s\right], s \in \left(0, \frac{1}{2}\right)$. By applying the Taylor series to $W(s)$ at a fixed points $s_0 \in \left(0, \frac{1}{2}\right)$ (say $s_0 = 0.25$), we get

$$\sec^{2\rho}\left[\frac{\pi}{2}s\right] = \sum_{k=0}^{\infty} a_k (s-s_0)^k = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r} a_k s^r (-1)^{k-r} s_0^{k-r}, \quad (3.8)$$

where $a_k = \frac{W^{(k)}(s)|_{s=s_0}}{k!}$. Using Equation (3.8) in Equation (3.7) the expression for $[f(x;\xi)]^\rho$ becomes

$$[f(x,\xi)]^\rho = \left(\frac{\pi}{2}\right)^\rho (g(x;\xi))^\rho \sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r} a_k (-1)^{k-r} s_0^{k-r} (G(x,\xi))^r (1+G(x,\xi))^{-(2\rho+r)}. \quad (3.9)$$

Further expanding Equation (3.9) using generalized binomial series expansion. The expression for $[f(x;\xi)]^\rho$ becomes

$$[f(x,\xi)]^\rho = \left(\frac{\pi}{2}\right)^\rho \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty} a_k (-1)^{m+k-r} s_0^{k-r}\binom{k}{r}\binom{(2\rho+r)+m-1}{m}(G(x,\xi))^{r+m}(g(x;\xi))^\rho. \quad (3.10)$$

Substituting Equation (3.10) in $R_\rho(X)$, the Rényi's entropy for NCT-G family of distribution is given by

$$R_\rho(X) = \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}\Psi_{ij}\int_{-\infty}^{\infty}(g(x;\xi))^{\rho}(G(x,\xi))^{r+m}\,dx\right]. \tag{3.11}$$

where $\Psi_{kmr} = a_k\,(-1)^{m+k-r}\,s_0^{k-r}\left(\frac{\pi}{2}\right)^{\rho}\binom{k}{r}\binom{(2\rho+r)+m-1}{m}$.

ii) q-Entropy

The q-entropy is given by

$$H(\rho) = \frac{1}{1-\rho}\log\left[1-\int_{-\infty}^{\infty}\{f(x)\}^{\rho}\,dx\right],$$

where $\rho > 0$ and $\rho \neq 1$. Substituting $[f(x,\xi)]^{\rho}$ from Equation (3.10) into the expression for $H(\rho)$, the q-Entropy for NCT-G family of distribution is given by

$$H(\rho) = \frac{1}{1-\rho}\log\left[1-\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}\Psi_{kmr}\int_{-\infty}^{\infty}(g(x;\xi))^{\rho}(G(x,\xi))^{r+m}\,dx\right]$$

where $\rho > 0$ and $\rho \neq 1$.

iii) Shannon's Entropy

The Shannon's entropy with PDF $f(x)$ is a particular case of the Rényi's entropy when $\rho \uparrow 1$. Shannon entropies are defined as $\eta_X = E(-\log f(x))$. For the NCT-G family of distribution is given by

$$\eta_X = E\left[-\log\left\{\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}g(x,\xi)\left(G(x,\xi)\right)^{2i+j-2}\right\}\right].$$

4. Estimation Method

4.1. Maximum Likelihood Estimation (MLE)

The parameters of the NCT-G family are estimated in this segment using the method of MLE. Given random sample $x_1, ..., x_n$ of size n with parameters vector ξ from the NCT-G family of distribution and let $u = \xi^T$ be $(p\times 1)$ parameter vectors, then the log density and total log-likelihood function respectively are expressed as

$$l(x;\xi) = \log\left(\frac{\pi}{2}\right) + 2\log\left[\sec\left\{\frac{\pi}{2}\frac{G(x;\xi)}{1+G(x;\xi)}\right\}\right] - 2\log\left(1+G(x;\xi)\right) + \log g(x;\xi).$$

and

$$l(\underline{x},\xi) = n\log\left(\frac{\pi}{2}\right) + 2\sum_{i=1}^{n}\log\left[\sec\left\{\frac{\pi}{2}\frac{G(x_i;\xi)}{1+G(x_i;\xi)}\right\}\right] - 2\sum_{i=1}^{n}\log\left(1+G(x_i;\xi)\right) + \sum_{i=1}^{n}\log g(x_i;\xi). \tag{4.1}$$

Differentiating Equation (4.1), we get

$$\frac{\partial l}{\partial \xi} = \frac{\pi}{2}\sum_{i=1}^{n}\tan\left\{\pi\frac{G(x_i;\xi)}{1+G(x_i;\xi)}\right\}\frac{G_k^{'}(x_i;\xi)}{\left(1+G(x_i;\xi)\right)^2} - 2\sum_{i=1}^{n}\frac{G_k^{'}(x_i;\xi)}{\left(1+G(x_i;\xi)\right)} + \sum_{i=1}^{n}\frac{g_k^{'}(x_i;\xi)}{g(x_i;\xi)},$$

where $g_k^{'}(x_i;\xi) = \frac{dg(x_i;\xi)}{d\xi}$, $g_k^{''}(x_i;\xi) = \frac{d^2 g(x_i;\xi)}{d^2\xi}$, $G_k^{'}(x_i;\xi) = \frac{dG(x_i;\xi)}{d\xi}$ and $G_k^{''}(x_i;\xi) = \frac{d^2 G(x_i;\xi)}{d^2\xi}$.

4.2. Method of Least Square Estimation (LSE)

This method was introduced by (Swain et al., 1988). Let $x_{(1)},...,x_{(n)}$ be the random sample of size n from $F(x,\xi)$ and the LSEs for the NCT-G FD can be obtained by minimizing

$$K(X;\xi) = \sum_{i=1}^{n}\left[F(x_{(i)};\xi) - \frac{i}{n+1}\right]^2. \tag{4.2}$$

with respect to ξ. The least-square estimates for the NCT-G family of distribution also become

$$K(X;\xi) = \sum_{i=1}^{n}\left[\tan\left\{\frac{\pi}{2}\frac{G(x_{(i)},\xi)}{1+G(x_{(i)},\xi)}\right\} - \frac{i}{n+1}\right]^2. \tag{4.3}$$

Differentiating Equation (4.3) with respect to ξ we get

$$\frac{\partial K}{\partial \xi} = \pi\sum_{i=1}^{n}\left[\tan\left\{\frac{\pi}{2}\frac{G(x_{(i)};\xi)}{1+G(x_{(i)};\xi)}\right\} - \frac{i}{n+1}\right] \times \sec^2\left[\frac{\pi}{2}\frac{G(x_{(i)};\xi)}{1+G(x_{(i)};\xi)}\right] \times \frac{G_k^{'}(x_{(i)};\xi)}{\left(1+G(x_{(i)};\xi)\right)^2},$$

where $G_k^{'}(x_i;\xi) = \frac{dG(x_i;\xi)}{d\xi}$. By solving $\frac{dK}{d\xi} = 0$, we will get the LSEs of any particular distribution.

4.3. Cramer-von Mises Estimator (CVME)

CVMEs belong to the class of minimum distance estimators and exhibit lower bias compared to other estimators in this category. They involve calculating the difference between the estimates of the CDF and the empirical CDF. For the NCT-G family of distribution parameters, the CVMEs are obtained by minimizing

$$C(X;\xi) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{(i)};\xi) - \frac{2i-1}{2n}\right]^2$$

with respect to ξ. The LSEs for the NCT-G family of distribution also becomes

$$C(X;\xi) = \sum_{i=1}^{n}\left[\tan\left\{\frac{\pi}{2}\frac{G(x_{(i)},\xi)}{1+G(x_{(i)},\xi)}\right\} - \frac{2i-1}{2n}\right]^2. \tag{4.4}$$

Differentiating Equation (4.4) with respect to ξ we get

$$\frac{\partial C}{\partial \xi} = \pi\sum_{i=1}^{n}\left[tan\left\{\frac{\pi}{2}\frac{G(x_{(i)},\xi)}{1+G(x_{(i)},\xi)}\right\} - \frac{2i-1}{2n}\right] \times \sec^2\left[\frac{\pi}{2}\frac{G(x_{(i)},\xi)}{1+G(x_{(i)},\xi)}\right] \times \frac{G_k'(x_{(i)};\xi)}{\left(1+G(x_{(i)};\xi)\right)^2}.$$

where $G_k'(x_i;\xi) = \frac{dG(x_i;\xi)}{d\xi}$. By solving $\frac{dC}{d\xi} = 0$, we will get the CVMEs.

5. Special member of NCT-G FD

Generalization of several distributions can be made using the NCT-G FD. The special distribution, a new class tangent IW distribution is introduced in this section.

5.1. A New Class Tan Inverse Weibull (NCT-IW) Distribution

The CDF and PDF of the Inverse Weibull distribution are respectively given by

$$G(x;\alpha,\delta) = exp(-\alpha x^{-\delta}); x > 0, \alpha, \delta > 0,$$

and

$$g(x;\alpha,\delta) = \alpha\delta x^{-\delta-1} exp(-\alpha x^{-\delta}); x > 0, \alpha, \delta > 0.$$

The CDF and PDF of the NCT-IW distribution is given by

$$F(x;\alpha,\delta) = \tan\left[\frac{\pi}{2}\frac{\exp(-\alpha x^{-\delta})}{1+\exp(-\alpha x^{-\delta})}\right]; x > 0. \tag{5.1}$$

$$f(x;\alpha,\delta)=\frac{\pi}{2}\alpha\delta x^{-(\delta+1)}\sec^2\left[\frac{\pi}{2}\frac{\exp(-\alpha x^{-\delta})}{1+\exp(-\alpha x^{-\delta})}\right]\frac{\exp(-\alpha x^{-\delta})}{(1+\exp(-\alpha x^{-\delta}))^2};x>0. \tag{5.2}$$

$$R(x;\alpha,\delta)=1-\tan\left[\frac{\pi}{2}\frac{\exp(-\alpha x^{-\delta})}{1+\exp(-\alpha x^{-\delta})}\right];x>0.$$

and

$$H(x;\alpha,\delta)=\frac{\pi}{2}\alpha\delta x^{-(\delta+1)}\frac{\exp(-\alpha x^{-\delta})}{(1+\exp(-\alpha x^{-\delta}))^2}\sec^2\left[\frac{\pi}{2}\frac{\exp(-\alpha x^{-\delta})}{1+\exp(-\alpha x^{-\delta})}\right]$$
$$\left[1-\tan\left(\frac{\pi}{2}\frac{\exp(-\alpha x^{-\delta})}{1+\exp(-\alpha x^{-\delta})}\right)\right]^{-1};x>0.$$

The QF and random deviate generation for the NCT-IW distribution respectively given by

$$Q_X(p)=\left[-\frac{1}{\alpha}\log\left(\frac{2\tan^{-1}p}{\pi-2\tan^{-1}p}\right)\right]^{-\frac{1}{\delta}}. \tag{5.3}$$

and

$$x=\left[-\frac{1}{\alpha}\log\left(\frac{2\tan^{-1}p}{\pi-2\tan^{-1}p}\right)\right]^{-\frac{1}{\delta}}.$$

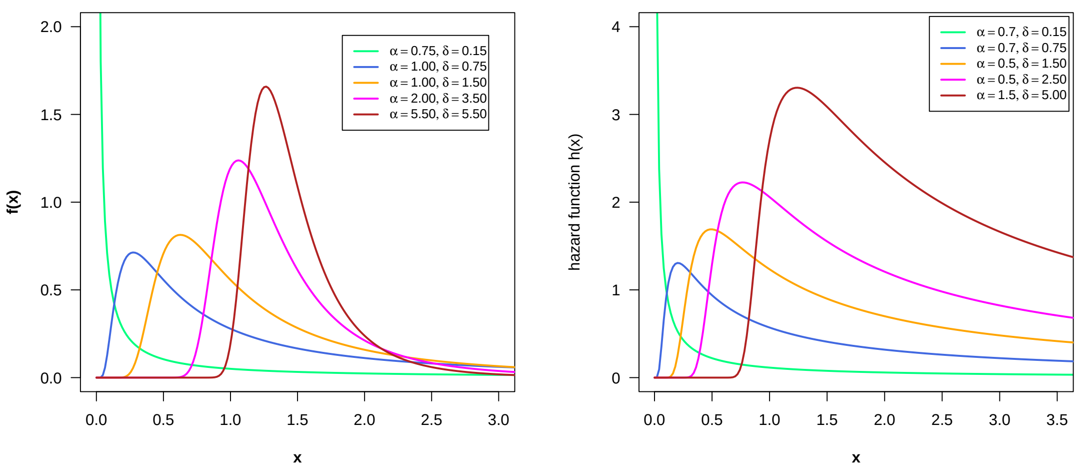

Figure 1: Shapes of PDF and HRF of NCT-IW distribution.

5.2. Linear Expansion NCT-IW distribution

Using Equation (3.5), Equation (5.2) can be presented in the linear form as

$$f(x;\alpha,\delta)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}x^{-(\beta+1)}\exp\left\{-(2i+j+2)\alpha x^{-\delta}\right\}, \tag{5.3}$$

where $\Delta_{ij}=\alpha\delta\left(\frac{\pi}{2}\right)^{2i-1}\frac{B_{2i}(-4)^i(1-4^i)(2i+j-1)}{(2i)!}\binom{2i-1}{j}$.

5.3. Moments

Using the PDF defined in Equation (5.3), the r^{th} order non-central moment $(\mu_r^{'})$ for the NCT-IW distribution can be presented as

$$\mu_r^{'}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\Gamma\left(\frac{\delta-r}{\delta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\delta-r}{\delta}}};\quad \forall\delta>r.$$

where $\Delta_{ij}^{*}=\alpha\left(\frac{\pi}{2}\right)^{2i-1}\frac{B_{2i}(-4)^i(1-4^i)(2i+j-1)}{(2i)!}\binom{2i-1}{j}$ and $\Gamma(.)$ is the gamma function.

5.4. Skewness and Kurtosis

Using the formula (5.3) to obtain the first four non-central moments, we can use the following expressions

$$Mean=\mu_1^{'}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\Gamma\left(\frac{\delta-1}{\delta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\delta-1}{\delta}}};\quad \forall\delta>1.,$$

$$\mu_2^{'}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\Gamma\left(\frac{\delta-2}{\delta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\delta-2}{\delta}}};\quad \forall\delta>2,$$

$$\mu_3^{'}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\Gamma\left(\frac{\delta-3}{\delta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\delta-3}{\delta}}};\quad \forall\delta>3,$$

$$\mu_4^{'}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\Gamma\left(\frac{\delta-4}{\delta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\delta-4}{\delta}}};\quad \forall\delta>4.$$

To calculate the central moments one can use the following formulae $\mu_1=\mu_1^{'}$; $\mu_2=\mu_2^{'}-\mu_1^{'2}$; $\mu_3=\mu_3^{'}-3\mu_1^{'}\mu_2^{'}+2\mu_1^{'3}$ and $\mu_4=\mu_4^{'}-4\mu_3^{'}\mu_2^{'}+6\mu_2^{'}\mu_1^{'2}-2\mu_1^{'4}$. Therefore skewness and kurtosis for the NCT-IW distribution can be calculated as

$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ and $\beta_2 = \frac{\mu_4}{\mu_2^2}$.

5.5. MGF

The MGF ($M_X(t)$) for the NCT-IW distribution is

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^k \Delta_{ij}^*}{k!} \frac{\Gamma\left(\frac{\delta-r}{\delta}\right)}{[\alpha\{(2i+j)+2\}]^{\frac{\delta-r}{\delta}}}; \quad \forall \delta > r.$$

5.6. Incomplete moments

The incomplete moment for NCT-IW distribution is given by

$$\begin{aligned} M_r(y) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Delta_{ij} \int_0^y x^{r-(\delta+1)} \exp\left\{-(2i+j+2)\alpha x^{-\delta}\right\} dx \\ &= \frac{1}{\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Delta_{ij} \frac{\gamma\left(\frac{\delta-r}{\delta}, (2i+j+2)\alpha y^{-\delta}\right)}{\{(2i+j+2)\alpha\}^{\frac{\delta-r}{\delta}}} \end{aligned}$$

where $\gamma(.)$ incomplete gamma function.

5.7. Mean residual life function

The Mean residual life for NCT-IW distribution is given by

$$\begin{aligned} \bar{M}(y) &= \frac{1}{F(y)}\left[\mu - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Delta_{ij} \int_0^y x^{-\delta} \exp\left\{-(2i+j+2)\alpha x^{-\delta}\right\}\right] - y \\ &= \frac{1}{F(y)}\left[\mu - \frac{1}{\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Delta_{ij} \frac{\gamma\left(\frac{\delta-1}{\delta}, (2i+j+2)\alpha y^{-\delta}\right)}{\{(2i+j+2)\alpha\}^{\frac{\delta-1}{\delta}}}\right] - y. \end{aligned}$$

where $\gamma(.)$ incomplete gamma function.

5.8. Entropy

i) Rényi's Entropy

The Rényi's entropy for NCT-IW distribution is given by

$$\begin{aligned} R_\rho(X) &= \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty} \Psi_{kmr}\,(\alpha\delta)^\rho \int_0^\infty x^{-\rho(\beta+1)} \exp(-(r+m+\rho)\alpha x^{-\delta}) dx\right] \\ &= \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty} \Psi_{kmr} \frac{(\alpha\delta)^\rho}{\delta} \frac{\Gamma\left(\left\{\frac{(\rho-1)(\delta+1)}{\delta}+1\right\}\right)}{\{(r+m+\rho)\alpha\}^{\frac{(\rho-1)(\delta+1)}{\delta}+1}}\right]. \end{aligned}$$

where $\Psi_{kmr} = a_k\,(-1)^{m+k-r}\,s_0^{k-r}\left(\frac{\pi}{2}\right)^{\rho}\binom{k}{r}\binom{(2\rho+r)+m-1}{m}$.

ii) q-Entropy

The q-Entropy for NCT-IW distribution is given by

$$H(\rho) = \frac{1}{1-\rho}\log\left[1-\Psi_{kmr}\,(\alpha\delta)^{\rho}\int_{0}^{\infty} x^{-\rho(\delta+1)}\exp(-(r+m+\rho)\alpha x^{-\delta})dx\right]$$
$$= \frac{1}{1-\rho}\log\left[1-\Psi_{kmr}\frac{(\alpha\delta)^{\rho}}{\delta}\frac{\Gamma\left(\left\{\frac{(\rho-1)(\delta+1)}{\delta}+1\right\}\right)}{\{(r+m+\rho)\alpha\}^{\frac{(\rho-1)(\delta+1)}{\delta}+1}}\right]$$

where $\rho > 0$, $\rho \neq 1$ and $\Psi_{kmr} = a_k\,(-1)^{m+k-r}\,s_0^{k-r}\left(\frac{\pi}{2}\right)^{\rho}\binom{k}{r}\binom{(2\rho+r)+m-1}{m}$.

iii) Shannon's Entropy

The Shannon entropy for the NCT-IW distribution is given by

$$\eta_X = E\left[-\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}x^{-(\delta+1)}\exp\left\{-(2i+j+2)\alpha x^{-\delta}\right\}\right\}\right].$$

5.9. Inequality Measure

i) Lorentz Curve

The Lorenz curve for NCT-IW distribution is given by

$$L_{F(y)} = \frac{\alpha\delta}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_{-\infty}^{y} x^{-\delta}\exp(-\alpha(2i+j+2)x^{-\delta})dx$$
$$= \frac{\alpha}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\frac{\gamma\left(\frac{\delta-1}{\delta},(2i+j+2)\alpha y^{-\delta}\right)}{\{(2i+j+2)\alpha\}^{\frac{\delta-1}{\delta}}},$$

where $\gamma(.)$ incomplete gamma function.

ii) Bonferroni Curve

The Bonferroni curve for the NCT-IW distribution is given by

$$B_{F(y)} = \frac{1}{\mu F(y)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_{-\infty}^{y} x^{-\delta}\exp(-\alpha(2i+j+1)x^{-\delta})dx$$
$$= \frac{1}{\delta\mu F(y)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\frac{\gamma\left(\frac{\delta-1}{\delta},(2i+j+2)\alpha y^{-\delta}\right)}{\{(2i+j+2)\alpha\}^{\frac{\delta-1}{\delta}}},$$

where $\gamma(.)$ incomplete gamma function.

5.10. MLE for NCT-IW distribution

The log-likelihood function is given by

$$l(x;\alpha,\delta) = n\log\left(\frac{\pi}{2}\alpha\delta\right) - (\delta+1)\sum_{i=1}^{n}\log x_i + 2\sum_{i=1}^{n}\text{logsec}\left[\pi\frac{\exp(-\alpha x_i^{-\delta})}{1+\exp(-\alpha x_i^{-\delta})}\right]$$
$$-2\sum_{i=1}^{n}\log\left(1+\exp(-\alpha x_i^{-\delta})\right) - \alpha\sum_{i=1}^{n}x_i^{-\delta}. \tag{5.4}$$

Differentiating the Equation (5.4) we get

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \frac{\pi}{2}\sum_{i=1}^{n}\frac{x_i^{-\delta}exp(-\alpha x_i^{-\delta})}{\left(1+\exp(-\alpha x_i^{-\delta})\right)^2}\tan\left[\frac{\pi}{2}\frac{\exp(-\alpha x_i^{-\delta})}{1+\exp(-\alpha x_i^{-\delta})}\right]$$
$$+2\sum_{i=1}^{n}\frac{x_i^{-\delta}exp(-\alpha x_i^{-\delta})}{\left(1+\exp(-\alpha x_i^{-\delta})\right)} - \sum_{i=1}^{n}x_i^{-\delta}$$

and

$$\frac{\partial l}{\partial \delta} = \frac{n}{\delta} - \sum_{i=1}^{n}\log x_i + \frac{\pi\alpha}{2}\sum_{i=1}^{n}\frac{x_i^{-\delta}\log(x_i)\exp(-\alpha x_i^{-\delta})}{\left(1+\exp(-\alpha x_i^{-\delta})\right)^2}\tan\left[\frac{\pi}{2}\frac{\exp(-\alpha x_i^{-\delta})}{1+\exp(-\alpha x_i^{-\delta})}\right]$$
$$+2\alpha\sum_{i=1}^{n}\frac{x_i^{-\delta}\log(x_i)\exp(-\alpha x_i^{-\delta})}{\left(1+\exp(-\alpha x_i^{-\delta})\right)} + \alpha\sum_{i=1}^{n}x_i^{-\delta}\log(x_i).$$

The MLEs of α and δ are obtained by solving simultaneously the equations $\frac{\partial l}{\partial \alpha} = 0$ and $\frac{\partial l}{\partial \delta} = 0$.

5.11. Simulation study

We created samples from the quantile function for various parameter combinations of the NCT-IW distribution. Using the maxLik R package developed by (Henningsen & Toomet, 2011) we have calculated the MLEs for each sample using the maxLik() function with the BFGS algorithm. This enables us to test parameter estimation problems, such as the flatness or sharpness of the likelihood function and provides estimates for the direction and size (overestimate or underestimate) of the MLEs bias. The simulation employs 25 samples of sizes 10 to 250. The procedure is repeated 1000 times, and the bias and mean square error (MSE) are calculated. Also, we have provided the lower bound (LB) and upper bound (UB) for estimated values with a 5% level of significance.

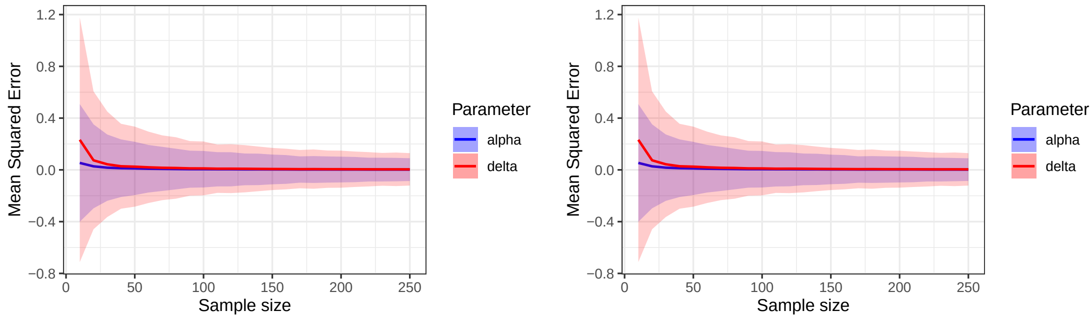

Figure 2. MSE plots for α and δ.

Table 2: LB, UB for MLEs, Bias, and MSEs with initial values $\alpha = 0.5$ and $\delta = 1.25$.

n	$\alpha = 0.5$				δ = **1.25**			
	LB	UB	Bias	MSE	LB	UB	Bias	MSE
10	0.1285	1.0131	-0.0163	0.0538	0.8598	2.4879	0.2224	0.2324
20	0.2222	0.8734	-0.0054	0.0274	0.9348	1.9224	0.0934	0.0739
30	0.2725	0.7893	0.0025	0.0171	0.9822	1.7593	0.0565	0.0432
40	0.2959	0.7380	0.0020	0.0129	0.9971	1.6540	0.0417	0.028
50	0.3091	0.7247	-0.0023	0.0110	1.0285	1.6330	0.0367	0.0249
60	0.3268	0.6954	-0.0061	0.0088	1.0360	1.5781	0.0358	0.0198
70	0.3397	0.6761	-0.0037	0.0076	1.0610	1.5251	0.0308	0.0163
80	0.3547	0.6653	-0.0047	0.0064	1.0586	1.5194	0.0229	0.0147
90	0.3505	0.6396	-3.00E-04	0.0053	1.0837	1.4989	0.0211	0.0116
100	0.3700	0.6389	-0.0039	0.0052	1.0870	1.4938	0.0218	0.0113
110	0.3749	0.6419	-4.00E-04	0.0046	1.0985	1.4707	0.0156	0.0091
120	0.3685	0.6300	-0.0012	0.0045	1.0960	1.4766	0.0143	0.0093
130	0.3730	0.6227	-0.0034	0.0039	1.0997	1.4553	0.0144	0.0087
140	0.3758	0.6185	-4.00E-04	0.0038	1.1095	1.4522	0.0143	0.0077
150	0.3885	0.6196	0.0021	0.0034	1.1056	1.4367	0.0084	0.0069
160	0.3923	0.6170	-4.00E-04	0.0032	1.1085	1.4279	0.0083	0.0064
170	0.3989	0.5984	-0.002	0.0027	1.1267	1.4119	0.0112	0.0057
180	0.4001	0.6008	-0.0024	0.0028	1.1177	1.4228	0.0116	0.006
190	0.3972	0.6020	-0.0011	0.0027	1.1273	1.4095	0.0106	0.0054
200	0.4090	0.6071	0.0021	0.0026	1.1222	1.4069	0.0074	0.0053
210	0.4049	0.5995	4.00E-04	0.0025	1.1341	1.4078	0.0086	0.0049
220	0.4077	0.5931	-0.0016	0.0022	1.1404	1.4118	0.0091	0.0046
230	0.4101	0.5955	-5.00E-04	0.0022	1.1443	1.3860	0.0099	0.0042
240	0.4120	0.5960	0.0018	0.0021	1.1384	1.3967	0.0055	0.0044
250	0.4118	0.5880	-0.0025	0.0020	1.1435	1.3839	0.0088	0.0041

The experiment is summarized in Tables 2 and 3, which show the bias and MSEs for each parameter along with LB and UB for MLEs. As can be seen, the MLE method consistently estimates the parameters of the proposed model. Further, as sample size increases, MLEs gradually approach the actual values of α and δ.

Table 3: LB, UB for MLEs, Bias, and MSEs with initial values $\alpha = 0.25$ and $\delta = 1.75$.

n	$\alpha = 0.25$				$\delta = 1.75$			
	LB	UB	Bias	MSE	LB	UB	Bias	MSE
10	0.0313	0.6271	-0.0049	0.024	1.1781	3.5613	0.3023	0.468
20	0.0715	0.4984	-0.0014	0.012	1.3271	2.753	0.1266	0.1556
30	0.1071	0.4366	-0.0065	0.0073	1.3929	2.4465	0.0908	0.0873
40	0.1157	0.4103	-0.0017	0.0057	1.4022	2.3857	0.0618	0.0656
50	0.1354	0.3911	-0.0039	0.0043	1.4548	2.2815	0.0559	0.0456
60	0.1404	0.3664	-0.0033	0.0035	1.4819	2.1821	0.0441	0.0349
70	0.1452	0.3674	-5.00E-04	0.0033	1.4854	2.1487	0.0368	0.0319
80	0.153	0.3632	0.0021	0.0029	1.4924	2.1183	0.0210	0.0264
90	0.1561	0.3524	-0.0012	0.0026	1.5201	2.1211	0.0288	0.0247
100	0.1663	0.3457	-3.00E-04	0.0022	1.5123	2.0659	0.0178	0.0205
110	0.1673	0.3422	-3.00E-04	0.0021	1.5202	2.0578	0.0227	0.0194
120	0.1735	0.3383	3.00E-04	0.0018	1.5479	2.0514	0.0186	0.0167
130	0.1750	0.3388	-0.0011	0.0018	1.5496	2.0119	0.0221	0.0153
140	0.1797	0.3357	2.00E-04	0.0016	1.5439	2.0256	0.013	0.0150
150	0.1790	0.3284	-0.0019	0.0015	1.5557	2.0171	0.0191	0.0142
160	0.1870	0.3262	-0.0011	0.0013	1.5665	1.986	0.0184	0.0119
170	0.1773	0.3176	-0.0012	0.0013	1.5632	2.0074	0.016	0.0121
180	0.1830	0.3263	-7.00E-04	0.0013	1.5672	1.9738	0.0132	0.0119
190	0.1836	0.3182	-0.0011	0.0012	1.5689	1.9828	0.0145	0.0110
200	0.1931	0.3185	9.00E-04	0.0011	1.5739	1.9571	0.007	0.0100
210	0.1905	0.3196	-2.00E-04	0.0011	1.5823	1.9547	0.0096	0.0095
220	0.1882	0.3179	2.00E-04	0.0011	1.5803	1.9534	0.009	0.0092
230	0.1899	0.3104	6.00E-04	9.00E-04	1.5883	1.9506	0.0072	0.0083
240	0.1938	0.3147	6.00E-04	9.00E-04	1.5959	1.9534	0.0068	0.0083
250	0.1972	0.3098	-2.00E-04	9.00E-04	1.5838	1.9401	0.0083	0.0083

5.12. Application

Employing two real data sets, we demonstrate the performance of the NCT-IW distribution in this section. The data sets employed for the application of the suggested distribution are given as follows

Dataset-I

The dataset from (Clark and Gross, 1975) contains information on the relief times of 20 patients who were administered an analgesic. An analgesic is a type of medication that is commonly used to reduce pain, and the relief time refers to the duration for which the patients experience relief from their pain after taking the medication. The data are 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, and 2.0.

Table 4: MLEs with SE (in parentheses) (dataset I)

Model	parameter	SE	parameter	SE	parameter	SE
NCT-IW(α, β)	6.6984	1.9636	3.8242	0.6632	–	–
IW(λ, θ)	4.0175	0.706	6.0224	2.0083	–	–
ArcTGE$(\alpha, \lambda, \theta)$	0.0000	1.2263	32.6645	4.397	2.1675	0.1147
ArcTLx(α, β, θ)	147.2664	44.0127	0.2871	0.2782	12.3869	9.6739
ASE(θ)	281.1502	4.8432	–	–	–	–
ASEW$(\lambda, \theta, \upsilon)$	1.0488	0.1284	104.561	19.0921	3.1656	0.1303
NCW(λ, θ)	0.2505	0.081	2.293	0.3402	–	–
TBXII(λ, ν, θ)	1.3952	0.1523	10.3398	4.4804	0.3949	0.2513
ECosW(β, λ, θ)	0.1902	0.0419	0.4638	0.1032	2.4299	0.0561
CosW(β, δ)	2.2183	0.3323	0.5655	0.0471	–	–
SinIW(δ, θ)	5.3385	1.4594	2.8386	0.4882	–	–

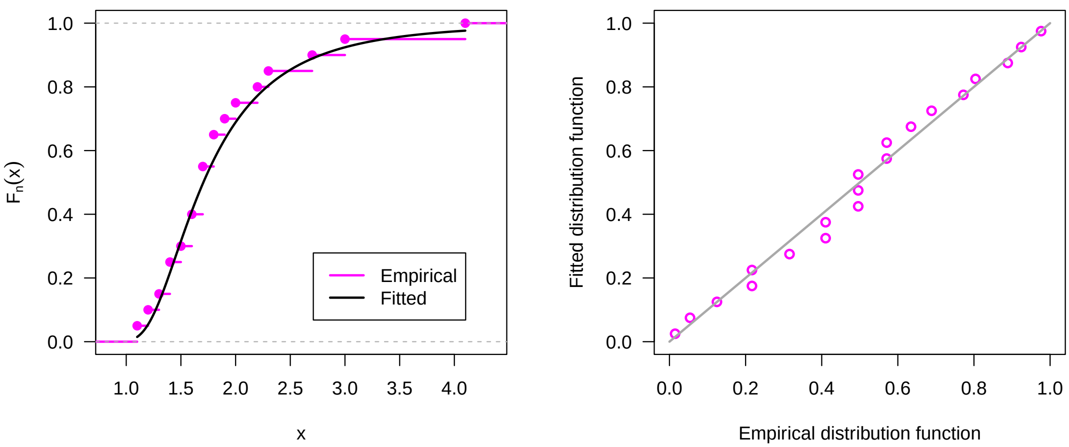

Figure 3. KS and P-P plots for the dataset-I.

Table 5: Some selection criteria and goodness-of-fit statistics (dataset-I).

Dist.	-2logL	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
NCT-IW	30.9715	34.9715	35.3603	0.1103	0.9682	0.0295	0.9803	0.1685	0.9969
IW	30.8174	34.8174	35.2062	0.102	0.9854	0.0266	0.988	0.1545	0.9984
ArcTGE	32.5499	38.5499	39.133	0.1367	0.8488	0.0501	0.8809	0.3109	0.929
ArcTLmx	35.6262	41.6262	42.2094	0.124	0.9182	0.0662	0.7806	0.5268	0.7175
ASE	170.4197	172.4197	172.6141	0.9232	0.0000	5.5995	0.0000	38.8183	0.0000
ASEW	31.1885	37.1885	37.7716	0.117	0.947	0.0363	0.9551	0.2096	0.9877
NCW	48.687	52.687	53.0757	0.1467	0.7829	0.1078	0.5521	0.78	0.494
Tan-BXII	31.0805	37.0805	37.6636	0.0917	0.996	0.0231	0.9944	0.1376	0.9994
ECosW	41.0062	47.0062	47.5893	0.1769	0.5586	0.179	0.3137	1.0983	0.3088
CosW	37.4854	41.4854	41.8742	0.177	0.5576	0.1279	0.4681	0.7563	0.5118
Sin-IW	31.1572	35.1572	35.546	0.1069	0.9763	0.0292	0.9813	0.1808	0.9949

We have compute goodness-of-fit statistics to analyze data sets under study and the fitted models are compared using the log-likelihood value (-2logL), Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (AD), Kolmogrov-Smirnov (KS) with p-values and Cramer-von Mises (CVM) for more detail (see Johnson et al., 1995). All the essential computations are carried out in R-software.

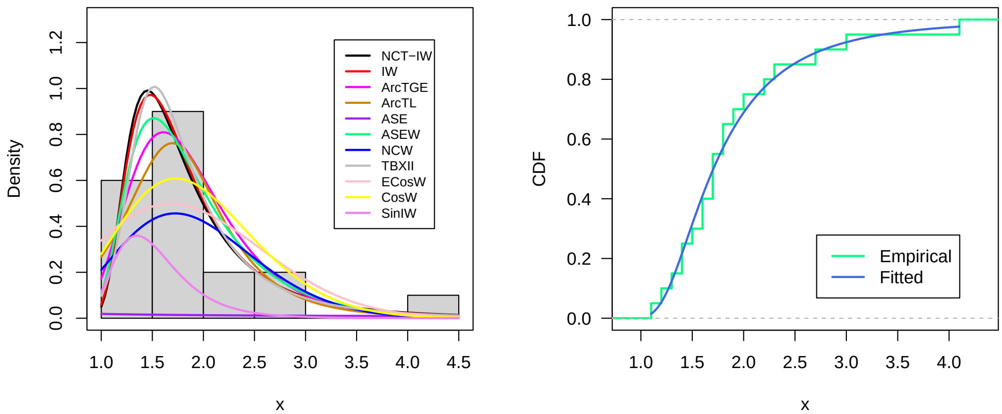

Figure 4. Estimated PDF (left) and empirical vs estimated CDF (right) (dataset-I).

Dataset-II

In this study, we consider the data used by (Tomer & Panwar, 2020), which represent the concentration of glycosaminoglycans (GAG) in urine, measured in milligrams per millimole of creatinine, and were collected from 40 children between the ages of 12 and 17 years.

"5.8, 5.4, 5.7, 3.1, 6.4, 7.0, 5.7, 3.9, 9.4, 4.4, 5.0, 15.9, 3.7, 9.1, 4.7, 3.6, 3.7, 4.1, 7.9, 3.3,6.6, 1.9, 3.0, 5.7, 3.2, 3.8, 5.3, 3.2, 4.2, 6.0, 9.7, 3.4, 3.2, 2.5, 2.0, 4.0, 4.3, 2.8, 2.2, 4.7"

For the comparison of fitting capability we have selected some models such as inverse Weibull (IW), arctan generalized exponential (ArcTGE) (Chaudhary et al., 2021), arctan Lomax (ArcTLx) (Chaudhary & Kumar, 2021), arcsine exponential (ASE) (Rahman, 2021), TBXII (Souza et al., 2021), NCW (Ahmad et al., 2023), EcosW (Muhammad et al., 2021), arcsine exponentiated Weibull (ASEW) (He et al., 2020), CosW (Souza, 2019b) and Sin-IW (Souza et al., 2021). In Tables 4 and 6, we have presented the estimated values of the parameters and their corresponding standard error (SE in parentheses) of the models under study using the MLE method. Similarly, in Tables 5 and 7, we have presented the model selection and goodness of fit statistics such as log-likelihood, AIC, HQIC, KS, AD, and CVM for both data sets. It has been observed that the suggested model NCT-IW has the least statistics as compared to the models under study. Hence NCC-G is more flexible and provides a good fit. Also, we have displayed the graphical illustrations of the fitted models in Figures 3 and 5. These figures also verified that the NCT-IW model can perform well as compared to candidate models.

Table 6: MLEs with SE (in parentheses) (dataset II)

Model	**parameter**	**SE**	**parameter**	**SE**	**parameter**	**SE**
NCT-IW(α, β)	25.712	3.654	2.3722	0.1641	–	–
ArcTL(α, β, θ)	70.8335	7.3788	0.2679	0.1611	5.5074	2.1685
ASE(θ)	300.0999	7.3788	–	–	–	–
NCW(λ, θ)	0.0793	0.027	1.6641	0.1767	–	–
ECosW(β, λ, θ)	0.2824	0.0516	0.0603	0	1.9897	0.0615
CosW(β, δ)	1.6117	0.173	0.228	0.0186	–	–

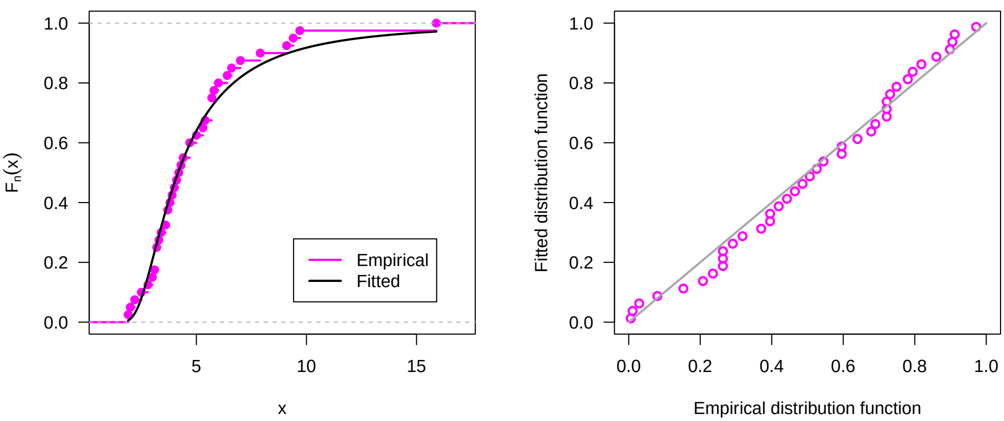

Figure 5. KS and P-P plots (dataset-II).

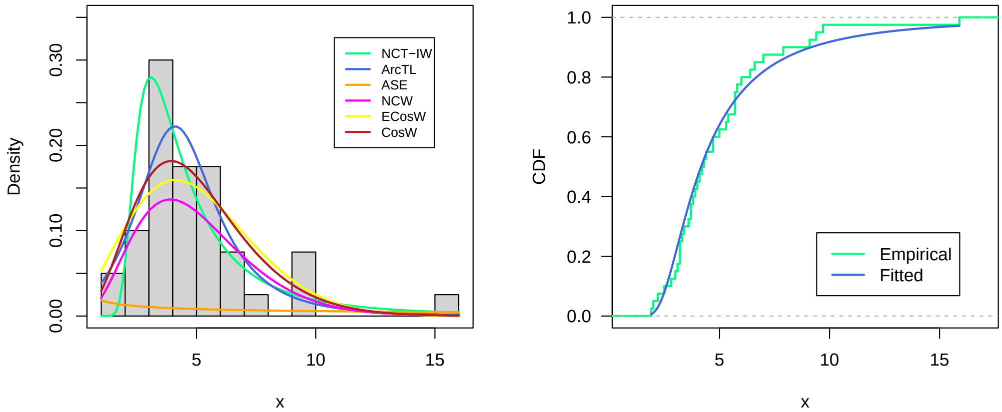

Figure 6. Estimated PDF (left) for all models and empirical vs estimated CDF of NCT-IW (right) (dataset-II).

Table 7: Some selection criteria and goodness-of-fit statistics (dataset-II).

Dist.	-2logL	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
NCT-IW	170.3992	174.3992	175.6205	0.0885	0.9129	0.0561	0.8412	0.432	0.8156
ArcTL	172.3715	178.3715	180.2035	0.0899	0.9032	0.0493	0.8834	0.4402	0.8073
ASE	380.1219	382.1219	382.7326	0.8609	0.0000	10.0105	0.0000	60.1362	0.0000
NCW	196.8155	200.8155	202.0368	0.1388	0.4239	0.1178	0.5064	0.8929	0.4179
ECosW	179.81	185.81	187.642	0.1189	0.624	0.1681	0.3397	1.109	0.3045
CosW	174.3874	178.3874	179.6087	0.0963	0.8518	0.1015	0.5797	0.6701	0.5833

6. Conclusions and Future Work

We have developed a new class of distributions, called the Tan-G family, by transforming the tangent function based on the ratio of CDF $G(x)$ and $1 + G(x)$ of a baseline distribution. The general properties of this family of distributions have been provided. To introduce a member of this family having a reverse-j or increasing or inverted bathtub-shaped hazard function, we used the Inverse Weibull distribution as a baseline distribution. We have explored some statistical properties of this member distribution, called NCT-IW. The parameters associated with the new distribution have been estimated using the MLE method. To assess the estimation procedure, we conducted a Monte Carlo simulation and found that the biases and mean square errors decrease as the sample size increases, even for small samples. We then applied the NCT-IW distribution to two real data sets. Using model selection criteria and goodness-of-fit test statistics, we have empirically demonstrated that the suggested model performs better than other existing models (most of which have more parameters) under study. Therefore, we expect that the suggested family and its member distribution can be applied in various fields such as medical science, reliability engineering, survival analysis, etc., and can be used to generate new models in the future.

Further, this work can be extended to bivariate distribution and it can be analyzed under the Bayesian approach. The estimates of the value of the best Hausdorff approximation d (for more details, see Kyurkchiev & Markov (2015)) can be used in practice as one possible additional criterion in the 'saturation' study of sigmoidal cumulative function.

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