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ON THE PRODUCT OF GEOMETRIC-ARITHMETIC, SOMBOR AND FIRST ZAGREB INDICES

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Abstract: Assume Ω is a connected and simple graph with edge set $E(\Omega)$ and vertex set $V(\Omega)$. In chemical graph theory, the Geometric-Arithmetic index, Sombor index, and first Zagreb index of graph are three well-defined and studied topological indices. In the present study, we introduce the new graph invariant, we call it as $GSM(\Omega)$ index, this new graph invariant is the product of Geometric-Arithmetic index, Sombor index, and first Zagreb index. Furthermore we discuss the effect on $GSM(\Omega)$ of inserting and deleting an edge into a graph Ω. In addition, we investigate the connections between the GSM(G) index and various well-studied topological indices.

Keywords and Phrases: Topological indices, maximum and minimum degree, degree (of vertex).

2020 Mathematics Subject Classification: 05C07, 05C09, 05C92.

1. Introduction

Let $\Omega = (V, E)$ be a simple graph of order n size m, with vertex set $V(\Omega)$ and edge set $E(\Omega)$. The Δ and δ stand for the maximum and minimum degrees of Ω, respectively. The δ_1 represents the minimum non-pendent vertex degree of the graph Ω. Additionally, p stands for the number of pendent vertices in Ω. The degree $d_\Omega(j)$ of a vertex j is the number of vertices adjacent to $j \in V(\Omega)$. The

topological indices are one of many useful tools discovered by graph theory for chemists these are type of a molecular descriptor that is calculated based on the molecular graph of a chemical compound. Molecular graphs are commonly used to model molecular and molecule compounds. Molecular graphs topological indices are one of the earliest and most often used descriptors in QSAR/QSPR research. The bounds of a topological indices are important information of a molecular graph in the sense that they establish the approximate range of the topological indices in terms of molecular structural parameters.
The below is a definition of the first Zagreb index [7].

$$M_1(\Omega) = \sum_{j \in V(\Omega)} d_\Omega(j)^2 = \sum_{jk \in E(\Omega)} (d_\Omega(j) + d_\Omega(k)).$$

The below is a definition of the second Zagreb index [8].

$$M_2(\Omega) = \sum_{jk \in E(\Omega)} (d_\Omega(j) \cdot d_\Omega(k)).$$

In 1975, Randić proposed the Randić to measure the degree of branching in the carbon-atom structure of saturated hydrocarbons. The Randić [15] is detailed below

$$R(\Omega) = \sum_{jk \in E(\Omega)} \frac{1}{\sqrt{d_\Omega(j) \cdot d_\Omega(k)}}.$$

The general Randić index [1] was defined as

$$R_\alpha = R_\alpha(\Omega) = \sum_{jk \in E(\Omega)} (d_\Omega(j) \cdot d_\Omega(k))^\alpha.$$

The geometric-arithmetic graph index was introduced in [19]

$$GA(\Omega) = \sum_{jk \in E(\Omega)} \frac{2\sqrt{d_\Omega(j) \cdot d_\Omega(k)}}{d_\Omega(j) + d_\Omega(k)}.$$

The harmonic graph index first appeared in [5], and was defined as [20]

$$H(\Omega) = \sum_{jk \in E(\Omega)} \frac{2}{d_\Omega(j) + d_\Omega(k)}.$$

The sombor index was introduced by I. Gutman [9] to be described as

$$SO(\Omega) = \sum_{jk \in E(\Omega)} \sqrt{d_\Omega(j)^2 + d_\Omega(k)^2}.$$

The Forgotten graph index was introduced in [6]

$$F(\Omega) = \sum_{j \in V(\Omega)} d_\Omega(j)^3 = \sum_{jk \in E(\Omega)} (d_\Omega(j)^2 + d_\Omega(k)^2).$$

Explanation of the Randić index and the bulk of its relevant mathematical properties can be found in [10, 13, 17]. Given the prominence of topological indices, various researchers are interested in researching the comparison or relationship between topological indices, for example, refer to [2-4, 12, 16]. Motivated by previous work on topological indices we introduce new degree based graph index which is the the product of Geometric-Arithmetic index, Sombor index and First Zagreb index, i.e., $(GA \cdot SO \cdot M_1)(\Omega)$, we denoted this new graph index as $GSM(\Omega)$ by taking first letter of $\mathbf{G}A(\Omega), \mathbf{S}O(\Omega)$ and $\mathbf{M}_1(\Omega)$ and is defined as

$$\mathbf{GSM}(\Omega) = \sum_{jk \in E(\Omega)} 2\sqrt{d_\Omega(j) \cdot d_\Omega(k)(d_\Omega(j)^2 + d_\Omega(k)^2)}. \tag{1.1}$$

For all standard notations and terminologies we refer to Harary [11].

2. Preliminaries

In this section, we recall some Lemma which will be used in the next section.

Lemma 2.1. [14] (Cauchy-Schwarz Inequality). *Let $X = \{x_1, x_2 \cdots x_n\}$ and $Y = \{y_1, y_2 \cdots y_n\}$ be two sequences of real numbers. Then*

$$\left(\sum_{t=1}^{n} x_t y_t\right)^2 \leq \sum_{t=1}^{n} x_t^2 \sum_{t=1}^{n} y_t^2.$$

With equality if and only if the sequences X and Y are proportional, i.e., there exists a constant c such that $x_t = cy_t$, for each $1 \leq t \leq n$.

Lemma 2.2. [14] (Čebyšev's Inequality). *Let $X = \{x_1, x_2 \cdots x_n\}$ and $Y = \{y_1, y_2 \cdots y_n\}$ are two real sequences such that, $x_1 \leq \cdots \leq x_n$ and $y_1 \leq \cdots \leq y_n$, or $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$, then the following inequality is true*

$$\left(\frac{1}{n}\sum_{t=1}^{n} x_t\right)\left(\frac{1}{n}\sum_{t=1}^{n} y_t\right) \leq \frac{1}{n}\sum_{t=1}^{n} x_t \cdot y_t.$$

Lemma 2.3. [14] (Abel's Inequality). *Let $X = \{x_1, x_2 \cdots x_n\}$ and $Y = \{y_1, y_2 \cdots y_n\}$, $(y_1 \geq \ldots y_n) \geq 0$ be two sequences of real numbers, and let*

$$s_k = x_1 + x_2 \cdots + x_k. \ \ (k = 1, \cdots, n).$$

$$If \quad m = \min_{1\le k\le n} s_k, \quad M = \max_{1\le k\le n} s_k, \quad then$$
$$my_1 \le x_1y_1 + \cdots + x_ny_n \le My_1.$$

3. Results and Discussion

In this section we examine the impact of addition of edge and removal of edge to the $GSM(\Omega)$ index of a simple graph Ω. Also, we establish maximum values of the $GSM(\Omega)$ index and we give some bounds on $GSM(\Omega)$ index of a graph Ω in terms of number of vertices n, number of edges m, maximum vetex degree Δ, minimum vetex degree δ, minimum non-pendent vertex degree δ_1, number of pendent vertices p, and the Forgotten index $F(G)$, second Zagreb index $M_2(\Omega)$, Sombor index $SO(\Omega)$, and the general randic index $R_\alpha(\Omega)$, for $\alpha = \frac{1}{2}$.

Theorem 3.1. *Let i and j be two non-adjacent vertices of graph Ω, and let $\Omega + ij$ be the graph obtained from Ω by adding edge ij to it. Then*

$$GSM(\Omega) < GSM(\Omega + ij).$$

Proof. Let $x, y > 0$, formulation of eqn. (1.1) in terms of x, y we get

$$\begin{aligned}
& 2\sqrt{(x+1)\cdot y\cdot((x+1)^2+y^2)} - 2\sqrt{x\cdot y\cdot(x^2+y^2)} \\
= & 2\sqrt{xy(x^2+1+x)+2xy(x+y^2+1)+y} - 2\sqrt{xy(x^2+y^2)} \\
= & 2\left(\frac{xy(x^2+1+x)+2xy(x+y^2+1)+y-x^3y-xy^3}{\sqrt{xy(x^2+1+x)+2xy(x+y^2+1)+y}-\sqrt{xy(x^2+y^2)}}\right) \\
= & 2\left(\frac{xy(x+1)+2xy(x+1)+y(1+y^2)}{\sqrt{xy(x^2+1+x)+2xy(x+y^2+1)+y}-\sqrt{xy(x^2+y^2)}}\right) > 0.
\end{aligned}$$

Now, let $x_1, x_2, x_3, \ldots x_t$ be the neighbors of x in Ω for $t = d_\Omega(i)$ and let $y_1, y_2, y_3, \ldots y_q$ be the neighbors of y for $q = d_\Omega(j)$.

$$\begin{aligned}
GSM(\Omega+ij) - GSM(\Omega) = & \; 2\sqrt{(d_\Omega(i)+1)(d_\Omega(j)+1)\big((d_\Omega(i)+1)^2+(d_\Omega(j)+1)^2\big)} \\
& + \sum_{r=1}^{t}\Big(2\sqrt{(d_\Omega(i)+1)(d_\Omega(i_r))\big((d_\Omega(i)+1)^2+(d_\Omega(i_r))^2\big)} \\
& -2\sqrt{(d_\Omega(i))(d_\Omega(i_r))\big((d_\Omega(i))^2+(d_\Omega(i_r))^2\big)}\Big) \\
& + \sum_{k=1}^{q}\Big(2\sqrt{(d_\Omega(j)+1)(d_\Omega(j_k))\big((d_\Omega(j)+1)^2+(d_\Omega(j_k))^2\big)} \\
& -2\sqrt{(d_\Omega(j))(d_\Omega(j_k))\big((d_\Omega(j))^2+(d_\Omega(j_k))^2\big)}\Big) \\
> & \; 0 \\
GSM(\Omega+ij) > & \; GSM(\Omega).
\end{aligned}$$

The first sum above represents the contribution of edge ij in $(\Omega + ij)$.

Corollary 3.2. *If Ω is not a tree, then*

$$GSM(T) < GSM(\Omega).$$

holds for each spanning tree T of Ω.

Theorem 3.3. *Let i and j be two adjacent vertices of graph Ω, and let $(\Omega - ij)$ be the graph obtained from Ω by deleting edge ij to it. Then*

$$GSM(\Omega - ij) < GSM(\Omega).$$

Proof. Note first that $x, y > 0$, formulation of eqn. (1.1) in terms of x, y we get

$$\begin{aligned}
& 2\sqrt{(x-1)y((x-1)^2+y^2)} - 2\sqrt{xy(x^2+y^2)} \\
= \; & 2\sqrt{(xy-y)(x^2+y^2-2x+1)} - 2\sqrt{xy(x^2+y^2)} \\
= \; & 2\sqrt{x^3y+xy^3-2x^2y+xy-x^2y-y^3+2xy-y} - 2\sqrt{xy(x^2+y^2)} \\
< \; & 0.
\end{aligned}$$

Now, let $x_1, x_2, x_3, \dots x_t$ be the neighbors of x in Ω for $t = d_\Omega(i) - 1$ and let $y_1, y_2, y_3, \dots y_q$ be the neighbors of y for $q = d_\Omega(j) - 1$.

$$\begin{aligned}
GSM(\Omega - ij) - GSM(\Omega) \; = \; & -2\sqrt{d_\Omega(i)d_\Omega(j)(d_\Omega(i)^2 + d_\Omega(j))^2} \\
& + \sum_{r=1}^{t} \Big(2\sqrt{(d_\Omega(i)-1)d_\Omega(i_r)((d_\Omega(i)-1)^2 + d_\Omega(i_r)^2)} \\
& - 2\sqrt{d_\Omega(i)d_\Omega(i_r)(d_\Omega(i)^2 + d_\Omega(i_r)^2)} \Big) \\
& + \sum_{k=1}^{q} \Big(2\sqrt{(d_\Omega(j)-1)d_\Omega(j_k)((d_\Omega(j)-1)^2 + d_\Omega(j_k)^2)} \\
& - 2\sqrt{d_\Omega(j)d_\Omega(j_k)(d_\Omega(j)^2 + d_\Omega(j_k)^2)} \Big) \\
< \; & 0 \\
GSM(\Omega - ij) \; < \; & GSM(\Omega).
\end{aligned}$$

By above equality we can observe the first summand is the removal of edge ij in $(\Omega - ij)$.

Our finding on the maximum value of the GSM index is a straightforward Corollary of Theorem 3.1, which states that the value of the GSM index grows as additional edges are added. Specifically, because any n-vertex graph Ω may be converted into a complete graph by joining all pairs of non-adjacent vertices, we find that

$$GSM(\Omega) \leq GSM(K_n) = 2n\sqrt{2}(n-1)^2.$$

Theorem 3.4. *Let Ω be a graph with m edges and the maximum degree Δ. Then*

$$GSM(\Omega) \leq m\sqrt[3]{2}\Delta^2.$$

With equality if and only if Ω is Δ-regular graph.

Proof. In a graph Ω, $d_\Omega(i) \leq \Delta$ for each vertex, $i \in V(\Omega)$, we have that

$$2\sqrt{d_\Omega(i)d_\Omega(j)(d_\Omega(i)^2 + d_\Omega(j)^2)} \leq 2\sqrt{\Delta\Delta(\Delta^2+\Delta^2)} \leq \sqrt[3]{2}\Delta^2.$$

Then directly from eqn. (1.1), we have that

$$GSM(\Omega) \leq m\sqrt[3]{2}\Delta^2.$$

Equality holds if and only if $d_\Omega(i) = \Delta$ for each vertex $i \in V(\Omega)$.
Theorem 3.4 directly implies the following corollary.

Corollary 3.5. *Let Ω be a graph with n vertices and the maximum degree Δ. Then*

$$GSM(\Omega) = \begin{cases} n\Delta\sqrt[3]{2}, & \text{if } n\Delta \text{ is even} \\ \sqrt{2}\Delta^3(n-1) + (\Delta-1)\sqrt{\Delta(\Delta-1)(2\Delta^2-2\Delta+1)}. & \text{if } n\Delta \text{ is odd} \end{cases}$$

Proof. Since $2|E|$ is equal to the sum of the vertex degrees in a graph. The fact that the maximum vertex degree is Δ represents both that $m \leq \frac{n\Delta}{2}$ and that the $GSM(\Omega)$ contribution of each edge is at most $\Delta^2\sqrt[3]{2}$. Then

$$GSM(\Omega) \leq m\sqrt[3]{2}\Delta^2 \leq n\Delta^3\sqrt{2}.$$

Equality above is attained only if $m = \frac{n\Delta}{2}$ and Ω is Δ-regular graph, which is possible only if $n\Delta$ is even.
If $n\Delta$ is odd, then $m \leq \frac{n\Delta-1}{2}$ and atleast one vertex of Ω has degree atmost $\Delta - 1$ while remaining vertices have degree atmost Δ. Hence, atmost $\frac{(n-1)\Delta}{2}$ edges have the $GSM(\Omega)$ contribution equal to $\Delta^2\sqrt[3]{2}$, while the remaining $m - \frac{(n-1)\Delta}{2}$ edges have the $GSM(\Omega)$ contribution at most $2\sqrt{2\Delta^4 - 2\Delta^3 + 3\Delta^2 - \Delta}$. In total

$$\begin{aligned} GSM(\Omega) &\leq \frac{(n-1)\Delta}{2} \cdot \Delta^2 2\sqrt{2} + 2\left(m - \frac{(n-1)\Delta}{2}\right)\sqrt{\Delta(\Delta-1)(2\Delta^2-2\Delta+1)} \\ &\leq \sqrt{2}\Delta^3(n-1) + (\Delta-1)\sqrt{\Delta(\Delta-1)(2\Delta^2-2\Delta+1)}. \end{aligned}$$

Theorem 3.6. *For any graph* Ω

$$GSM(\Omega) \leq 2\sqrt{M_2(\Omega) \cdot F(\Omega)}.$$

With equality if and only if Ω *is regular.*
Proof. By setting $x_t = 2\sqrt{d_\Omega(i) \cdot d_\Omega(j)}$ and $y_t = \sqrt{d_\Omega(i)^2 + d_\Omega(j)^2}$ in Lemma 2.1, we get

$$\begin{aligned}
\left(\sum_{t=1}^{m} 2\sqrt{d_\Omega(i) \cdot d_\Omega(j)}\sqrt{d_\Omega(i)^2 + d_\Omega(j)^2}\right)^2 &\leq \sum_{t=1}^{m}(2\sqrt{d_\Omega(i) \cdot d_\Omega(j)}) \sum_{t=1}^{m} \sqrt{d_\Omega(i)^2 + d_\Omega(j)^2} \\
(GSM(\Omega))^2 &\leq 4\sum_{t=1}^{m}(d_\Omega(i) \cdot d_\Omega(j)) \sum_{t=1}^{m}(d_\Omega(i)^2 + d_\Omega(j)^2) \\
(GSM(\Omega))^2 &\leq 4 \cdot M_2(\Omega) \cdot F(\Omega) \\
GSM(\Omega) &\leq 2\sqrt{M_2(\Omega) \cdot F(\Omega)}.
\end{aligned}$$

With equality if and only if Ω is a regular graph.

Theorem 3.7. *Let* Ω *be a graph with* n *vertices and the minimum degree* δ*. Then*

$$GSM(\Omega) = \begin{cases} n\delta^3\sqrt{2}, & \text{if } n\delta \text{ is even} \\ 2(\delta+1)\sqrt{\delta(\delta+1)(2\delta^2+2\delta+1)} + (n\delta - 2\delta - 1)\sqrt{2}\delta^2. & \text{if } n\delta \text{ is odd} \end{cases}$$

Proof. Since $2|E|$ is equal to the sum of the vertex degrees in a graph. The fact that the minimum vertex degree is δ represents both that $m \geq \frac{n\delta}{2}$ and that the GSM contribution of each edge is at most $\delta^2\sqrt[3]{2}$. Then

$$GSM(\Omega) \geq m\sqrt[3]{2}\delta^2 \geq n\delta^3\sqrt{2}.$$

As $2m \geq n\delta$ holds in a graph in which every vertex has degree atleast δ. Note however the equality $2m = n\delta$ may hold only if $n\delta$ is even.
If $n\delta$ is odd then atleast one vertex of Ω has degree atleast $\delta + 1$, so that there are atleast $\delta + 1$ edges its contribution is atleast $2\sqrt{\delta(\delta+1)(2\delta^2+2\delta+1)}$ and the total number of edges is atleast $\frac{n\delta+1}{2}$. Hence

$$\begin{aligned}
GSM(\Omega) &\geq 2(\delta+1)\sqrt{\delta(\delta+1)(2\delta^2+2\delta+1)} + \left(\frac{n\delta+1}{2} - (\delta+1)\right)2\sqrt{2}\delta^2 \\
&\geq 2(\delta+1)\sqrt{\delta(\delta+1)(2\delta^2+2\delta+1)} + \frac{(n\delta-2\delta-1)2\sqrt{2}\delta^2}{2} \\
&\geq 2(\delta+1)\sqrt{\delta(\delta+1)(2\delta^2+2\delta+1)} + (n\delta-2\delta-1)\sqrt{2}\delta^2.
\end{aligned}$$

Theorem 3.8. *Let* Ω *be a graph with* p *pendant vertices and minimal non-pendant vertex degree* δ_1*. Then*

$$2\sqrt{2}\delta_1(m-p) + 2p\sqrt{\delta_1(\delta_1^2+1)} \leq GSM(\Omega) \leq 2\sqrt{2}\Delta(m-p) + 2p\sqrt{\Delta(\Delta^2+1)}.$$

With equality if and only if Ω is regular.

Proof. From the definition of $GSM(\Omega)$ index we have

$$\begin{aligned} GSM(\Omega) &= \sum_{ij\in E(\Omega),d_\Omega(i)=1} 2\sqrt{d_\Omega(j)(d_\Omega(j)^2+1)} \\ &+ \sum_{ij\in E(\Omega),d_\Omega(i)\neq 1} 2\sqrt{d_\Omega(i)d_\Omega(j)(d_\Omega(i)^2+d_\Omega(j)^2)} \end{aligned}$$

The edge contribution in the second sum can be written as

$$2\sqrt{d_\Omega(i)d_\Omega(j)(d_\Omega(i)^2+d_\Omega(j)^2)},$$

then we have

$$2\sqrt{2}\delta_1 \leq 2\sqrt{d_\Omega(i)d_\Omega(j)(d_\Omega(i)^2+d_\Omega(j)^2)} \leq 2\sqrt{2}\Delta$$

regarding the contributions in the first sum

$$2\sqrt{\delta_1(\delta_1^2+1)} \leq 2\sqrt{d_\Omega(j)(d_\Omega(j)^2+1)} \leq 2\sqrt{\Delta(\Delta^2+1)}$$

By summing over all edges of Ω, we get

$$2\sqrt{2}\delta_1(m-p)+2p\sqrt{\delta_1(\delta_1^2+1)} \leq GSM(\Omega) \leq 2\sqrt{2}\Delta(m-p)+2p\sqrt{\Delta(\Delta^2+1)}.$$

Equalities hold above if and only if $d_\Omega(i)=\Delta=\delta_1$ for each non pendant vertex $i\in V(\Omega)$. This shows that Ω is $(1,\Delta)$-semiregular if $p>0$ and Ω is regular if $p=0$.

Theorem 3.8 directly implies the following corollary.

Corollary 3.9. *Let Ω be a graph without pendant vertices. Then*

$$2\sqrt{2}m\delta \leq GSM(\Omega) \leq 2\sqrt{2}m\Delta.$$

Proof. By inserting $p=0$ and $\delta_1=0$ in Theorem 3.8, we get the desired result.

Theorem 3.10. *For any graph Ω*

$$2\cdot\delta\cdot SO(\Omega) \leq GSM(\Omega) \leq 2\cdot\Delta\cdot SO(\Omega).$$

With equality if and only if Ω *is regular graph.*
Proof. By using definition of $GSM(\Omega)$ index we have

$$\begin{aligned} GSM(\Omega) &= \sum_{jk\in E(\Omega)} 2\sqrt{d_\Omega(j)\cdot d_\Omega(k)(d_\Omega(j)^2+d_\Omega(k)^2)} \\ &\leq \sum_{jk\in E(\Omega)} 2\sqrt{\Delta\cdot\Delta\cdot(d_\Omega(j)^2+d_\Omega(k)^2)} \\ &\leq 2\cdot\Delta \sum_{jk\in E(\Omega)} \sqrt{(d_\Omega(j)^2+d_\Omega(k)^2)} \\ &\leq 2\cdot\Delta\cdot SO(\Omega). \end{aligned} \tag{3.1}$$

By using definition of $GSM(\Omega)$ and similar argument for δ, we get

$$GSM(\Omega) \geq \quad 2\cdot\delta\cdot SO(\Omega). \tag{3.2}$$

By Eqn. (3.1) and (3.2), we get

$$2\cdot\delta\cdot SO(\Omega) \leq GSM(\Omega) \leq 2\cdot\Delta\cdot SO(\Omega).$$

Equality holds if and only if $\delta = \Delta$, i.e., Ω is a regular graph.

Theorem 3.11. *If* Ω *is a graph with* m *edges, then*

$$GSM(\Omega) \geq \frac{2R_{\frac{1}{2}}(\Omega)SO(\Omega)}{m}.$$

With equality if and only if Ω *is regular graph.*
Proof. By setting $x_t = 2\sqrt{d_\Omega(i)\cdot d_\Omega(j)}$ and $y_t = \sqrt{d_\Omega(i)^2+d_\Omega(j)^2}$ in Lemma. 2.2, we have

$$\begin{aligned} &\left(\frac{1}{m}\sum_{ij\in E(\Omega)} 2\sqrt{d_\Omega(i)\cdot d_\Omega(j)}\right)\left(\frac{1}{m}\sum_{ij\in E(\Omega)} \sqrt{d_\Omega(i)^2+d_\Omega(j)^2}\right) \\ &\leq \left(\frac{1}{m}\sum_{ij\in E(\Omega)} 2\sqrt{d_\Omega(i)\cdot d_\Omega(j)(d_\Omega(i)^2+d_\Omega(j)^2)}\right) \\ &\left(\frac{2R_{\frac{1}{2}}(\Omega)}{m}\right)\left(\frac{SO(\Omega)}{m}\right) \leq \frac{GSM(\Omega)}{m} \\ &\frac{2R_{\frac{1}{2}}(\Omega)\cdot SO(\Omega)}{m} \leq GSM(\Omega) \\ &GSM(\Omega) \geq \frac{2R_{\frac{1}{2}}(\Omega)\cdot SO(\Omega)}{m}. \end{aligned}$$

With equality if and only if Ω is regular graph.

Theorem 3.12. *If Ω is a graph with minimum vertex degree δ, then*

1. $2\delta SO(\Omega) \leq GSM(\Omega) \leq 2\Delta SO(\Omega)$,
2. $2\sqrt{2}\delta R_{\frac{1}{2}}(\Omega) \leq GSM(\Omega) \leq 2\sqrt{2}\Delta R_{\frac{1}{2}}$.

With equality if and only if Ω is regular graph.

Proof.

(1) The proof for this result is we proven in Theorem 3.10, here we obtain another way of proof, by setting $s_k = 2\sqrt{d_\Omega(i)\cdot d_\Omega(j)}$, $y_n = \sqrt{d_\Omega(i)^2 + d_\Omega(j)^2}$, $m = \min_{1\leq k\leq m} s_k = 2\delta, M = \max_{1\leq k\leq m} s_k = 2\Delta, (k = 1, 2, \ldots m)$ in Lemma 2.3, we get

$$\begin{aligned}
2\delta\sum_{ij\in\Omega}\sqrt{d_\Omega(i)^2 + d_\Omega(j)^2} &\leq \sum_{ij\in\Omega} 2\sqrt{d_\Omega(i)\cdot d_\Omega(j)}\sqrt{d_\Omega(i)^2 + d_\Omega(j)^2}\\
&\leq 2\Delta\sum_{ij\in\Omega}\sqrt{d_\Omega(i)^2 + d_\Omega(j)^2}\\
2\delta SO(\Omega) &\leq \sum_{ij\in\Omega} 2\sqrt{d_\Omega(i)\cdot d_\Omega(j)(d_\Omega(i)^2 + d_\Omega(j)^2)}\\
&\leq 2\Delta SO(\Omega)\\
2\delta SO(\Omega) &\leq GSM(\Omega) \leq 2\Delta SO(\Omega).
\end{aligned}$$

(2) By setting $s_k = \sqrt{d_\Omega(i)^2 + d_\Omega(j)^2}$, $y_n = 2\sqrt{d_\Omega(i)\cdot d_\Omega(j)}$, $m = \min_{1\leq k\leq m} s_k = \sqrt{2}\delta, M = \max_{1\leq k\leq m} s_k = \sqrt{2}\Delta, (k = 1, 2, \ldots m)$ in Lemma 2.3, we get

$$\begin{aligned}
\sqrt{2}\delta\sum_{ij\in E(\Omega)} 2\sqrt{d_\Omega(i)\cdot d_\Omega(j)} &\leq \sum_{ij\in\Omega}\sqrt{d_\Omega(i)^2 + d_\Omega(j)^2}2\sqrt{d_\Omega(i)\cdot d_\Omega(j)}\\
&\leq \sqrt{2}\Delta\sum_{ij\in E(\Omega)} 2\sqrt{d_\Omega(i)\cdot d_\Omega(j)}\\
2\sqrt{2}\delta R_{\frac{1}{2}}(\Omega) &\leq \sum_{ij\in\Omega} 2\sqrt{d_\Omega(i)\cdot d_\Omega(j)(d_\Omega(i)^2 + d_\Omega(j)^2)}\\
&\leq 2\sqrt{2}\Delta R_{\frac{1}{2}}(\Omega)\\
2\sqrt{2}\delta R_{\frac{1}{2}}(\Omega) &\leq GSM(\Omega) \leq 2\sqrt{2}\Delta R_{\frac{1}{2}}(\Omega).
\end{aligned}$$

Equality holds if and only if $d_\Omega(i) = d_\Omega(i)$ for each edge $ij \in \Omega$, i.e., if and only if Ω is a regular graph.

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