South East Asian J. of Mathematics and Mathematical Sciences
Vol. 19, No. 3 (2023), pp. 359-370

DOI: 10.56827/SEAJMMS.2023.1903.28 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

ABSOLUTE MEAN GRACEFUL LABELING IN THE CONTEXT OF m-SPLITTING AND DEGREE SPLITTING GRAPHS

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(**Received:** *May 07, 2023* **Accepted:** *Nov. 26, 2023* **Published:** *Dec. 30, 2023)*

Abstract: A graph G with q edges is said to be absolute mean graceful if there is a one-to-one function f from $V(G)$ to the set $\{0, \pm1, \pm2, \pm3, \ldots, \pm q\}$ such that when each edge xy is assigned the label $\lceil \frac{|f(x)-f(y)|}{2} \rceil$, then the resulting edge labels are distinct. In this paper, the absolute mean graceful labeling of m-splitting and degree splitting graphs of some graphs are investigated.

Keywords and Phrases: Absolute mean graceful labeling, m-splitting graph, splitting graph, degree splitting graph.

2020 Mathematics Subject Classification: 05C78, 05C76.

1. Introduction and Preliminaries

All the graphs $G = (V(G), E(G))$ considered in this paper have p vertices and q edges and are simple, finite, connected and undirected. We follow Harary [8] for terminologies and notations related to graph theory.

Assigning values to the vertices or edges of graphs under certain conditions is referred to as *graph labeling*. For various graph labeling problems and references we follow the dynamic survey by Gallian [6].

Labeled graphs have wide range of applications. Some applications of labeled graphs are found in [4, 11, 16, 17, 18]. The concept of labeled graphs is originated by Rosa [14] to counter the conjecture due to Ringel [13]. Rosa [14] introduced the β-valuation. It was named graceful by Golomb [7]. Several variants of graceful labeling are introduced like super graceful, odd graceful, k-graceful labeling etc. Kaneria and Chudasama [9] introduced absolute mean graceful labeling with the flavor of graceful labeling, which is defined as follows.

Definition 1.1. [9] *A function f is said to be an absolute mean graceful labeling of a graph G with q edges, if f is a one-to-one function from $V(G)$ to the set $\{0, \pm 1, \pm 2, \pm 3, \ldots, \pm q\}$ such that when each edge xy of G has assigned the label $f^*(xy) = \lceil \frac{|f(x)-f(y)|}{2} \rceil$, the set of resulting edge labels is $\{1, 2, 3, \ldots, q\}$. A graph G that admits absolute mean graceful labeling is called an absolute mean graceful graph.*

From the above definition we observe that,

- For any absolute mean graceful graph, the edge label q can be produced either when the labeled vertices q and $-q$ are adjacent or the labeled vertices q and $-(q-1)$ are adjacent.

- If a set of vertex labels $(x_1, x_2, \ldots, x_j)$ form an absolute mean graceful labeling for G, then so do the labels $(-x_1, -x_2, \ldots, -x_j)$, where $j = 1, 2, \ldots, p$.

Kaneria and Chudasama [5, 9] investigated absolute mean graceful labeling in the context of duplication of graph elements and some graphs families. Kaneria *et al.* [10] have proved various absolute mean graceful graphs in the context of path union of graphs. Akbari *et al.* [2] have proved several jelly fish and jewel related graphs are absolute mean graceful. While the same authors discussed absolute mean graceful labeling of subdivision graphs of various graphs in [3].

In this paper we discuss absolute mean graceful labeling in the context of m-splitting and degree splitting graphs of some graphs. First we will provide some definitions which are useful for the discussion.

Definition 1.2. [6] *The bistar is a graph formed by joining the apex vertices of two copies of $K_{1,n}$ by an edge, and it is denoted by $B_{n,n}$.*

Definition 1.3. [15] *For a graph G, the splitting graph denoted by $S'(G)$ is formed by adding a new vertex u' to each vertex u so that u' is adjacent to every vertex that is adjacent to u in G.*

Definition 1.4. [1] *The m-splitting graph of a graph G denoted by $Spl_m(G)$ is formed from G by adding new m vertices, say $w_1, w_2, \ldots, w_m$ to each vertex w of a graph G, so that w_j, $1 \leq j \leq m$ is adjacent to every vertex that is adjacent to w in G.*
By the above definition, 1-splitting graph is a splitting graph.

Following are some observations regarding m-splitting graphs analogous to splitting graphs [15].

- $Spl_m(G)$ has $p = p_1(1+m)$ vertices and $q = q_1(1+2m)$ edges, where p_1 and q_1 denotes the number of vertices and edges of G respectively.
- If G has k triangles then $Spl_m(G)$ has $(3m+1)k$ triangles.
- Let $H = Spl_m(G)$ and $w \in V(G)$ and $w_1, w_2, \ldots, w_m$ are added vertices corresponding to w. Then $d_H(w) = (m+1)d_G(w)$ and $d_H(w_j) = d_G(w)$, for each $j = 1, 2, \ldots, m$.

Definition 1.5. [12] *Consider a graph G with $V(G) = H_1 \cup H_2 \cup \ldots H_l \cup R$ where each set H_i is a set of all vertices having same degree with atleast two vertices and $R = V(G) \setminus \bigcup_{i=1}^{l} H_i$. The degree splitting graph of a graph G denoted by $DS(G)$ is formed from G by adding vertices $w_1, w_2, \ldots, w_l$ and joining to each vertex of H_i for $1 \le i \le l$.*

2. Main Results

Theorem 2.1. *$Spl_m(P_n)$ is an absolute mean graceful graph for all $m \ge 1$ and $n \ge 2$.*

Proof. Consider P_n with vertex set $\{w_i : 1 \le i \le n\}$. To obtain $Spl_m(P_n)$ add $w_1^j, w_2^j, \ldots, w_n^j$ vertices corresponding to $w_1, w_2, \ldots, w_n$, where $1 \le j \le m$. If $G = Spl_m(P_n)$ then $|V(G)| = p = n(1+m)$ and $|E(G)| = q = (n-1)(1+2m)$. To define vertex labeling function $f : V(G) \to \{0, \pm 1, \pm 2, \pm 3, \ldots, \pm q\}$, consider following two cases.

Case-1: n is odd

$$f(w_i) = (-1)^{i+1}(q-(i-1));\ 1 \le i \le n,$$

$$f(w_i^j) = \begin{cases} q - 2n + 3 - 3(i-1) - (4n-4)(j-1); & i = 1, 3, \ldots, n,\ 1 \le j \le m, \\ -q + 2n + 3(i-2) + (4n-4)(j-1); & i = 2, 4, \ldots, n-1,\ 1 \le j \le m. \end{cases}$$

Case-2: n is even

$$f(w_i) = (-1)^{i+1}(q-(i-1));\ 1 \le i \le n,$$

$$f(w_i^j) = \begin{cases} q - 2n + 3 - 3(i-1) - (4n-4)(j-1); & i = 1, 3, \ldots, n-1,\ 1 \le j \le m, \\ -q + 2n + 3(i-2) + (4n-4)(j-1); & i = 2, 4, \ldots, n,\ 1 \le j \le m. \end{cases}$$

The vertex labeling function f defined above is one-to-one in both cases and induced edge labels are $1, 2, 3, \ldots, q$. Hence $Spl_m(P_n)$ is an absolute mean graceful graph for all $m \geq 1$ and $n \geq 2$.

Illustration 2.1. Absolute mean graceful labeling of $Spl_3(P_6)$ is shown in the Figure 1.

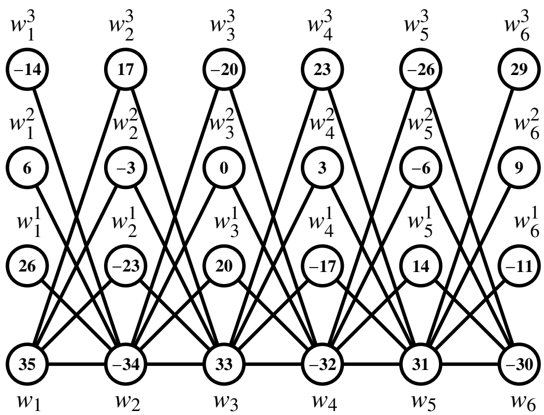

Figure 1: $Spl_3(P_6)$ and its absolute mean graceful labeling

Corollary 2.1. *$S'(P_n)$ is an absolute mean graceful graph for all $n \geq 2$.*
Proof. If we take $m = 1$ then by Theorem 2.1 the result holds.

Theorem 2.2. *$Spl_m(K_{1,n})$ is an absolute mean graceful graph for all $m, n \geq 1$.*
Proof. Consider $K_{1,n}$ with vertex set $\{u_0, u_i : 1 \leq i \leq n\}$, where u_0 is called apex vertex and $u_1, u_2, \ldots, u_n$ are pendant vertices. To obtain $Spl_m(K_{1,n})$ add $u_0^j, u_1^j, u_2^j, \ldots, u_n^j$ vertices corresponding to $u_0, u_1, u_2, \ldots, u_n$, where $1 \leq j \leq m$.
If $G = Spl_m(K_{1,n})$ then $|V(G)| = p = (n+1)(1+m)$ and $|E(G)| = q = n(1+2m)$.
We define vertex labeling function $f : V(G) \to \{0, \pm1, \pm2, \pm3, \ldots, \pm q\}$ by

$$\begin{aligned} f(u_0) &= q, \\ f(u_{1+i}) &= -n + 2i; \quad 0 \leq i \leq n-1, \\ f(u_0^j) &= n + 2n(j-1); \quad 1 \leq j \leq m, \\ f(u_i^j) &= -2nm + 2nj - 3n + 2i - 2; \quad 1 \leq i \leq n, 1 \leq j \leq m. \end{aligned}$$

The vertex labeling function f defined above is one-to-one and induced edge labels are $1, 2, 3, \ldots, q$. Hence $Spl_m(K_{1,n})$ is an absolute mean graceful graph for all $m, n \geq 1$.

Illustration 2.2. Absolute mean graceful labeling of $Spl_3(K_{1,4})$ is shown in the Figure 2.

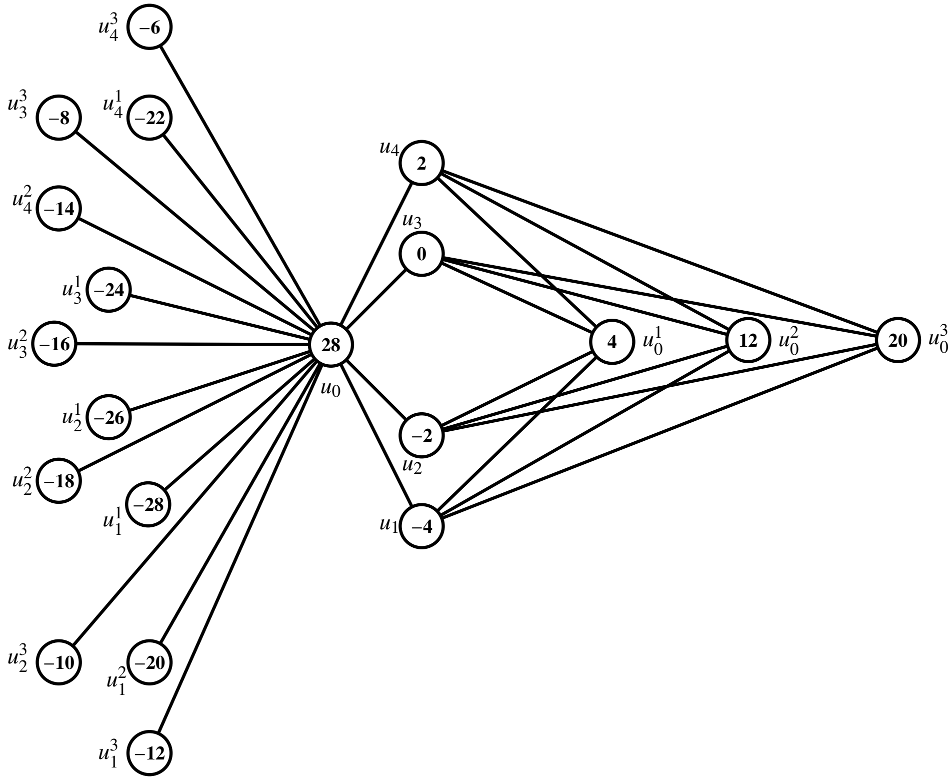

Figure 2: $Spl_3(K_{1,4})$ and its absolute mean graceful labeling

Corollary 2.2. *$S'(K_{1,n})$ is an absolute mean graceful graph for all $n \geq 1$.*
Proof. If we take $m = 1$ then by Theorem 2.2 the result holds.

Theorem 2.3. *$S'(B_{n,n})$ is an absolute mean graceful graph for all $n \geq 2$.*
Proof. Consider $V(B_{n,n}) = \{u_0, u_i, v_0, v_i : 1 \leq i \leq n\}$, where v_i and u_i are pendant vertices, where $1 \leq i \leq n$. Suppose u_0', u_i', v_0' and v_i' be the vertices corresponding to u_0, u_i, v_0 and v_i, where $1 \leq i \leq n$, which are added to obtain $S'(B_{n,n})$.
If $G = S'(B_{n,n})$ then $|V(G)| = p = 4 + 4n$ and $|E(G)| = q = 3 + 6n$.
To define vertex labeling function $f : V(G) \to \{0, \pm 1, \pm 2, \pm 3, \ldots, \pm q\}$, consider following two cases.

Case-1: n is even

$$\begin{aligned}
f(u_0) &= 2n+5,\\
f(v_0) &= 6n+3,\\
f(u'_0) &= 2n+3,\\
f(v'_0) &= 6n+1,\\
f(u'_1) &= -2n+3,\\
f(u'_{2+i}) &= 4n+3-2i; \quad 0 \le i \le n-2,\\
f(u_{1+i}) &= -2n+11+4i; \quad 0 \le i \le \frac{n-4}{2},\\
f(u_{\frac{n}{2}+i}) &= -4n-1+4i; \quad 0 \le i \le \frac{n}{2},\\
f(v'_i) &= -6n-2+2(i-1); \quad 1 \le i \le n,\\
f(v_i) &= -4n-3+4(i-1); \quad 1 \le i \le n.
\end{aligned}$$

Case-2: n is odd

$$\begin{aligned}
f(u_0) &= 2n+5,\\
f(v_0) &= 6n+3,\\
f(u'_0) &= 2n+3,\\
f(v'_0) &= 6n+1,\\
f(u'_1) &= f(v'_0)-2,\\
f(u'_{2+i}) &= 4n+3-2i; \quad 0 \le i \le n-2,\\
f(u_{1+i}) &= -2n+13+4i; \quad 0 \le i \le \frac{n-5}{2},\\
f(u_{\frac{n-1}{2}+i}) &= -4n-1+4i; \quad 0 \le i \le \frac{n+1}{2},\\
f(v'_i) &= -6n-2+2(i-1); \quad 1 \le i \le n,\\
f(v_i) &= -4n-3+4(i-1); \quad 1 \le i \le n.
\end{aligned}$$

The vertex labeling function f defined above is one-to-one in both cases and induced edge labels are $1, 2, 3, \ldots, q$. Hence $S'(B_{n,n})$ is an absolute mean graceful graph for all $n \ge 2$.

Illustration 2.3. Absolute mean graceful labeling of $S'(B_{5,5})$ is shown in the Figure 3.

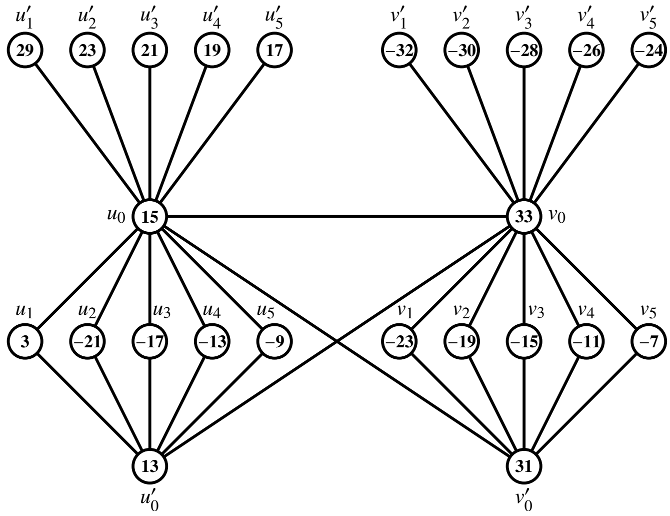

Figure 3: $S'(B_{5,5})$ and its absolute mean graceful labeling

Theorem 2.4. *$DS(P_n)$ is an absolute mean graceful graph for all $n \geq 4$.*
Proof. Consider P_n with vertex set $\{x_i : 1 \leq i \leq n\}$. Here $V(P_n) = H_1 \cup H_2$,where $H_1 = \{x_1, x_n\}$ and $H_2 = \{x_i : 2 \leq i \leq n-1\}$. To construct $DS(P_n)$ from P_n, we add vertices w_1 and w_2 corresponding to H_1 and H_2 respectively. If $G = DS(P_n)$ then $|V(G)| = p = 2 + n$ and $|E(G)| = q = 2n - 1$.
To define vertex labeling function $f : V(G) \to \{0, \pm1, \pm2, \pm3, \ldots, \pm q\}$, consider following two cases.

Case-1: n is odd

$$f(w_1) = 2n - 1,$$
$$f(w_2) = -3,$$
$$f(x_i) = \begin{cases} 2n - 1 - 2i; & i = 1, 3, \ldots, n, \\ -2n + 2i - 3; & i = 2, 4, \ldots, n-1. \end{cases}$$

Case-2: n is even

$$f(w_1) = 2n - 1,$$
$$f(w_2) = -5,$$
$$f(x_i) = \begin{cases} 2n - 1 - 2i; & i = 1, 3, \ldots, n-1, \\ -2n + 2i - 3; & i = 2, 4, \ldots, n. \end{cases}$$

The vertex labeling function f defined above is one-to-one in both cases and induced edge labels are $1, 2, 3, \ldots, q$. Hence $DS(P_n)$ is an absolute mean graceful graph for all $n \geq 4$.

Illustration 2.4. Absolute mean graceful labeling of $DS(P_7)$ is shown in the Figure 4.

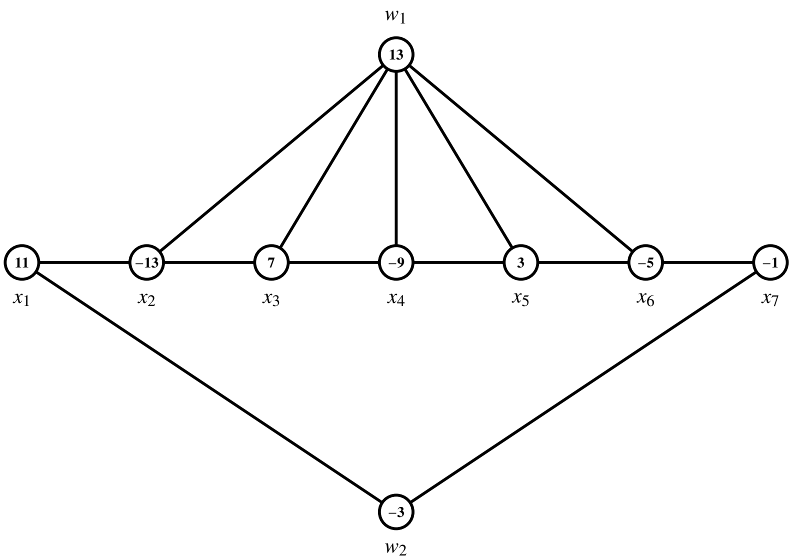

Figure 4: $DS(P_7)$ and its absolute mean graceful labeling

Theorem 2.5. *$DS(B_{n,n})$ is an absolute mean graceful graph for all $n \geq 2$.*
Proof. Let $V(B_{n,n}) = H_1 \cup H_2$, where $H_1 = \{u, v\}$ and $H_2 = \{v_i, u_i : 1 \leq i \leq n\}$. To construct $DS(B_{n,n})$ from $B_{n,n}$, we add vertices w_1 and w_2 corresponding to H_1 and H_2 respectively. If $G = DS(B_{n,n})$ then $|V(G)| = p = 4+2n$ and $|E(G)| = q = 3 + 4n$.
We define vertex labeling function $f : V(G) \to \{0, \pm 1, \pm 2, \pm 3, \ldots, \pm q\}$ by

$$\begin{aligned} f(u) &= 4n + 1, \\ f(v) &= 4n - 3, \\ f(w_1) &= 4n + 3, \\ f(w_2) &= -5, \\ f(u_i) &= -4n + 4i - 7; \quad 1 \leq i \leq n, \\ f(v_i) &= 4n - 4i - 1; \quad 1 \leq i \leq n. \end{aligned}$$

The vertex labeling function f defined above is one-to-one and induced edge labels are $1, 2, 3, \dots, q$. Hence $DS(B_{n,n})$ is an absolute mean graceful graph for all $n \geq 2$.

Illustration 2.5. Absolute mean graceful labeling of $DS(B_{6,6})$ is shown in the Figure 5.

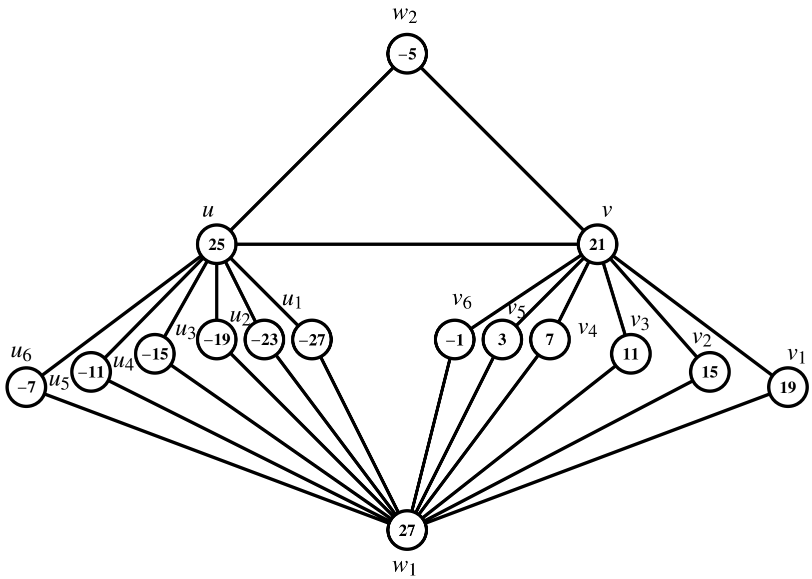

Figure 5: $DS(B_{6,6})$ and its absolute mean graceful labeling

Theorem 2.6. *$DS(K_{2,n})$ is an absolute mean graceful graph for all $n \geq 2$.*
Proof. Let $V(K_{2,n}) = H_1 \cup H_2$, where $H_1 = \{u_1, u_2\}$ and $H_2 = \{v_i : 1 \leq i \leq n\}$. Now, we consider following two cases.

Case-1: $n = 2$

In order to obtain $DS(K_{2,2})$ from $K_{2,2}$, we add a vertex w.

We define vertex labeling function $f : V(G) \to \{0, \pm1, \pm2, \pm3, \dots, \pm q\}$ by

$$f(w) = -6,$$
$$f(u_1) = 8,$$
$$f(u_2) = -8,$$
$$f(v_1) = 0,$$
$$f(v_2) = 4.$$

Case-2: $n > 2$

In order to obtain $DS(K_{2,n})$ from $K_{2,n}$, we add vertices w_1 and w_2 corresponding to H_1 and H_2 respectively. If $G = DS(K_{2,n})$ then $|V(G)| = p = 4 + n$ and $|E(G)| = q = 2 + 3n$.

We define vertex labeling function $f : V(G) \to \{0, \pm1, \pm2, \pm3, \ldots, \pm q\}$ by

$$\begin{aligned} f(w_1) &= 3n + 2, \\ f(u_1) &= -f(w_1), \\ f(u_2) &= -3n, \\ f(v_i) &= 3n + 2 - 4i; \quad 1 \le i \le n, \\ f(w_2) &= n - 1. \end{aligned}$$

The vertex labeling function f defined above is one-to-one in both cases and induced edge labels are $1, 2, 3, \ldots, q$. Hence $DS(K_{2,n})$ is an absolute mean graceful graph for all $n \ge 2$.

Illustration 2.6. Absolute mean graceful labeling of $DS(K_{2,4})$ is shown in the Figure 6.

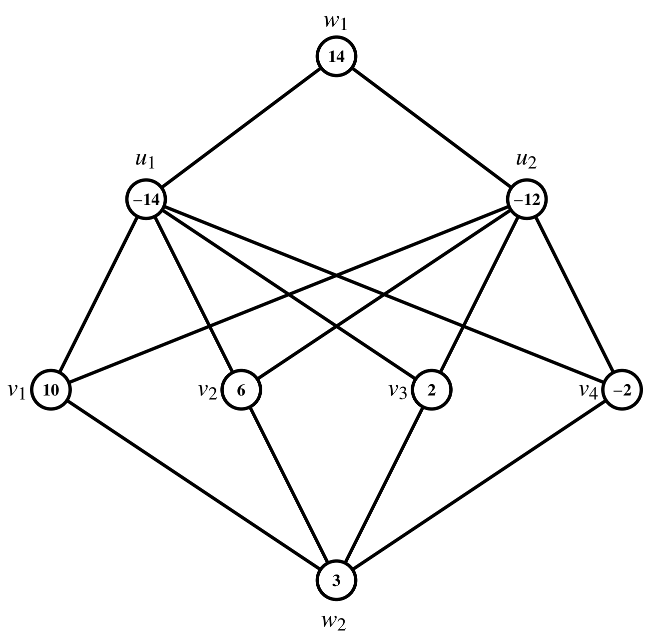

Figure 6: $DS(K_{2,4})$ and its absolute mean graceful labeling

3. Conclusion

We have investigated new results on absolute mean graceful labeling in this paper. Obtaining similar results for other graph labeling techniques and in the context of different graph operations remains an open area of research.

Acknowledgement

Thanks are owed to the anonymous referees for their kind and constructive suggestions.

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